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Hodge-to-singular correspondence for reduced curves

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Abstract. We study the summands of the decomposition theorem for the Hitchin system for GL_n, in arbitrary degree, over the locus of reduced spectral curves. A key ingredient is a new correspondence between these summands and the topology of hypertoric quiver varieties. In contrast to the case of meromorphic Higgs fields, the intersection cohomology groups of moduli spaces of regular Higgs bundles depend on the degree. We describe this dependence.

Keywords: Hitchin fibration, Higgs bundles, perverse sheaf, decomposition theorem, quiver varieties, compactified Jacobians.

Let C be a compact Riemann surface of genus g with canonical bundle ω_C. Let d and n be integers with $n > 1$.

The *Dolbeault moduli space* $M(n,d)$ is the coarse moduli space which parametrises S-equivalence classes of semistable Higgs bundles on C of rank n and degree d, i.e. polystable pairs $(\mathcal{E}, \phi)$ consisting of a vector bundle $\mathcal{E}$ of rank n and degree d and a section $\phi \in H^0(C, \mathcal{E}nd(\mathcal{E}) \otimes \omega_C)$, called a *Higgs field*; see [79]. The Dolbeault moduli space is equipped with a projective fibration called the *Hitchin fibration*,

$$\begin{aligned} \chi(n,d)\colon M(n,d) &\to A_n := \bigoplus_{i=1}^{n} H^0(C, \omega_C^{\otimes i}), \\ (\mathcal{E}, \phi) &\mapsto \mathrm{char}(\phi) = (\mathrm{tr}(\phi), \mathrm{tr}(\Lambda^2 \phi), \dots, \mathrm{tr}(\Lambda^i \phi), \dots, \det(\phi)), \end{aligned} \tag{1}$$

which assigns to $(\mathcal{E}, \phi)$ the characteristic polynomial $\mathrm{char}(\phi)$ of the Higgs field ϕ.

The cohomology of $M(n,d)$ has been extensively studied in the literature, especially under the assumption that n and d are coprime, when $M(n,d)$ is smooth; see for instance [17, 20, 33, 36, 38, 39, 41, 43, 63, 63]. Recently, studies about the intersection cohomology $IH^*(M(n,d), \mathbb{Q})$ of singular Dolbeault moduli spaces have started to emerge; see for

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instance [13, 14, 27, 28, 48, 57–60, 75].[1] From this viewpoint, the decomposition theorem for the Hitchin map is a key tool to investigate the intersection cohomology of $M(n,d)$: it allows us to decompose $IH^*(M(n,d),\mathbb{Q})$ into building blocks, which are cohomology of some perverse sheaves on A_n.

In this article, we establish a correspondence – which we find surprising – between the summands of the decomposition theorem for the Hitchin map of $M(n,d)$ and the singularities of $M(n,0)$; see Theorem 1.1. We call it the *Hodge-to-singular correspondence*. This correspondence offers a new perspective on the cohomology of Dolbeault moduli spaces, even in the smooth case. What is surprising is that this relation is not realised via a direct geometric correspondence between $M(n,d)$ and $M(n,0)$, like a deformation or an alteration, but, as we explain below, by comparing the Hodge theory of $M(n,d)$ and the singularity theory of $M(n,0)$ on the common base of the Hitchin fibration.

The Hodge-to-singular correspondence gives a formula expressing the dependence of the intersection cohomology of $M(n,d)$ on the degree d. This should be compared with the degree independence of the cohomology of spaces of meromorphic Higgs bundles [58, Theorems 0.1, 0.4]. It also extends the main theorem of [18] to arbitrary degree, and provides a conceptual and elegant explanation for the occurrence of the cographic matroid in [18, Section 6]. Further, it provides new evidence for Davison's conjectures [13, Conjecture 5.6, Remark 5.10], now proved in [15, Theorems 1.42, 1.43, Corollary 14.9], which however appeared after the first version of this paper.

Our correspondence suggests a unifying approach to the study of smooth and singular Dolbeault moduli spaces that may be useful to tackle the topological mirror symmetry conjecture [38, 59] and the P = W conjecture [17, 19] simultaneously in both settings.

In a different context, similar ideas have already been successfully employed to compute the Hodge numbers of the exceptional O'Grady 10 example of compact hyperkähler manifolds, out of the Hodge numbers of the Hilbert scheme of five points on a K3 surface; see [24]. Indeed, both spaces can be interpreted as special moduli spaces of sheaves on K3 surfaces corresponding to different degrees d (or rather Euler characteristics), and the computation of their Betti numbers can be reduced to determining how their cohomology depends on d. In fact, the Hodge-to-singular correspondence, formulated here for Dolbeault moduli spaces, can also be stated for Mukai systems on the moduli spaces of sheaves on K3 or abelian surfaces; see Section 10.3.

1. Main results

1.1. Hodge-to-singular correspondence

In order to state our main results, we need to recall some features of the singularities of $M(n,d)$. A point of $M(n,d)$ corresponds to a strictly polystable Higgs bundle

$$(\mathcal{E}_1,\phi_1)^{\oplus m_1}\oplus\cdots\oplus(\mathcal{E}_r,\phi_r)^{\oplus m_r},$$

[1]The lists of works are not intended to be exhaustive at all. We limit ourselves to listing a series of selected papers that inspired the current article.

where $(\mathcal{E}_i, \phi_i)$ are distinct stable Higgs bundles of slope $\deg \mathcal{E}_i/\operatorname{rank} \mathcal{E}_i = d/n$, and it is singular if $(r, m_r) \neq (1,1)$. In particular, the r-uples of positive integers $\underline{m} = \{m_i\}$ and $\underline{n} = \{n_i\} := \{\operatorname{rank} \mathcal{E}_i\}$ satisfy the relations

$$n = \sum_{i=1}^{r} m_i n_i \qquad n_i \cdot \frac{d}{n} = \deg \mathcal{E}_i \in \mathbb{Z}. \tag{2}$$

For $\underline{m}, \underline{n} \in \mathbb{Z}^r_{>0}$ satisfying (2), define $M^\circ_{\underline{m},\underline{n}}(d)$ to be the locus of polystable Higgs bundles in $M(n,d)$ of multirank $\underline{n}$ and multiplicity $\underline{m}$, and denote by $M_{\underline{m},\underline{n}}(d)$ its closure in $M(n,d)$. The loci $M^\circ_{\underline{m},\underline{n}}(d)$ are the strata of a complex Whitney stratification of $M(n,d)$,

$$M(n,d) = \bigsqcup_{\underline{m},\underline{n}} M^\circ_{\underline{m},\underline{n}}(d), \tag{3}$$

and an analytic normal slice $W_{\underline{m},\underline{n}}$ through $M^\circ_{\underline{m},\underline{n}}(d)$ is isomorphic to a Nakajima quiver variety; see [60, proof of Proposition 2.10]. The Whitney stratification (3) induces a filtration of the base A_n of the Hitchin fibration by closed subsets $S_{\underline{m},\underline{n}} := \chi(n,0)(M_{\underline{m},\underline{n}}(0))$, and a stratification

$$A_n = \bigsqcup S^\circ_{\underline{m},\underline{n}} \quad \text{with} \quad S^\circ_{\underline{m},\underline{n}} := S_{\underline{m},\underline{n}} \setminus \bigcup_{S_{\underline{m}',\underline{n}'} \subsetneq S_{\underline{m},\underline{n}}} S_{\underline{m}',\underline{n}'}. \tag{4}$$

Alternatively, $S^\circ_{\underline{m},\underline{n}}$ is the locus of characteristic polynomials whose irreducible factors have degree n_i and multiplicity m_i, i.e. $a \in S^\circ_{\underline{m},\underline{n}}$ decomposes as $a = \prod_{i=1}^r a_i^{m_i}$, where a_i is an irreducible polynomial of degree n_i. The locus $S_{\underline{m},\underline{n}}$ is the closure of $S^\circ_{\underline{m},\underline{n}}$ in A_n. Let $A_n^{\mathrm{red}} := \bigsqcup_{\underline{m}=(1,\ldots,1)} S^\circ_{\underline{m},\underline{n}} \subset A_n$ be the open subset of reduced characteristic polynomials. Define

$$M(n,d)^{\mathrm{red}} := \chi(n,d)^{-1}(A_n^{\mathrm{red}}) \quad \text{and} \quad M_{\underline{m},\underline{n}}(d)^{\mathrm{red}} := M_{\underline{m},\underline{n}}(d) \cap M(n,d)^{\mathrm{red}}.$$

When $\underline{m} = \{1, \ldots, 1\}$, we abbreviate the notation by omitting the subscript $\underline{m}$, e.g. $S_{\underline{n}} := S_{\underline{m},\underline{n}}$. Remarkably, $W_{\underline{n}}$ admits diffeomorphic symplectic resolutions $\widetilde{W}_{\underline{n}}$, whose top cohomology $H^{\dim W_{\underline{n}}}(\widetilde{W}_{\underline{n}}, \mathbb{Q})$ is a representation of the fundamental group of $S^\circ_{\underline{n}}$; see Proposition 4.11 and Theorem 7.4.[2] Denote by $\mathcal{L}_{\underline{n}}$ the corresponding local system on $S^\circ_{\underline{n}}$ (which carries a pure Hodge structure of weight zero).

The Hodge-to-singular correspondence is a splitting of the complex of mixed Hodge modules $R\chi(n,e)_*\mathbb{Q}_{M(n,e)}|_{A_n^{\mathrm{red}}}$ whose direct summands are controlled by the topology of the singularities $W_{\underline{n}}$.

For an integer d let $\mathcal{P}_d$ be the set of partitions $\underline{n} = \{n_i\}$ of n such that $n_i d/n \in \mathbb{Z}$ for every i. Note that this set is in bijection with the partitions of $q = \gcd(n,d)$, where the correspondence exchanges the partition $\{a_i\}$ of q with the partition $\{a_i n/q\} \in \mathcal{P}_d$.

[2] In the following, we identify $\widetilde{W}_{\underline{n}}$ with the hypertoric quiver variety $Y(\Gamma^{\pm}_{\underline{n}}, \theta)$. For convenience, we do not introduce hypertoric quiver varieties here, and we postpone their definition to Section 4.

Theorem 1.1 (Hodge-to-singular correspondence). *Let $\langle\bullet\rangle = [-2\bullet](-\bullet)$. Let d and e be integers, with e coprime to n. There is an isomorphism in $D^b MHM_{\mathrm{alg}}(A_n^{\mathrm{red}})$ or in $D^b(A_n^{\mathrm{red}})$ (ignoring the Tate shifts)*

$$R\chi(n,e)_* \mathbb{Q}_{M(n,e)}|_{A_n^{\mathrm{red}}} \simeq \bigoplus_{\underline{n} \in \mathcal{P}_d} R\chi(n,d)_* \operatorname{IC}(M_{\underline{n}}(d), \chi(n,d)^* \mathcal{L}_{\underline{n}}) \langle \operatorname{codim} S_{\underline{n}} \rangle|_{A_n^{\mathrm{red}}}. \tag{5}$$

Taking cohomology, we obtain an isomorphism of mixed Hodge structures

$$H^k(M(n,e)^{\mathrm{red}}, \mathbb{Q}) \simeq \bigoplus_{\underline{n} \in \mathcal{P}_d} IH^{k-2\operatorname{codim} S_{\underline{n}}}(M_{\underline{n}}(d)^{\mathrm{red}}, \chi(n,d)^* \mathcal{L}_{\underline{n}})(-\operatorname{codim} S_{\underline{n}}). \tag{6}$$

which respects the perverse filtrations[3] *carried by these vector spaces.*

Note that the summand of the RHS of (5) corresponding to the trivial partition $\underline{n} = \{n\}$ is $R\chi(n,d)_* \operatorname{IC}(M(n,d), \mathbb{Q})$. This means that (6) measures the difference between $H^k(M(n,e)^{\mathrm{red}}, \mathbb{Q})$ and $IH^k(M(n,d)^{\mathrm{red}}, \mathbb{Q})$.

Singular Dolbeault moduli spaces for $d = 0$ were introduced to study the topology of character varieties, to which they are homeomorphic via the non-abelian Hodge correspondence. Historically, to avoid dealing with singular moduli spaces, the moduli problem has been slightly modified or twisted (cf. the "bait-and-switch" in [39, Section 1]), and the Dolbeault moduli spaces with d coprime to n have received more attention in the literature. The Hodge-to-singular correspondence suggests that the historical order, say from smooth to singular spaces, should be reversed at least conceptually: the intersection cohomology of singular Dolbeault moduli spaces should be considered as "primitive" objects into which the cohomology of smooth Dolbeault moduli spaces decomposes.

Maulik and Shen [58, Theorem 0.1] proved an analogue of Theorem 1.1 for twisted Higgs bundles, i.e. pairs $(\mathcal{E}, \phi)$ where now ϕ is a global section of $\mathcal{E}nd(\mathcal{E}) \otimes \mathcal{O}_C(D)$ for some effective divisor D with $\deg D > 2g - 2$, the so-called "Fano" condition. Unfortunately, the estimates in [58, Section 4] are not useful in our setting, when $\mathcal{O}_C(D) \simeq \omega_C$, i.e. under the "Calabi–Yau" condition; see [58, Section 0.4], [10, Section 11] and Remark 9.6. This accounts for the existence of proper supports for the complex $R\chi(n,e)_* \operatorname{IC}(M(n,e), \mathbb{Q})$ in A_n, which on the contrary do not appear under the "Fano" assumption by [58, Theorem 0.4].

We can suggestively summarise the Hodge-to-singular correspondence in the following table.

HODGE THEORY VS SINGULARITIES OF $M(n,d)$

$\gcd(n,d) = 1$	$\gcd(n,d) \neq 1, d \neq 0$	$d = 0$
no singularities, more summands of dec. thm	intermediate behaviour	more singular strata, no proper summands of dec. thm

[3] We recall the definition of perverse filtration in (52).

1.2. Dependence of $IH^*(M(n,d),\mathbb{Q})$ on degree

The Hodge-to-singular correspondence asserts that the intersection cohomology $IH^k(M(n,d)^{\mathrm{red}},\mathbb{Q})$ for arbitrary $d \in \mathbb{Z}$ decomposes into the direct sum of isotypic components of tensor products of $IH^k(M(n',0)^{\mathrm{red}},\mathbb{Q})$ for $n' \leq n$ under the action of a symmetric group; see Remark 10.1. We can refine the result by showing that $IH^k(M(n',0)^{\mathrm{red}},\mathbb{Q})$ is determined only by the smooth locus of the Hitchin fibration, which in particular implies that it cannot be further decomposed according to the decomposition theorem for the Hitchin fibration. Remarkably, this is in contrast to the smooth case, i.e. when d is coprime to n, where $R\chi(n,d)_*\mathbb{Q}_{M(n,d)}$ admits summands properly supported on A_n which cannot be detected on the smooth locus of the Hitchin fibration.

To this end, recall that any $a \in A_n$ can be viewed as a monic polynomial of degree n with coefficient of the degree $n-i$ term in $H^0(C,\omega_C^{\otimes i})$, or equivalently as the curve $C_a \subset \mathrm{Tot}(\omega_C)$ of degree n over C cut by this monic equation, called a *spectral curve*. So an alternative definition of $S^\circ_{\underline{n}}$ is the locus parametrising spectral curves $C_a = \bigcup_{i=1}^r C_i$ such that the reduced irreducible component C_i has degree n_i over C. We also define the following spaces:

(i) $S^\times_{\underline{n}} \subseteq S^\circ_{\underline{n}}$ is the locus parametrising reducible nodal curves having smooth irreducible components of degree n_i over C;

(ii) $\pi_{\underline{n}}\colon \mathcal{C}^\times_{\underline{n}} \to S^\times_{\underline{n}}$ is the universal spectral curve over $S^\times_{\underline{n}}$;

(iii) $\mathrm{Pic}^0(\mathcal{C}^\times_{\underline{n}}) \to S^\times_{\underline{n}}$ is the (relative) Jacobian of the universal spectral curve over $S^\times_{\underline{n}}$;

(iv) $g_{\underline{n}} : A^\times_{\underline{n}} \to S^\times_{\underline{n}}$ is the maximal abelian proper quotient of $\mathrm{Pic}^0(\mathcal{C}^\times_{\underline{n}})$, or equivalently the Jacobian of the normalisation of $\mathcal{C}^\times_{\underline{n}}$;

(v) $\Lambda^l_{\underline{n}} := R^l(g_{\underline{n}})_*\mathbb{Q}_{A^\times_{\underline{n}}} = \Lambda^l R^1(g_{\underline{n}})_*\mathbb{Q}_{A^\times_{\underline{n}}} = \Lambda^l R^1(\pi_{\underline{n}})_*\mathbb{Q}_{A^\times_{\underline{n}}}$ (see [17, Lemma 1.3.5]).

Definition 1.2. The *Ngô strings*[4] for the partition $\underline{n}$ of n is the complex of mixed Hodge modules supported on $S_{\underline{n}}$ given by

$$\mathscr{S}_{\underline{n}} := \bigoplus_{l=0}^{2\dim S_{\underline{n}}} \mathrm{IC}(S_{\underline{n}}, \Lambda^l_{\underline{n}} \otimes \mathscr{L}_{\underline{n}})[-l]\langle \mathrm{codim}\, S_{\underline{n}}\rangle. \tag{7}$$

Note that $\mathscr{S}_{\underline{n}}$ is independent of d, since $\mathscr{L}_{\underline{n}}$ and $\Lambda^l R^1(g_{\underline{n}})_*\mathbb{Q}_{A^\times_{\underline{n}}}$ are so.

In the coprime case $\gcd(e,n)=1$, the summands of the decomposition theorem of the Hitchin fibration on the locus of reduced spectral curves have been described in [18]. As a first application of the Hodge-to-singular correspondence, we recover [18, Proposition 4.1, Theorem 6.12].

Theorem 1.3 (Ngô strings in degree e coprime to n). *Let e be an integer coprime to n. There is an isomorphism in $D^bMHM_{\mathrm{alg}}(A_n^{\mathrm{red}})$ or in $D^b(A_n^{\mathrm{red}})$ (ignoring the Tate shifts)*

$$R\chi(n,e)_*\mathbb{Q}_{M(n,e)}|_{A_n^{\mathrm{red}}} \simeq \bigoplus_{\underline{n}\vdash n} \mathscr{S}_{\underline{n}}|_{A_n^{\mathrm{red}}}. \tag{8}$$

[4]The definition of Ngô strings differs from [24, Definition A.0.4] by a shift and a Tate twist. This convention avoids writing the shifts all the time.

Note that the proof of Theorem 1.3 below is not independent of [18]. What is new is the conceptual interpretation of that result: the rather mysterious occurrence of the cohomology of the cographic matroid in [18, Theorem 6.12] can now be interpreted in geometric terms as the local intersection cohomology of the singularities of $M(n,0)$, or the top cohomology of their resolutions, in symbols $H^{\dim W_{\underline{n}}}(\widetilde{W}_{\underline{n}}, \mathbb{Q})$. In the follow-up paper [61], we provide an alternative combinatorial proof of Theorems 1.3 and 1.4, which is independent of [18].

In view of [18], where the locus of reduced spectral curves carries a plethora of proper summands of the decomposition theorem, it could appear bizarre that instead there is none if $d = 0$.

Theorem 1.4 (Ngô strings in degree 0). *The complex $R\chi(n,0)_* \operatorname{IC}(M(n,0), \mathbb{Q})$ has full support on the reduced locus A_n^{red}, i.e.*

$$R\chi(n,0)_* \operatorname{IC}(M(n,0), \mathbb{Q})|_{A_n^{\mathrm{red}}} \simeq \mathscr{S}_{\{n\}}|_{A_n^{\mathrm{red}}}.$$

Question 1.5 (Full support). *Does $R\chi(n,0)_* \operatorname{IC}(M(n,0), \mathbb{Q})$ have full support on the whole Hitchin base A_n, i.e.*

$$R\chi(n,0)_* \operatorname{IC}(M(n,0), \mathbb{Q}) \simeq \mathscr{S}_{\{n\}}?$$

The multiplicative group $\mathbb{G}_{\mathrm{m}}$ acts on $M(n,d)$ by scaling the Higgs fields. Instead, the additive group $H^0(C, \omega_C)$ translates the Higgs fields, sending $(\mathcal{E}, \phi)$ to $(\mathcal{E}, \phi + \omega \cdot \mathrm{id}_E)$ for all $\omega \in H^0(C, \omega_C)$. The $(H^0(C, \omega_C) \rtimes \mathbb{G}_{\mathrm{m}})$-actions descend to A_n, making the Hitchin fibration $(H^0(C, \omega_C) \rtimes \mathbb{G}_{\mathrm{m}})$-equivariant. In particular, the supports of $R\chi(n,d)_* \operatorname{IC}(M(n,d), \mathbb{Q})$ are $(H^0(C, \omega_C) \rtimes \mathbb{G}_{\mathrm{m}})$-invariant. Then its minimal potential support is $H^0(C, \omega_C) \hookrightarrow \bigoplus_{i=1}^n H^0(C, \omega_C^{\otimes i}) = A_n$, and it lies outside A_n^{red}.

As a partial positive answer to Question 1.5, we observe that no summand of $R\chi(n,d)_* \operatorname{IC}(M(n,d), \mathbb{Q})$ is supported on $H^0(C, \omega_C)$. This generalises a result by Heinloth [41, Theorem 1] to arbitrary degree. The proof relies on a result by Kinjo and Koseki [48] and Davison [14] asserting that $R\chi(n,d)_* \operatorname{IC}(M(n,d), \mathbb{Q})$ is a direct summand $R\chi(n,e)_* \mathbb{Q}_{M(n,e)}$ for $\gcd(n,e) = 1$; see Proposition 3.9.

Proposition 1.6 (Vanishing of intersection form). *The intersection form on the intersection cohomology $IH^*(M(n,d), \mathbb{Q})$ is trivial. Equivalently, the forgetful map $IH_c^*(M(n,d), \mathbb{Q}) \to IH^*(M(n,d), \mathbb{Q})$ is zero.*

In Theorem 1.7 we answer Question 1.5 positively in rank 2 and illustrate its far-reaching consequences. Note that the intersection cohomology groups of $M(2,d)$ have been computed in [43, Theorem 7.6] and [60, Theorems 1.8, 1.9].

Theorem 1.7 (Ngô strings in rank 2). *There exist isomorphisms*

$$R\chi(2,1)_* \mathbb{Q}_{M(2,1)} \simeq \mathscr{S}_{\{2\}} \oplus \mathscr{S}_{\{1,1\}}, \quad R\chi(2,0)_* \operatorname{IC}(M(2,0), \mathbb{Q}) \simeq \mathscr{S}_{\{2\}},$$

in the bounded derived category $D^b MMH(A_n)$ of Hodge modules on A_n. As a result, there exists a surjective map

$$H^*(M(2,1), \mathbb{Q}) \twoheadrightarrow IH^*(M(2,0), \mathbb{Q})$$

strictly filtered by the perverse filtrations associated to $\chi(2,d)$.

The Hodge-to-singular correspondence also provides a recursive strategy to express $IH^*(M^{\mathrm{red}}(n,d),\mathbb{Q})$ in any degree in terms of $IH^*(M^{\mathrm{red}}(n',0),\mathbb{Q})$ for $n' \leq n$. We outline it in Section 10.1. A closed non-recursive formula appears in [61, Theorems 1.1, 1.6, 1.8]. In loc. cit. the authors and Pagaria determine explicitly the local systems in the Ngô strings for $R\chi(n,d)_* \operatorname{IC}(M(n,d),\mathbb{Q})|_{A_n^{\mathrm{red}}}$. This completes the description of the summands of the decomposition theorem for the Hitchin fibration in arbitrary rank and degree over the locus of reduced spectral curves.

An extension of the Hodge-to-singular correspondence to the whole A_n would provide a recursive formula to compute $IH^*(M(n,d),\mathbb{Q})$ for arbitrary degree d. After this paper has become available, Davison, Hennecart and Schlegel Mejia achieved this latter goal; see [15, Section 14.1]. Despite the different representation-theoretic flavour of [15], the current paper and [15] share ultimately the same central idea: to reduce to a local statement regarding Nakajima quiver varieties. The Hodge-to-singular correspondence has the virtue of offering a more geometric characterization of the key object in [15], namely the BPS sheaf, as well as a closed formula for the BPS sheaf on each moduli space $M(n,d)$ rather than a generating formula; see Theorems 4.19 and 7.5, and [61]. On the other hand, so far the Hodge-to-singular correspondence has been established only on the locus of reduced characteristic polynomials, while the results of [15] hold over the whole Hitchin base. Observe however that the work of Davison, Hennecart and Schlegel Mejia does not prescribe the shape of the summands of the decomposition theorem of the Hitchin fibration (for instance, it does not answer Question 1.5), which would be of independent interest for applications to the P = W and topological mirror symmetry conjectures.

Moreover, the Hodge-to-singular correspondence implies that $IH^*(M^{\mathrm{red}}(n,d),\mathbb{Q})$ only depends on $\gcd(n,d)$; see Corollary 1.8. The degree independence of the cohomology ring $H^*(M(n,e),\mathbb{Q})$ for $\gcd(n,e)=1$ is a classical corollary of the non-abelian Hodge correspondence, observed for instance in [35, Remark 4.8]; see also [33, Theorem 1.7], [64, Corollary 1.2] and [82]. The independence of the complex $R\chi(n,e)_* \operatorname{IC}(M(n,e),\mathbb{Q})|_{A_n^{\mathrm{red}}}$ for $\gcd(n,e)=1$ has been proved using vanishing cycle techniques in [56, Theorem 0.5, Remark 4.9]; see also [48, Theorem 1.1, Example 5.18] and the comment [49, Section 1.5 (2)]. The latest and most refined algebraic proof [21, Theorem 0.1] asserts that a canonical isomorphism $H^*(M(n,e),\mathbb{Q}) \simeq H^*(M(n,e'),\mathbb{Q})$ for $\gcd(n,e)=\gcd(n,e')=1$ preserves simultaneously the ring structure, the tautological classes, and the perverse filtrations.

In the singular case, the same argument as in [35, Remark 4.8] (cf. [29, Section 0.4]) shows that

$$IH^*(M(n,d),\mathbb{Q}) \simeq IH^*(M(n,d'),\mathbb{Q}) \tag{9}$$

for any integers d and d' with $\gcd(n,d)=\gcd(n,d')$. Then Corollary 1.8 promotes the independence (9) at the sheaf level on the locus of reduced characteristic polynomials. As a corollary, this makes the isomorphism $IH^*(M^{\mathrm{red}}(n,d),\mathbb{Q}) \simeq IH^*(M^{\mathrm{red}}(n,d'),\mathbb{Q})$ filtered with respect to the perverse filtration.

Corollary 1.8. *Let d and d' be integers with $\gcd(d,n)=\gcd(d',n)$. Then*

$$R\chi(n,d)_* \operatorname{IC}(M(n,d),\mathbb{Q})|_{A_n^{\mathrm{red}}} \simeq R\chi(n,d')_* \operatorname{IC}(M(n,d'),\mathbb{Q})|_{A_n^{\mathrm{red}}}.$$

Note that Corollary 1.8 has recently been extended onto the whole Hitchin base in [15, Corollary 14.9].

1.3. *Some applications of the Hodge-to-singular correspondence*

1.3.1. Stabilisation. Coskun and Woolf [12] have proposed a conjecture about the stabilisation of the cohomology of moduli space of sheaves on compact complex surface. As a final remark, we observe that $IH^*(M(n,d), \mathbb{Q})$ does stabilise as the rank n grows, independently of the degree d.

Proposition 1.9 (Stabilisation). *The intersection Betti and Hodge numbers of $M(n,d)$ stabilise to the stable intersection Betti and Hodge numbers of $M(n,0)$ as n tends to infinity, i.e. the limit*

$$\lim_{n\to\infty} \dim IH^*(M(n,d))^{p,q} \tag{10}$$

is finite and independent of d.

Analogous stabilisation results for the Mukai systems on the moduli space of sheaves on K3 surfaces are discussed in Section 10.3.

1.3.2. Restriction to smooth fibres. Let M be a projective irreducible holomorphic symplectic variety of dimension $2k$ equipped with a Lagrangian fibration $f : M \to B$. Felisetti, Shen and Yin [29] have proved that the restriction of $H^*(M, \mathbb{Q})$ to a smooth fibre of f is isomorphic to $H^*(\mathbb{P}^k, \mathbb{Q})$. In general, this is false for non-compact hyperkähler varieties, but it holds true for the Hitchin fibration for $n > 1$ by [1, Theorem 5], up to a copy of $H^\bullet(C, \mathbb{Q})$. Here we propose an alternative proof.

Proposition 1.10. *The restriction of $IH^*(M(n,d))$ to any smooth fibre $M_a \subset M(n,d)$ of the Hitchin fibration $\chi(n,d)$ is given by*

$$\mathrm{Im}\{IH^*(M(n,d), \mathbb{Q}) \to H^*(M_a, \mathbb{Q})\} = H^\bullet(C, \mathbb{Q}) \otimes \mathbb{Q}[\alpha|_{M_a}]/(\alpha|_{M_a}^{\dim M(n,d)+1}),$$

where α is a $\chi(n,d)$-relative ample class on $M(n,d)$.

Let $\check{M}(n,d)$ be a fibre of the determinant map $M(n,d) \to M(1,nd)$, given by $(\mathcal{E},\phi) \mapsto (\det \mathcal{E}, \mathrm{tr}\,\phi)$. The restriction of $IH^*(\check{M}(n,d))$ to any smooth fibre $\check{M}_a \subset \check{M}(n,d)$ of the Hitchin fibration $\check{\chi}(n,d)$ is given by

$$\mathrm{Im}\{IH^k(\check{M}(n,d), \mathbb{Q}) \to H^k(\check{M}_a, \mathbb{Q})\} = \begin{cases} \langle \alpha^{k/2}|_{\check{M}_a} \rangle & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

1.4. *Strategy*

In this paper we propose a geometric proof of the Hodge-to-singular correspondence. Here we outline the main steps. It is well-known that the Hitchin fibration is a family of compactified Jacobians of spectral curves; see [3, 42, 74]. In Section 5, we enlarge the Hitchin fibration $\chi(n,d)\colon M(n,d) \to A_n$ to a family $\pi\colon \bar{J}_{d,B} \to B$ of compactified Jacobians over a versal deformation of spectral curves:[5]

$$\begin{array}{ccc} M(n,d) & \longrightarrow & \bar{J}_{d,B} \\ {\scriptstyle \chi(n,d)}\downarrow & & \downarrow{\scriptstyle \pi} \\ A_n & \longrightarrow & B \end{array}$$

Note that the relative dimension of π is strictly smaller than $\dim B$. This is a crucial ingredient to show that $R\pi_* \operatorname{IC}(\bar{J}_{d,B}, \mathbb{Q})$ is independent of d, while $R\chi(n,d)_* \operatorname{IC}(M(n,d), \mathbb{Q})$ is not; see Theorem 9.5 and Remark 9.6. In particular, we have

$$R\pi_* \operatorname{IC}(\bar{J}_{e,B}, \mathbb{Q})|_{A_n} \simeq R\pi_* \operatorname{IC}(\bar{J}_{d,B}, \mathbb{Q})|_{A_n}. \tag{11}$$

If $\gcd(n,e) = 1$, the inclusion $M(n,e) \hookrightarrow \bar{J}_{e,B}$ is a regular embedding of smooth varieties, so

$$\operatorname{IC}(\bar{J}_{e,B}, \mathbb{Q})|_{M(n,e)} \simeq \mathbb{Q}_{\bar{J}_{e,B}}|_{M(n,e)} \simeq \mathbb{Q}_{M(n,e)}, \tag{12}$$

and the LHS of (11) is isomorphic to $R\chi(n,e)_*\mathbb{Q}_{M(n,e)}$. If instead $\gcd(n,d) \neq 1$, the failure of (12) is measured by the topology of the singularities of $M(n,d)$. Following [8], we identify the map $M(n,d) \hookrightarrow \bar{J}_{d,B}$ with the inclusion of the hypertoric quiver variety $Y(\Gamma_{\underline{n}}, 0)$ into the toric Lawrence variety $X(\Gamma^{\pm}_{\underline{n}}, 0)$,

$$\begin{array}{ccc} Y(\Gamma_{\underline{n}}, 0) & \overset{\iota_M}{\hookrightarrow} & X(\Gamma^{\pm}_{\underline{n}}, 0) \\ {\scriptstyle \chi_{\Gamma_{\underline{n}}}}\downarrow & & \downarrow{\scriptstyle \pi_{\Gamma_{\underline{n}}}} \\ \mathbb{C}^{b_1(\Gamma_{\underline{n}})} & \overset{\iota_A}{\hookrightarrow} & \mathbb{C}^s \end{array}$$

(see Theorem 7.4 for the proof, and Section 4 for details about the notation). Now a local version of the Hodge-to-singular correspondence for hypertoric quiver varieties (Theorem 4.19) shapes the RHS of (11), and gives the main result, Theorem 1.1; see also Theorem 7.5. The key geometric input is the following: $X(\Gamma^{\pm}_{\underline{n}}, 0)$ admits small resolutions of singularities that restrict to semismall resolutions of $Y(\Gamma_{\underline{n}}, 0)$; see Section 4.5.

1.5. Outline

- In Section 3 we prove a version of the Ngô support theorem for weakly abelian Lagrangian fibrations on singular symplectic varieties; see Theorem 3.8. We specialise the support theorem to the Hitchin fibration in Theorem 3.13.

[5] Actually, to make the diagram work we should slice the Hitchin base, restrict to the locus of nodal spectral curves, and take into consideration the automorphisms of the curves by working over $\bar{\mathcal{M}}_g$. For expository reasons, we allow this abuse here, and postpone the precise details to Section 5.

- In Section 4 we discuss toric Lawrence varieties and hypertoric quiver varieties. The main result is the local Hodge-to-singular correspondence, i.e. Theorem 4.19.
- In Section 5 we explain how locally étale the Dolbeault moduli space embeds into the universal compactified Jacobian.
- In Section 7 we describe local models for this embedding; see Theorem 7.4. To this purpose we need some auxiliary results: an explicit description of the image of the Kodaira–Spencer map for nodal spectral curves provided in Section 6; and the topological trivialisation of a tubular neighbourhood of an abelian variety in the universal compactified Jacobian in Section 8.
- In Section 9 we show the degree independence of the intersection cohomology of the universal compactified Jacobian; see Theorem 9.5.
- In Section 10 we collect the proofs of the main theorems stated in Section 1.

1.6. Notation

The intersection cohomology of a complex variety X with middle perversity and rational coefficients is denoted by $IH^*(X,\mathbb{Q}) := H^*(\mathrm{IC}(X,\mathbb{Q}))$, where $\mathrm{IC}(X,\mathbb{Q})$ is the perverse intersection cohomology complex of X shifted by $-\dim X$. Ordinary singular cohomology with rational coefficients is denoted by $H^*(X,\mathbb{Q})$. Recall that they all carry mixed Hodge structures. Intersection complexes and perverse sheaves on an algebraic stack are descents of perverse sheaves from a smooth atlas; see [44, 51, 52]. The formalism of six operations works in this context too. An adaptation of the theory of mixed Hodge modules to stacks is sketched in [14, Sections 2.2 and 2.3]. However, since we only deal with Deligne–Mumford stacks, several technical issues simplify drastically. One can proceed for instance along the lines of [47, Section III.15]. We use it in Section 7.

2. Stratifications

Recall that a *stratification* of a complex variety X is a finite collection of locally closed smooth subvarieties $X_i \subseteq X$, called *strata*, such that X is the disjoint union of X_i with $i = 1, \ldots, r$, i.e. $X = \bigsqcup_{i=1}^{r} X_i$. A stratification is called a *Whitney stratification* if all pairs of strata satisfy the Whitney conditions A and B. We omit the precise definition of these conditions (see for instance [32]), since in the following we will only use the stronger property of Definition 2.1, which implies Whitney conditions A and B by Lemma 2.2.

Definition 2.1. A stratification $X = \bigsqcup_i X_i$ is *analytically trivial in the normal direction to each stratum* if for any $x \in X_i$ there exists a normal slice N_x through X_i at x, and a neighbourhood of x in X which is locally analytically isomorphic to $N_x \times T_x X_i$ at $(x, 0)$.

Lemma 2.2. *A stratification analytically trivial in the normal direction to each stratum is a Whitney stratification.*

Proof. We follow the argument in [62, Section 4.7]. Whitney conditions A and B are local, so it is enough to check them for $N_x \times T_x X_i$ at $(x, 0)$. Since $T_x X_i$ is a smooth factor, it is enough to check them for N_x at x. But this follows from [81, Lemma 19.3]. ∎

3. Ngô theorem for Lagrangian fibrations on singular spaces

The goal of this section is to generalise the description of Ngô strings in [24, Theorem 7.0.3] to the case of weakly abelian Lagrangian fibrations on singular symplectic varieties.

We will closely follow Maulik and Shen [58]. Let B be a quasi-projective complex variety. Let $g\colon P \to B$ be a smooth B-group scheme with connected fibres, and let $f\colon X \to B$ be a proper morphism. Assume that the group scheme P acts on X via

$$\mathrm{act}\colon P \times_B X \to X. \tag{13}$$

Definition 3.1 (Weak abelian fibration). The triple (X, P, B) is a *weak abelian fibration* of relative dimension c if

(1) every fibre of the map g is pure of dimension c, and X has pure dimension

$$\dim X = c + \dim B,$$

(2) the action (13) of P on X has affine stabilisers,

(3) the Tate module $T_{\overline{\mathbb{Q}}_l}(P)$ associated with the group scheme P is polarisable.

For a closed point $b \in B$, we define $\delta(b)$ as the dimension of the maximal affine and connected subgroup of P_b. For any closed subvariety $Z \subseteq B$, we denote by δ_Z the minimum value of the (upper semicontinuous) function δ on Z.

Theorem 3.2. *Let (X, P, B) be a weak abelian fibration of relative dimension c. Assume that*

(1) (δ-regularity) $\operatorname{codim} Z_\delta \geq \delta$ *for any* $Z_\delta := \{b \in B \mid \delta(b) = \delta\}$*;*

(2) $\tau_{>2c}(Rf_* \operatorname{IC}(X, \mathbb{Q})) = 0$ *for the standard truncation functor* $\tau_{>*}(-)$.

Then every support Z of $Rf_* \operatorname{IC}(X, \mathbb{Q})$ satisfies

$$\operatorname{codim} Z = \delta(Z).$$

Proof. [58, Theorem 1.8] says that if $f\colon X \to B$ is part of the datum of a weak abelian fibration with property (2), then any support Z of $Rf_* \operatorname{IC}(X, \mathbb{Q})$ satisfies $\operatorname{codim} Z \leq \delta_Z$. Together with δ-regularity, we obtain $\operatorname{codim} Z = \delta_Z$. ∎

Proposition 3.3 (Relative dimension bound). *Let $f\colon X \to B$ be an equidimensional proper morphism of complex algebraic varieties of relative dimension c. Suppose that there exists a Whitney stratification $X = \bigsqcup_i X_i$ such that for any $b \in B$ we have*

$$\operatorname{codim}_X X_i \geq 2 \operatorname{codim}_{f^{-1}(b)}(X_i \cap f^{-1}(b)). \tag{14}$$

Then

- $\tau_{>2c}(Rf_* \operatorname{IC}(X, \mathbb{Q})) = 0$;
- $R^{2c} f_* \operatorname{IC}(X, \mathbb{Q}) = R^{2c} f_* \mathbb{Q}_X$.

Proof. Via proper base change we have

$$\mathcal{H}^*(Rf_* \operatorname{IC}(X, \mathbb{Q}))_b = H^*(f^{-1}(b), \operatorname{IC}(X, \mathbb{Q})|_{f^{-1}(b)}) \quad \text{for any } b \in B.$$

These cohomology groups are the limit of the Grothendieck spectral sequence

$$E_2^{p,q} = H^{p-q}(f^{-1}(b), \mathcal{H}^q(\operatorname{IC}(X, \mathbb{Q}))|_{f^{-1}(b)}) \Rightarrow H^p(f^{-1}(b), \operatorname{IC}(X, \mathbb{Q})|_{f^{-1}(b)}).$$

Let $Z_q \subseteq X$ be the support of the constructible sheaf $\mathcal{H}^q(\operatorname{IC}(X, \mathbb{Q}))$. In particular,

$$E_2^{p,q} = 0 \quad \text{for } p - q > 2\dim(Z_q \cap f^{-1}(b)).$$

The irreducible components of Z_q are the closures of some X_i with $q < \operatorname{codim}_X X_i$, or X itself if $q = 0$. Indeed, the strong support condition for intersection cohomology implies that $\mathcal{H}^q(\operatorname{IC}(X, \mathbb{Q}))_x = 0$ for any $x \in X_i$ and $q \geq \max\{\operatorname{codim}_X X_i, 1\}$; see [22, (12)]. By (14), we conclude that $E_2^{p,q} = 0$ for

$$p > q + 2\dim(Z_q \cap f^{-1}(b)) \geq \max\{\operatorname{codim}_X X_i, 1\} + 2\dim(Z_q \cap f^{-1}(b)),$$

i.e. for $p > 2c$ and arbitrary q, or for $p = 2c$ and $q \neq 0$. This yields

$$\mathcal{H}^{>2c}(Rf_* \operatorname{IC}(X, \mathbb{Q})) = 0, \quad \mathcal{H}^{2c}(Rf_* \operatorname{IC}(X, \mathbb{Q})) = \mathcal{H}^{2c}(Rf_* \mathbb{Q}_X). \qquad \blacksquare$$

Definition 3.4 (Symplectic variety). A normal variety X is *symplectic* if it has rational singularities, and it admits a holomorphic symplectic form on its smooth locus $X^{\mathrm{reg}} \subset X$, i.e. a non-degenerate holomorphic (closed) 2-form $\omega \in H^0(X^{\mathrm{reg}}, \Omega^2_{X^{\mathrm{reg}}})$.

By [46, Corollary 1.8], this is equivalent to requiring that a holomorphic symplectic form ω on X^{reg} extends to a (possibly degenerate) holomorphic 2-form $\widetilde{\omega}$ on a resolution $\widetilde{X} \to X$. We say that X admits a *symplectic resolution* if $\omega_{\widetilde{X}}$ is non-degenerate.

Definition 3.5 (Lagrangian fibration). Let X be a symplectic variety. An irreducible subvariety $Y \subset X$, not contained in the singular locus of X, is *isotropic* if $\omega|_{Y^{\mathrm{reg}} \cap X^{\mathrm{reg}}}$ vanishes. Note that $\dim Y \leq \frac{1}{2} \dim X$, and Y is *Lagrangian* if equality holds.

A *Lagrangian fibration* is a proper surjective morphism $f\colon X \to B$ with connected fibres onto a normal variety B whose general fibre is Lagrangian.

Recall that a Lagrangian fibration $f\colon X \to B$ is equidimensional, and any irreducible component of a fibre of X is not contained in the singular locus of X and is Lagrangian; see [76, Theorem 17] where the projectivity assumption on X and B can be dropped.

Proposition 3.6. *Let X be a symplectic variety endowed with a Lagrangian fibration $f\colon X \to B$. Then X admits a complex Whitney stratification*

$$X = \bigsqcup X_i \tag{15}$$

by locally closed disjoint algebraic symplectic submanifolds X_i such that the restriction map $f|_{X_i}\colon X_i \to B_i := f(X_i)$ is Lagrangian. In particular, for any $b \in B$ we have

$$\operatorname{codim}_X X_i = 2\operatorname{codim}_{f^{-1}(b)}(X_i \cap f^{-1}(b)). \tag{16}$$

Proof. By [45, Theorem 2.3, Proposition 3.1], X admits a stratification by locally closed subsets X_i, with $i = 1, \dots, r$, such that:

- each X_i is smooth and symplectic;
- each X_i is an open subset of the singular locus of the closure of some X_j;
- the stratification is analytically trivial in the normal direction to each stratum, so it is a Whitney stratification by Lemma 2.2.

Applying [55, Theorem 3.1] iteratively, we find that the restrictions $f|_{X_i}$ are Lagrangian.

Since both f and $f|_{X_i}$ are Lagrangian, we obtain $\dim X - \dim X_i = 2\dim f^{-1}(b) - 2\dim(X_i \cap f^{-1}(b))$. ■

Example 3.7. The Dolbeault moduli space $X = M(n,d)$ is a symplectic variety,[6] and the Hitchin system is a Lagrangian fibration; see for instance [42,43] and [53, Main Thm]. The Whitney stratification (15) for $M(n,d)$ is the stratification (3). In particular, Proposition 3.3 gives

$$\tau_{>\dim M(n,d)}(R\chi(n,d)_* \operatorname{IC}(M(n,d),\mathbb{Q})) = 0,$$
$$R^{\dim M(n,d)} f_* \operatorname{IC}(M(n,d),\mathbb{Q}) = R^{\dim M(n,d)} f_* \mathbb{Q}_{M(n,d)}.$$

Now let (X, P, B) be a δ-regular weak abelian fibration. We denote by I the set of supports $Z_\alpha \subseteq B$ for $Rf_* \operatorname{IC}(X,\mathbb{Q})$. There exist open dense subsets $Z_\alpha^\times$ such that the restriction P_α of the group scheme P admits a Chevalley decomposition

$$0 \to R_\alpha \to P_\alpha \to A_\alpha \to 0,$$

where $R_\alpha \to Z_\alpha^\times$ is an affine group scheme and $g_\alpha\colon A_\alpha \to Z_\alpha^\times$ is a proper abelian group scheme. Let $\Lambda_\alpha^l := R^l g_{\alpha *}\mathbb{Q}_{A_\alpha}$.

Theorem 3.8 (Ngô strings from symplectic varieties). *Let (X, P, B) be a δ-regular weak abelian Lagrangian fibration on a symplectic variety X of dimension $2c$. Then there exists an isomorphism in $D^b MHM_{\mathrm{alg}}(B)$ (resp. $D^b(B,\mathbb{Q})$)*

$$Rf_* \operatorname{IC}(X,\mathbb{Q}) \simeq \bigoplus_{\alpha\in I} \bigoplus_{l=0}^{2\dim Z_\alpha} \operatorname{IC}(Z_\alpha, \Lambda_\alpha^l \otimes \mathscr{L}_\alpha)[-l]\langle \operatorname{codim} Z_\alpha\rangle, \tag{17}$$

[6] The Dolbeault moduli space $M(n,d)$ has rational singularities. A direct Riemann–Roch computation, analogous to [5, Lemma 2.2], shows that the singular locus of $M(n,d)$ has codimension at least 4, except for $(g,n,d) = (2,2,0)$ and $g = 1$. In the general case, this implies that $M(n,d)$ has terminal rational singularities by [30] and [50, Theorem 5.22]. In the special cases $(g,n,d) = (2,2,0)$ and $g = 1$, the existence of a symplectic resolution implies the rationality of the singularities; see for instance [80, Theorem 1], [2, Section 6] and [28, Proposition 6.1].

where, for every α, $\mathscr{L}_\alpha$ is a local system on an open subset of Z_α. Moreover,

(i) $\operatorname{codim} Z_\alpha = \delta(Z_\alpha)$*;*

(ii) $R^{2c} f_* \mathbb{Q}_X$ *admits the direct summand* $\bigoplus_{\alpha \in I} i_{\alpha *}\mathscr{L}_\alpha$*, where* $i_\alpha \colon Z_\alpha^\times \to B$ *is the natural immersion.*

Proof. (i) is a combination of Theorem 3.2 and Propositions 3.3 and 3.6. Maulik and Shen [58, Proposition 1.5] provided a generalisation of Ngô freeness [68, Proposition 7.4.10] to the singular context. This gives the special form (17) to the decomposition theorem for $Rf_* \operatorname{IC}(X, \mathbb{Q})$. Taking cohomology of (17), together with the isomorphism $R^{2c} f_* \operatorname{IC}(X, \mathbb{Q}) = R^{2c} f_* \mathbb{Q}_X$ of Proposition 3.3, we obtain (ii). ■

3.1. Ngô strings for Dolbeault moduli spaces

We specialise Theorem 3.8 to the Hitchin fibration $\chi(n,d)\colon M(n,d) \to A_n$. We rely on the recent key result by Kinjo and Koseki [48], building on the work of Davison [14].

Proposition 3.9. *Let d and e be integers, with e coprime to n. Then the complex $R\chi(n,d)_* \operatorname{IC}(M(n,d), \mathbb{Q})$ is a direct summand of $R\chi(n,e)_* \mathbb{Q}_{M(n,e)}$.*

Proof. See [14, Theorem 6.6] and [48, Corollary 5.15, Example 5.18]. ■

Proposition 3.9 allows us to extend [18, Theorem 4.1] and [41, Theorem 1] to Dolbeault moduli spaces of arbitrary degree.

Corollary 3.10. *If Z is a support of $R\chi(n,d)_* \operatorname{IC}(M(n,d), \mathbb{Q})$ with $Z \cap A_n^{\mathrm{red}} \neq \emptyset$, then $Z = S_{\underline{n}}$ for some partition $\underline{n}$ of n. In particular, the generic point of any support Z of $R\chi(n,d)_* \operatorname{IC}(M(n,d), \mathbb{Q})$ with $Z \cap A_n^{\mathrm{red}} \neq \emptyset$ lies in the locus $A_n^\times$ of nodal spectral curves.*

Proof. See [18, Proposition 4.1] and Proposition 3.9. Alternatively, the same proof of [18, Proposition 4.1] works verbatim as long as the number of irreducible components of $\chi(n,d)^{-1}(a)$ is constant for $a \in S_{\underline{n}}^\circ$, which is proved in [61, Proposition 7.6 (3)]. ■

Corollary 3.11 (Proposition 1.6). *The intersection form on $IH^*(M(n,d), \mathbb{Q})$ is trivial. Equivalently, the forgetful map $IH_c^*(M(n,d), \mathbb{Q}) \to IH^*(M(n,d), \mathbb{Q})$ is zero.*

Proof. By [40, Proposition 4] and the translation action of $H^0(C, \omega_C)$ in Section 1.2, the vanishing of the intersection form is equivalent to $H^0(C, \omega_C) \subset A_n$ not being a support of $R\chi(n,d)_* \operatorname{IC}(M(n,d), \mathbb{Q})$. In the coprime case, this is proved by Heinloth [41, Theorem 1]. Together with Proposition 3.9, we obtain the result in the non-coprime case. ■

Remark 3.12. Corollary 3.11 holds for the moduli spaces of PGL_n- and SL_n-Higgs bundles too.

Theorem 3.13 (Ngô strings from Dolbeault moduli spaces in arbitrary degree). *There exists an isomorphism in $D^b MHM_{\mathrm{alg}}(A_n^{\mathrm{red}})$ (resp. $D^b(A_n^{\mathrm{red}}, \mathbb{Q})$)*

$$R\chi(n,d)_* \operatorname{IC}(M(n,d),\mathbb{Q})|_{A_n^{\mathrm{red}}} \simeq \bigoplus_{\underline{n}\in I_d} \mathscr{S}(\mathscr{L}_{\underline{n}}(d))|_{A_n^{\mathrm{red}}}, \tag{18}$$

where I_d is the set of partitions $\underline{n}$ of n, and the Ngô string $\mathscr{S}(\mathscr{L}_{\underline{n}}(d))$ is the complex of mixed Hodge modules supported on $S_{\underline{n}}$ given by

$$\mathscr{S}(\mathscr{L}_{\underline{n}}(d)) := \bigoplus_{l=0}^{2\dim S_{\underline{n}}} \operatorname{IC}(S_{\underline{n}}, \Lambda^l_{\underline{n}} \otimes \mathscr{L}_{\underline{n}}(d))[-l]\langle \operatorname{codim} S_{\underline{n}}\rangle$$

for certain local systems $\mathscr{L}_{\underline{n}}(d)$ supported on an open subset of $S_{\underline{n}}$.

Proof. This follows from Theorem 3.8 and Corollary 3.10. ∎

The rest of the paper is devoted to the correspondence between the local systems $\mathscr{L}_{\underline{n}}(d)$ and the singularities of $M(n,d)^{\mathrm{red}}$.

4. Hypertoric quiver varieties

The singularities of $M(n,d)^{\mathrm{red}}$ are modelled on hypertoric quiver varieties. In this section we introduce these varieties and establish a local version of the Hodge-to-singular correspondence (Theorem 4.19).

4.1. Affine toric varieties associated to a quiver

Let $Q = (V, E)$ be a connected directed graph (a *quiver*) with r vertices $V = \{v_1, \dots, v_r\}$ and s oriented edges E. Given $e \in E$, let $s(e), t(e) \in V$ be the source and target of e, so that $e\colon s(e) \to t(e)$. The underlying unoriented graph, denoted by $|Q|$, has first Betti number $b_1 := s - r + 1$.

We consider the group of all $\mathbb{Z}$-linear combinations of V whose coefficients sum to zero. We identify this group with $\mathbb{Z}^{r-1}$, by fixing the basis $\{v_1 - v_2, \dots, v_1 - v_r\}$. We also identify $\mathbb{Z}^s$ with the group of $\mathbb{Z}$-linear combinations of E. The boundary map of the quiver Q,

$$A : \mathbb{Z}^s \to \mathbb{Z}^{r-1}, \quad e \mapsto s(e) - t(e),$$

induces an embedding of algebraic tori

$$\mathbb{T}^{r-1} := \operatorname{Spec} \mathbb{C}[\mathbb{Z}^{r-1}] \hookrightarrow \mathbb{T}^s := \operatorname{Spec} \mathbb{C}[\mathbb{Z}^s].$$

The torus $\mathbb{T}^s$ acts on the affine complex space $\mathbb{C}^s = \operatorname{Spec} \mathbb{C}[z_e\colon e \in E]$, and so does $\mathbb{T}^{r-1}$ by restriction. Explicitly, $\lambda = (\lambda_i)_{i=1}^r \in \mathbb{T}^r/\mathbb{T} = \mathbb{T}^{r-1}$ acts as $\lambda \cdot z_e = \lambda_{s(e)} z_e \lambda_{t(e)}^{-1}$.

Definition 4.1. The *affine toric variety* $X(Q,0)$ is the affine GIT quotient of $\mathbb{C}^s$ by $\mathbb{T}^{r-1}$,

$$X(Q,0) := \mathbb{C}^s /\!\!/_0 \, \mathbb{T}^{r-1} = \operatorname{Spec} \mathbb{C}[z_e : e \in E]^{\mathbb{T}^{r-1}}.$$

The cone of the toric variety $X(Q,0)$ is generated by the rows $\mathscr{B} = \{\beta_1, \dots, \beta_s\} \subset \mathbb{Z}^{b_1}$ of a *Gale dual* of A, i.e. an $(s \times b_1)$-matrix B whose columns form a base of $\ker(A : \mathbb{Z}^s \to \mathbb{Z}^{r-1}) \simeq \mathbb{Z}^{b_1}$, or equivalently a matrix that sits in the exact sequence

$$0 \to \mathbb{Z}^{b_1} \xrightarrow{B} \mathbb{Z}^s \xrightarrow{A} \mathbb{Z}^{r-1} \to 0.$$

Recall that a *loop* in a quiver is an edge whose head and tail coincide.

Lemma 4.2. *Let Q be a quiver with m loops, and Q' be the quiver obtained from Q by deleting all the loops. Then*

$$X(Q,0) \simeq X(Q',0) \times \mathbb{C}^m.$$

Proof. Let $L \subset E$ be the loops of Q. We identify the edges E' of Q' with $E \setminus L$. For any $l \in L$, the coordinates z_l are $\mathbb{T}^{r-1}$-invariant by construction, so we have

$$\mathbb{C}[z_e : e \in E]^{\mathbb{T}^{r-1}} \simeq \mathbb{C}[z_e : e \in E']^{\mathbb{T}^{r-1}} \otimes \mathbb{C}[z_l : l \in L]. \qquad \blacksquare$$

4.2. Semiprojective toric varieties associated to a quiver

In general, $X(Q,0)$ is a singular variety. A toric resolution of singularities can be obtained via projective GIT as follows.

The ring of polynomials $S := \mathbb{C}[z_e : e \in E]$ is graded by $\mathbb{Z}^{r-1}$ in such a way that $\deg(z_e) = Ae$. For $\theta \in \mathbb{Z}^{r-1}$, let S_θ be the $\mathbb{C}$-vector space of homogenous polynomials of degree θ. For instance, S_0 is the coordinate ring of $X(Q,0)$.

Definition 4.3. The *semiprojective toric variety* $X(Q,\theta)$ is the projective GIT quotient of $\mathbb{C}^s$ by the torus $\mathbb{T}^{r-1}$ with respect to the *stability* $\theta \in \mathbb{Z}^{r-1}$, i.e.

$$X(Q,\theta) := \mathbb{C}^s /\!\!/_\theta \, \mathbb{T}^{r-1} := \operatorname{Proj}\Big(\bigoplus_{k=0}^{\infty} t^k S_{k\theta}\Big).$$

Consider now the coarsest complete fan $\Gamma(\mathscr{A})$ in $\mathbb{R}^{r-1} \simeq \mathbb{Z}^{r-1} \otimes \mathbb{R}$ that refines all complete simplicial fans in $\mathbb{R}^{r-1}$ whose rays lie in $\mathscr{A} := \{Ae : e \in E\}$. We say that $\theta \in \mathbb{Z}^{r-1}$ is *generic* if it lies in the interior of the maximal cones of $\Gamma(\mathscr{A})$.

Proposition 4.4 ([37, Proposition 8.2]). *For generic $\theta \in \mathbb{Z}^{r-1}$, the affinisation morphism*

$$\pi_X = \pi_X(Q,\theta) : X(Q,\theta) \to \operatorname{Spec} H^0(X(Q,\theta), \mathcal{O}_{X(Q,\theta)}) = X(Q,0)$$

is a resolution of singularities.

4.3. Lawrence varieties and hypertoric quiver varieties

We specialise the previous constructions to doubled quivers. Let Q be a quiver with r vertices, s oriented edges E, and first Betti number b_1.

Definition 4.5. The *double* of Q, denoted by $Q^{\pm}$, is the quiver obtained from Q by replacing each edge e of Q with a pair of edges e^{+} and e^{-} having the same endpoints as e but opposite orientations.

Definition 4.6. The *affine Lawrence toric variety* associated to Q is the affine GIT quotient

$$X(Q^{\pm}, 0) = \mathbb{C}^{2s} /\!\!/_0 \mathbb{T}^{r-1}.$$

The *Lawrence toric variety* associated to Q is the projective GIT quotient

$$X(Q^{\pm}, \theta) = \mathbb{C}^{2s} /\!\!/_\theta \mathbb{T}^{r-1} = \operatorname{Proj}\Big(\bigoplus_{k=0}^{\infty} t^k S_{k\theta}\Big).$$

The action of $\mathbb{T}^{r-1}$ on $\mathbb{C}^{2s}$ is prescribed by the boundary map of the double quiver $Q^{\pm}$ as in Section 4.1.

Notation 4.7. For convenience, we relabel the coordinates z_{e^+} and z_{e^-} of $\mathbb{C}^{2s}$ by z_e and w_e, indexed by $e \in E$. Note that the definition of $X(Q^{\pm}, \theta)$ is independent of the choice of an orientation of Q: a different choice corresponds to exchanging some coordinates z_e with the corresponding w_e.

Suppose that the matrix $A = (a_{ie})$ corresponds to the boundary map for Q. We consider the homogeneous ideal

$$\operatorname{Circ}(\mathscr{B}) := \Big\langle \sum_{e \in E} a_{ie} z_e w_e : i = 2, \dots, r \Big\rangle \subset S. \tag{19}$$

Definition 4.8. The *hypertoric quiver variety* $Y(Q, 0)$ (resp. $Y(Q, \theta)$) is the irreducible subvariety of the affine Lawrence variety $X(Q^{\pm}, 0)$ (resp. of the Lawrence variety $X(Q^{\pm}, \theta)$) cut by the homogeneous ideal $\operatorname{Circ}(\mathscr{B})$.

Remark 4.9. Hypertoric quiver varieties are both hypertoric varieties in the sense of [6, 34], and *Nakajima quiver varieties* whose dimension vector has all coordinates equal to 1; see [67]. In [37], $Y(Q, 0)$ is called a *toric quiver variety*, but note that it is only rarely toric; see [37, Section 10]. Moreover, Altmann and Hille defined a toric quiver variety to be the fibre over 0 of the morphism $\pi_Y : Y(Q, \theta) \to Y(Q, 0)$. To avoid confusion, we will refer to $Y(Q, 0)$ and $Y(Q, \theta)$ as hypertoric quiver varieties.

Since the product $z_e w_e$ is $\mathbb{T}^{r-1}$-invariant, the injective morphism of $\mathbb{C}$-algebras

$$\mathbb{C}[z_e w_e : e \in E] \hookrightarrow \mathbb{C}[z_e, w_e : e \in E]^{\mathbb{T}^{r-1}}$$

induces the fibre product square

$$\begin{array}{ccc} Y(Q,0)=\operatorname{Spec}\left(\dfrac{\mathbb{C}[z_e,w_e\colon e\in E]^{\mathbb{T}^{r-1}}}{\mathrm{Circ}(\mathscr{B})}\right) & \overset{\iota_M}{\hookrightarrow} & X(Q^{\pm},0)=\operatorname{Spec}(\mathbb{C}[z_e,w_e\colon e\in E]^{\mathbb{T}^{r-1}}) \\ \chi_Q\downarrow & & \downarrow\pi_Q \\ \mathbb{C}^{b_1}=\operatorname{Spec}\left(\dfrac{\mathbb{C}[z_e w_e\colon e\in E]}{\mathrm{Circ}(\mathscr{B})}\right) & \overset{\iota_A}{\hookrightarrow} & \mathbb{C}^s=\operatorname{Spec}(\mathbb{C}[z_e w_e\colon e\in E]) \end{array} \tag{20}$$

Hence, $Y(Q,0)$ is the inverse image under the map π_Q of the linear subspace $\mathbb{C}^{b_1}$.

4.4. *Cohomology of hypertoric quiver varieties*

The cohomology of hypertoric quiver varieties can be expressed in terms of the cohomology of the cographic matroid. This permits comparing the Hodge-to-singular correspondence with [18, Theorem 6.11].

Definition 4.10. The *cographic matroid* $\mathscr{C}_Q$ of the unoriented graph $|Q|$ is the matroid whose independent subsets are the subsets I of unoriented edges of $|Q|$ such that $|Q|\setminus I$ is connected. The set $\mathscr{C}_Q$ is partially ordered by inclusion and we denote by $|\mathscr{C}_Q|$ the associated simplicial complex, i.e. the complex whose k-dimensional faces are the independent subsets of cardinality $k+1$.

We now recall some facts about the affinisation morphisms

$$\pi_X\colon X(Q^{\pm},\theta)\to X(Q^{\pm},0),\quad \pi_Y\colon Y(Q,\theta)\to Y(Q,0).$$

In particular, we observe that the top cohomology of $Y(Q,\theta)$ can be identified with the top cohomology of $|\mathscr{C}_Q|$, which explains the occurrence of the cographic matroid in [18, Section 6] via the Hodge-to-singular correspondence; see Sections 7 and 10.

In the following all the stabilities $\theta,\theta'\in\mathbb{Z}^{r-1}$ are generic.

Proposition 4.11. *The following facts hold:*

(1) ([37, Proposition 8.2]) *$X(Q^{\pm},\theta)$ is a smooth toric variety of dimension b_1+s.*

(2) ([37, Proposition 8.2]) *$Y(Q,\theta)$ is a smooth symplectic variety of dimension $2b_1$.*

(3) ([37, Lemma 6.4]) $\pi_X^{-1}(0)=\pi_Y^{-1}(0)=: C(Q^{\pm},\theta)$.

(4) *The irreducible components of $C(Q^{\pm},\theta)$ are Lagrangian in $Y(Q,\theta)$. In particular, they all have dimension b_1.*

(5) ([37, Remark after Lemma 6.5]) *The spaces $C(Q^{\pm},\theta)\subset Y(Q,\theta)\subset X(Q^{\pm},\theta)$ are deformation retracts in one another.*

(6) ([34, Lemma 2.1], [37, Theorem 6.3]) *The symplectic resolutions $Y(Q,\theta)$ and $Y(Q,\theta')$ are diffeomorphic.*

(7) *By* (5) *and* (6)*, the cohomology*

$$H^*(C(Q^\pm,\theta),\mathbb{Q}) \simeq H^*(Y(Q,\theta),\mathbb{Q}) \simeq H^*(X(Q^\pm,\theta),\mathbb{Q})$$

is independent of θ.

(8) ([7, Theorems 7.3.3, 7.8.1]) *There exists an isomorphism*

$$H^{2b_1}(Y(Q,\theta),\mathbb{Q}) \simeq \widetilde{H}^{2b_1-1}(|\mathscr{C}_Q|,\mathbb{Q}).$$

Proof. We just comment on (4). Let F be an irreducible component of $C(Q^\pm,\theta)$. Since $C(Q^\pm,\theta)$ is a fibre of $Y(Q,\theta)\to Y(Q,0)\xrightarrow{\chi_Q}\mathbb{C}^{b_1}$, we have $\dim F \geq \dim Y(Q,\theta) - \dim \mathbb{C}^{b_1} = b_1$ by semicontinuity. On the other hand, the symplectic form vanishes along F^{reg}. Indeed, F is a proper toric variety by (1) and (3). As all toric varieties, it has rational singularities, and its resolution $\widetilde{F}$ has no global differential forms, so $H^0(F^{\mathrm{reg}},\Omega^2_{F^{\mathrm{reg}}}) = H^0(\widetilde{F},\Omega^2_{\widetilde{F}}) = 0$, for instance by [46, Corollary 1.8]. So F is isotropic as in Definition 3.5, and $\dim F \leq b_1$. ∎

4.5. Decomposition theorem for hypertoric quiver varieties

The goal of this section is to describe the restriction $\mathrm{IC}(X(Q^\pm,0),\mathbb{Q})|_{Y(Q,0)}$; see Theorem 4.19. To this end, we briefly recall the statement of the decomposition theorem for semismall maps.

Let $f\colon X\to Y$ be a proper morphism of irreducible varieties. A *stratification* of f is a collection of finitely many locally closed subsets Y_k such that $f^{-1}(Y_k)\to Y_k$ are topologically locally trivial fibrations. A stratum Y_k is *relevant* if $2\dim f^{-1}(Y_k) - \dim Y_k = \dim X$.

Definition 4.12 (Semismall maps). The map f is said to be *semismall* if

$$2\dim f^{-1}(Y_k) - \dim Y_k \leq \dim X \quad \text{for any } k\geq 0.$$

Further, if the stronger inequalities

$$2\dim f^{-1}(Y_k) - \dim Y_k < \dim X \quad \text{for any } k > 0$$

hold, we say that f is *small*.

Theorem 4.13 (Decomposition theorem for semismall maps). *Let $f\colon X\to Y$ be a semismall map from a smooth variety X. Then there exists a canonical isomorphism in $D^bMHM_{\mathrm{alg}}(Y)$ (resp. $D^b(Y,\mathbb{Q})$)*

$$Rf_*\mathbb{Q}_X \simeq \bigoplus_{Y_k} \mathrm{IC}(\overline{Y}_k, R^{\operatorname{codim} Y_k} f_*\mathbb{Q}_{f^{-1}(Y_k)})\langle \operatorname{codim} Y_k/2\rangle,$$

where the summation runs over all the relevant strata of a stratification of f.

In particular, if f is small, then $Rf_*\mathbb{Q}_X \simeq \mathrm{IC}(Y,\mathbb{Q})$.

We note that since, for every k, $f^{-1}(Y_k) \to Y_k$ is a topologically locally trivial fibration, the sheaves $R^{\operatorname{codim} Y_k} f_* \mathbb{Q}_{f^{-1}(Y_k)}$ are in fact local systems. The stalk at a point $y_k \in Y_k$ is the vector space generated by the irreducible components of $f^{-1}(y_k)$, therefore these local systems have finite monodromy.

Given a quiver $Q = (V, E)$, we describe complex Whitney stratifications of the toric Lawrence variety $X(Q^{\pm}, 0)$ and of the hypertoric quiver variety $Y(Q, 0)$.

Definition 4.14. A partition $\underline{V}$ of the set V is a set $\{V_1, \ldots, V_r\}$ of disjoint subsets $V_i \subset V$ such that $V = \bigsqcup_{i=1}^{r} V_i$. The quiver $Q_{\underline{V}}$ is obtained from Q by identifying all vertices in V_i, for any $i = 1, \ldots, r$, and by deleting all loops.

Proposition 4.15. *The toric Lawrence variety $X(Q^{\pm}, 0)$ admits a complex Whitney stratification*

$$X(Q^{\pm}, 0) = \bigsqcup_{\underline{V} \vdash V} X_{\underline{V}} \tag{21}$$

by toric-invariant strata $X_{\underline{V}}$ whose analytic normal slices are isomorphic to $X(Q^{\pm}_{\underline{V}}, 0)$. This induces a complex Whitney stratification of $Y(Q, 0)$,

$$Y(Q, 0) = \bigsqcup_{\underline{V} \vdash V} Y_{\underline{V}} \quad \textit{with } Y_{\underline{V}} := X_{\underline{V}} \cap Y(Q, 0), \tag{22}$$

whose normal slices are isomorphic to $Y(Q_{\underline{V}}, 0)$.

Proof. Let N be the lattice of one-parameter subgroups of the torus $\mathbb{T}^{b_1+s} = N \otimes \mathbb{C}^*$. The affine toric variety $X(Q^{\pm}, 0)$ corresponds to the cone $\sigma \subset N \otimes \mathbb{R}$ generated by the columns of a matrix $B^{\pm}$ Gale dual to the boundary map $A^{\pm} := (A, -A)\colon \mathbb{Z}^{2s} \to \mathbb{Z}^{r-1}$ of the quiver $Q^{\pm}$; see Section 4.1. Let τ be a face of σ, N_τ be the sublattice of N spanned by $\tau \cap N$, and $N(\tau) := N/N_\tau$. The sets of rays of σ, τ, and those of σ not in τ are denoted respectively $\sigma(1)$, $\tau(1)$ and $\tau(1)^c$. The exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^{\tau(1)} & \longrightarrow & \mathbb{Z}^{\sigma(1)} & \longrightarrow & \mathbb{Z}^{\tau(1)^c} & \longrightarrow & 0 \\
 & & \big\downarrow (B^{\pm}_{\tau})^T & & \big\downarrow (B^{\pm})^T & & \big\downarrow & & \\
0 & \longrightarrow & N_\tau & \longrightarrow & N & \longrightarrow & N(\tau) & \longrightarrow & 0
\end{array}$$

are dual to

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M(\tau) & \longrightarrow & \mathbb{Z}^{\tau(1)^c} & \xrightarrow{A^{\pm}|_{\tau(1)^c}} & \mathbb{Z}^{r-1} & & \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}^{\sigma(1)} & \xrightarrow{A^{\pm}} & \mathbb{Z}^{r-1} & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & M_\tau & \longrightarrow & \mathbb{Z}^{\tau(1)} & \xrightarrow{A^{\pm}_{\tau}} & \mathbb{Z}^{r_\tau-1} & \longrightarrow & 0
\end{array}$$

In particular, the resulting map $A_\tau^\pm$ is the boundary map of a quiver $Q(\tau)$ obtained by contracting the edges corresponding to $\tau(1)^c$. Up to loops, $Q(\tau)$ is isomorphic to $Q_{\underline{V}}$ for some partition $\underline{V}$ of V. Therefore, $X(Q^\pm, 0)$ is covered by toric charts of the form

$$X(Q(\tau), 0) \times N(\tau) \otimes \mathbb{C}^* \simeq X(Q_{\underline{V}}^\pm, 0) \times \text{smooth factor}, \tag{23}$$

where the isomorphism follows from Lemma 4.2. We have obtained the stratification (21) which is Whitney by Lemma 2.2. By inspecting the restriction of the equations (19) to each chart (23), we get the Whitney stratification (22) too; see also [71, Section 2]. ■

Remark 4.16. A face τ of the cone defining the toric variety $X(Q^\pm, 0)$ corresponds to a toric orbit X_τ of $X(Q^\pm, 0)$. By the proof of Proposition 4.15 and adopting the notation used there, $X_{\underline{V}}$ is the union of the toric strata X_τ whose corresponding quiver $Q(\tau)$ is isomorphic to $Q_{\underline{V}}$ up to loops.

Proposition 4.17. *The map $\pi_X \colon X(Q^\pm, \theta) \to X(Q^\pm, 0)$ is small.*

Proof. The fan of $X(Q^\pm, \theta)$ is a regular triangulation of the cone $\sigma \subset N \otimes \mathbb{R}$ of $X(Q^\pm, 0)$; see [37, Section 2]. Since any regular triangulation of σ restricts to a regular triangulation of its face τ, the restriction π_X to the open set $X(Q_{\underline{V}}^\pm, 0) \times X_{\underline{V}}$ is the resolution

$$X(Q_{\underline{V}}^\pm, [\theta]) \times X_{\underline{V}} \to X(Q_{\underline{V}}^\pm, 0) \times X_{\underline{V}}.$$

Therefore, given $y \in X_{\underline{V}} \subset X(Q^\pm, 0) \times X_{\underline{V}}$ we obtain

$$\begin{aligned} 2 \dim \pi_X^{-1}(y) &= 2 \dim C(Q_{\underline{V}}^\pm, [\theta]) = 2 b_1(Q_{\underline{V}}) \\ &< b_1(Q_{\underline{V}}) + s = \dim X(Q_{\underline{V}}^\pm, 0) \\ &= \dim X(Q^\pm, 0) - \dim X_{\underline{V}}, \end{aligned}$$

i.e. π_X is small. ■

Proposition 4.18. *The map $\pi_Y \colon Y(Q, \theta) \to Y(Q, 0)$ is semismall.*

Proof. By Proposition 4.11 (2), $Y(Q, \theta)$ is a smooth symplectic variety. Any symplectic resolution of singularities is semismall by [45, Lemma 2.11]. ■

Theorem 4.19 (Local Hodge-to-singular correspondence). *We have a canonical isomorphism in $D^b MHM_{\mathrm{alg}}(Y)$ (resp. $D^b(Y, \mathbb{Q})$)*

$$\begin{aligned} &\mathrm{IC}(X(Q^\pm, 0), \mathbb{Q})|_{Y(Q,0)}[2b_1] \\ &\qquad \simeq \bigoplus_{\underline{V} \vdash V} \mathrm{IC}(Y_{\underline{V}}, \mathbb{Q}) \otimes H^{b_1(Q_{\underline{V}})}(Y(Q_{\underline{V}}, \theta), \mathbb{Q})[2b_1]\langle b_1(Q_{\underline{V}})\rangle. \end{aligned}$$

In particular, the restriction $\mathrm{IC}(X(Q^\pm, 0), \mathbb{Q})|_{Y(Q,0)}[2b_1]$ is pure and perverse.

Proof. Consider the cartesian square

$$\begin{array}{ccc} Y(Q,\theta) & \xhookrightarrow{\iota_{M,\theta}} & X(Q^{\pm},\theta) \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi_X} \\ Y(Q,0) & \xhookrightarrow{\iota_A} & X(Q^{\pm},0) \end{array}$$

The map π_X is small, π_Y is semismall and $\iota_{M,\theta}$ is a regular embedding. Therefore, the decomposition theorem for small maps gives $\mathrm{IC}(X(Q^{\pm},0),\mathbb{Q}) \simeq R\pi_{X,*}\mathbb{Q}_{X(Q^{\pm},\theta)}$. By proper base change, we obtain $R\pi_{X,*}\mathbb{Q}_{X(Q^{\pm},\theta)}|_{Y(Q,0)} \simeq R\pi_{Y,*}\mathbb{Q}_{Y(Q,\theta)}$. The statement now follows from the decomposition theorem for semismall maps. In this case, the local systems $R^{\dim X - \dim Y_k} f_*\mathbb{Q}_{f^{-1}(Y_k)}$ of Theorem 4.13 are trivial, supported on the relevant strata $Y_{\underline{V}}$ and with fibre $H^{\mathrm{top}}(\pi_Y^{-1}(y),\mathbb{Q}) \simeq H^{b_1(Q_{\underline{V}})}(Y(Q_{\underline{V}},\theta),\mathbb{Q})$ for all $y \in Y_{\underline{V}}$; see [71, Section 5]. ∎

5. Universal compactified Jacobians

The goal of this section is to enlarge the Hitchin fibration to a universal family of compactified Jacobians. Fix integers $g > 1$, $n > 1$, d as above. Let X be a projective curve with ample canonical bundle ω_X and planar singularities.

Definition 5.1. A rank 1 torsion free sheaf $\mathcal{I}$ on X is *semistable* with respect to the canonical polarisation ω_X if for any proper subcurve $Y \subset X$ we have

$$\frac{\chi(X,\mathcal{I})}{\deg(\omega_X)} \le \frac{\chi(Y,\mathcal{I}_Y)}{\deg(\omega_X|_Y)}, \tag{24}$$

where $\mathcal{I}_Y$ is the biggest torsion free quotient of the restriction $\mathcal{I}|_Y$. The *compactified Jacobian* $\overline{J}_d(X)$ is the projective coarse moduli space of semistable rank 1 torsion free sheaves on X of degree $d + (n - n^2)(1 - g)$ with respect to the canonical polarisation; see [78].

The BNR correspondence presents the Dolbeault moduli space $M(n,d)$ as the relative compactified Jacobian of the spectral curve family; see [3, 42, 74]. It says that given a rank 1 torsion free sheaf $\mathcal{I}$ on the spectral curve $\pi_a\colon C_a \to C$, the sheaf $\mathcal{E} \simeq \pi_{a,*}\mathcal{I}$ is a vector bundle of rank n equipped with a Higgs field ϕ, induced from the $\mathcal{O}_{C_a}$-module structure. Conversely, any Higgs bundle $(\mathcal{E},\phi)$ is the pushforward of some rank 1 torsion free sheaf on C_a.

Proposition 5.2 ([3, 42, 65, 74]). *The fibre of the Hitchin fibration $\chi^{-1}(n,d)(a)$ with $a \in A_n^{\mathrm{red}}$ is isomorphic to the compactified Jacobian $\overline{J}_d(C_a)$ of the spectral curve C_a.*

Remark 5.3. The stability condition for rank 1 torsion free sheaves on reduced curves used in [65, (10.5)] and [18] coincides with (24). Suppose that X is a reduced spectral curve $C_a = \bigcup_{i=1}^r C_i$ whose component C_i has degree n_i, and $n = \sum_{i=1}^r n_i$. Let $\mathcal{I}$ be a rank 1 torsion free sheaf of degree $d + (n - n^2)(1 - g)$ on C_a. By Riemann–Roch, (24) reads as follows: for any subcurve $C_I = \bigcup_{i \in I} C_i$, with $I \subset \{1, \dots, r\}$, we have

$$\chi(Y, \mathcal{I}_Y) \geq (d + n(1-g)) \frac{\sum_{i \in I} n_i}{n},$$

which is the stability condition [65, (10.5)].

Following Simpson, Caporaso, Pandharipande and Esteves [26, 69, 78] (cf. also [8, Section 2.3, Facts 2.7, 2.12] and [18]) we can enlarge the Hitchin fibration to a relative compactified Jacobian of a versal deformation of C_a. To this end, let $\bar{\mathcal{M}}_{g'}$ be the Deligne–Mumford stack of semistable curves of genus $g' := n^2(g-1) + 1$, and $A_n^{\times} := \bigsqcup_{\underline{n} \vdash n} S_{\underline{n}}^{\times}$. By [18, Lemma 2.1] the flat universal spectral curve $\mathcal{C}_n^{\times} \to A_n^{\times}$ induces a morphism

$$f^{\times} \colon A_n^{\times} \to \bar{\mathcal{M}}_{g'}.$$

Proposition 5.4 (Universal compactified Jacobian). *There exists an open substack $\mathcal{B} \subseteq \bar{\mathcal{M}}_{g'}$ containing $f^{\times}(A_n^{\times})$, and an irreducible* (*not necessarily smooth*) *Deligne–Mumford stack $\pi \colon \bar{J}_{d,\mathcal{B}} \to \mathcal{B}$ that étale locally is a relative compactified Jacobian parametrising semistable rank* 1 *torsion free sheaves of degree $d + (n - n^2)(1 - g)$ with respect to the canonical polarisation.*

Proof. The first paragraph of the proof of [18, Proposition 5.9] does not require the coprimality of n and d, and works verbatim in our context. ∎

Proposition 5.5. *For every partition $\underline{n}$ of n, let $a \in S_{\underline{n}}^{\times}$. There exists a local multisection A_a of $f^{\times} \colon A_n^{\times} \to f^{\times}(A_n^{\times}) \subseteq \mathcal{B}$ passing through a, i.e. A_a is a smooth locally closed subset of $A_n^{\times}$, passing through a, such that $f^{\times}(A_a)$ is an open subset in $f^{\times}(A_n^{\times})$, and the restriction $f^{\times}|_{A_a} \colon A_a \to f^{\times}(A_a)$ is étale. Furthermore, we have a cartesian diagram*

$$\begin{array}{ccccc} \chi(n,d)^{-1}(A_a) & \xrightarrow{\ \simeq\ } & \bar{J}_{d,\mathcal{B}} \times_{\mathcal{B}} A_a & \xrightarrow{\ i_M\ } & \bar{J}_{d,\mathcal{B}} \\ & \searrow^{\chi(n,d)} & \downarrow & & \downarrow \pi \\ & & A_a & \xrightarrow{\ f^{\times}|_{A_a}\ } & \mathcal{B} \end{array}$$

Proof. The proof of [18, Corollary 5.10] works verbatim. ∎

Remark 5.6. The $\mathbb{G}_{\mathrm{m}}$-action on A_n and the translation action $H^0(C, \omega_C) \otimes A_n \to A_n$ lift to the universal spectral curve $\mathcal{C}_n \to A_n$ and so to $M(n,d)$. Therefore, the group action induces an analytic isomorphism between an open neighbourhood $U_a \subseteq A_n$ of A_a and a neighbourhood of the zero-section of the trivial vector bundle $p \colon A_a \times (H^0(C, \mathcal{O}_C) \oplus H^0(C, \omega_C)) \to A_a$ such that there exists a homeomorphism

$$\chi(n,d)^{-1}(U_a) \simeq \chi(n,d)^{-1}(A_a) \times (H^0(C, \mathcal{O}_C) \oplus H^0(C, \omega_C)),$$

and so

$$R\chi(n,d)_* \operatorname{IC}(M(n,d), \mathbb{Q})|_{U_a} \simeq p^*(R\chi(n,d)_* \operatorname{IC}(\chi(n,d)^{-1}(A_a), \mathbb{Q})).$$

6. Kodaira–Spencer map for spectral curves

The differential of the map $f^{\times}\colon A_n \to \mathcal{B}$ in Proposition 5.5 can be identified with the Kodaira–Spencer map; see for instance [18, Section 5.2]. Its kernel is isomorphic to $H^0(C, \mathcal{O}_C) \oplus H^0(C, \omega_C)$; see again [18, Section 5.2]. In Proposition 6.3 we study its image.

6.1. *Kodaira–Spencer map*

We start with some recollection about the deformation of nodal spectral curves.

Definition 6.1. The *Kodaira–Spencer map* is the map

$$KS_a\colon T_a A_n \simeq H^0(C_a, N_{C_a/T^*C_a}) \to \operatorname{Ext}^1(\Omega^1_{C_a}, \mathcal{O}_{C_a}) \tag{25}$$

obtained by applying the functor $\operatorname{Hom}(-, \mathcal{O}_{C_a})$ to the relative cotangent sequence

$$0 \to N^*_{C_a/T^*C} \to \Omega^1_{T^*C}|_{C_a} \to \Omega^1_{C_a} \to 0. \tag{26}$$

Applying the functor $\mathcal{H}om(-, \mathcal{O}_{C_a})$ to (26), we obtain the exact sequence of coherent sheaves

$$0 \to T_{C_a} \to T_{T^*C}|_{C_a} \to N_{C_a/T^*C} \to \mathbb{T}^1 := \mathcal{E}xt^1(\Omega^1_{C_a}, \mathcal{O}_{C_a}) \to 0.$$

The first tangent sheaf $\mathbb{T}^1$ is a skyscraper sheaf supported at the nodes $E(C_a)$ of C_a and it parametrises smoothings of the nodes. The kernel of $N_{C_a/T^*C} \to \mathbb{T}^1$ is easily identified to $N_{C_a/T^*C} \otimes \mathcal{I}_{E(C_a)}$, where $\mathcal{I}_{E(C_a)}$ is the product of the maximal ideals of the nodes $E(C_a)$; see [77, Example 4.7.1]. Hence, there exists a short exact sequence

$$0 \to N_{C_a/T^*C} \otimes \mathcal{I}_{E(C_a)} \to N_{C_a/T^*C} \to \mathbb{T}^1 \to 0$$

inducing the exact sequence in cohomology

$$0 \to H^0(C_a, N_{C_a/T^*C} \otimes \mathcal{I}_{E(C_a)}) \to T_a A_n = H^0(C_a, N_{C_a/T^*C}) \to \mathbb{T}^1(C_a).$$

Note that the last map $T_a A_n \to \mathbb{T}^1(C_a)$ factors as

$$\begin{aligned} T_a A_n = H^0(C_a, N_{C_a/T^*C}) \xrightarrow{KS_a} &\operatorname{Ext}^1(\Omega^1_{C_a}, \mathcal{O}_{C_a}) \\ &\to H^0(C_a, \mathcal{E}xt^1(\Omega^1_{C_a}, \mathcal{O}_{C_a})) = \mathbb{T}^1(C_a), \end{aligned} \tag{27}$$

where the latter map comes from the local-to-global Ext spectral sequence.

6.2. *Image of the Kodaira–Spencer map for spectral curves*

Let $\underline{n}'$ be a partition of n. Then $\chi(n,d)^{-1}(S^{\times}_{\underline{n}'})$ admits a stratification by type

$$\chi(n,d)^{-1}(S^{\times}_{\underline{n}'}) = \bigsqcup_{\underline{n} \vdash n} \chi(n,d)^{-1}(S^{\times}_{\underline{n}'}) \cap M^{\circ}_{\underline{n}}(d).$$

Given the partition $\underline{n} = (n_1, \dots, n_r)$, the condition $(\mathcal{E}, \phi) \in \chi(n, d)^{-1}(S^{\times}_{\underline{n}'}) \cap M^{\circ}_{\underline{n}}(d)$ means that the Higgs bundle $(\mathcal{E}, \phi)$ splits as

$$(\mathcal{E}, \phi) = \bigoplus_{i=1}^{r} (\mathcal{E}_i, \phi_i), \tag{28}$$

where $(\mathcal{E}_i, \phi_i)$ are stable Higgs bundles of rank n_i and degree $d_i = dn_i/n$. Further, the spectral curve C_a of $(\mathcal{E}, \phi)$ is nodal and the union of the (possibly reducible) spectral curves C_{a_i} of $(\mathcal{E}_i, \phi_i)$,

$$C_a = \bigcup_{i=1}^{r} C_{a_i}. \tag{29}$$

A normal slice $N_{\underline{n}} \subseteq A_a$ to $S_{\underline{n}}$ through a parametrises spectral curves obtained by smoothing the nodes $E(\Gamma_{\underline{n}}) := \bigcup_{i,j=1}^{r} C_{a_i} \cap C_{a_j}$ of C_a.

Notation 6.2. For a partition $\underline{n}$ of n we will denote by $\Gamma_{\underline{n}}$ the dual graph of any spectral curve given by a point of $S^{\times}_{\underline{n}}$, i.e. $\Gamma_{\underline{n}}$ is the graph with vertices $[r] := \{1, \dots, r\}$ corresponding to the irreducible components of the curve and $n_i n_j (2g-2)$ edges between the vertices i, j, corresponding to the intersection points of the components. By abuse of notation we continue to denote by $E(\Gamma_{\underline{n}})$ the set of edges of $\Gamma_{\underline{n}}$.

Proposition 6.3. *The image of the map*

$$T_a N_{\underline{n}} \hookrightarrow T_a A_a \xrightarrow{df^{\times}} \operatorname{Ext}^1(\Omega^1_{C_a}, \mathcal{O}_{C_a}) \twoheadrightarrow \bigoplus_{e \in E(\Gamma_{\underline{n}})} H^0(C_a, \mathcal{E}xt^1(\Omega^1_{C_a}, \mathcal{O}_{C_a})_e) \simeq \operatorname{Spec} \mathbb{C}[T_e : e \in E(\Gamma_{\underline{n}})]$$

is cut by the linear equations

$$\operatorname{Circ}(\mathscr{B}) = \Big\langle \sum_{e \in E} a_{ie} T_e : i = 2, \dots, r \Big\rangle,$$

where $A = (a_{ie})$ is the boundary map of a quiver underlying the graph $\Gamma_{\underline{n}}$ (cf. Section 4).

Proof. Let $\nu_{\underline{n}} : C_{\underline{n},a} := \bigsqcup_{i=1}^{r} C_{a_i} \to C_a$ be the partial normalisation obtained by blowing-up C_a at $E(\Gamma_{\underline{n}})$. Applying the functor $\mathcal{H}om(-, {}_{C_a})$ to the short exact sequence

$$0 \to \mathcal{O}_{C_a} \to \nu_{\underline{n},*}\mathcal{O}_{C_{\underline{n},a}} \to \bigoplus_{e \in E(\Gamma_{\underline{n}})} \mathcal{O}_e \to 0,$$

we obtain the short exact sequence

$$0 \to N_{C_a/T^*C} \otimes \mathcal{I}_{E(\Gamma_{\underline{n}})} \to N_{C_a/T^*C} \to \mathbb{T}^1_{\underline{n}} := \bigoplus_{e \in E(\Gamma_{\underline{n}})} \mathcal{E}xt^1(\mathcal{O}_e, \omega_{C_a}) \to 0,$$

where $\mathcal{I}_{E(\Gamma_{\underline{n}})}$ is the product of the maximal ideals of the nodes $E(\Gamma_{\underline{n}})$, and ω_{C_a} is identified with N_{C_a/T^*C} by adjunction. Taking global cohomology, we get the long exact sequence

$$0 \to H^0(C_a, N_{C_a/T^*C} \otimes \mathcal{I}_{E(\Gamma_{\underline{n}})}) \to T_a A_n = H^0(C_a, N_{C_a/T^*C}) \to \mathbb{T}^1_{\underline{n}}(C_a) \\ \to H^1(C_a, N_{C_a/T^*C} \otimes \mathcal{I}_{E(\Gamma_{\underline{n}})}) \to H^1(C_a, N_{C_a/T^*C}) \to 0. \quad (30)$$

By Serre duality we get

$$H^0(C_a, N_{C_a/T^*C} \otimes \mathcal{I}_{E(\Gamma_{\underline{n}})}) \simeq H^0(C_a, \mathcal{H}om(\nu_{\underline{n},*}\mathcal{O}_{C_{\underline{n},a}}, \omega_{C_a})) \simeq H^1(C_a, \nu_{\underline{n},*}\mathcal{O}_{C_{\underline{n},a}})^\vee \\ \simeq \bigoplus_{i=1}^r H^1(C_i, \mathcal{O}_{C_i})^\vee \simeq \bigoplus_{i=1}^r H^0(C_i, \omega_{C_i}) \simeq T_a S_{\underline{n}}. \quad (31)$$

Combining (27), (30) and (31), we see that the image of the map

$$df^\times \colon T_a N_{\underline{n}} \simeq T_a A_n / T_a S_{\underline{n}} \to \mathbb{T}^1_{\underline{n}}(C_a)$$

coincides with the kernel of

$$\mathbb{T}^1_{\underline{n}}(C_a) \to \ker\{H^1(C_a, N_{C_a/T^*C} \otimes \mathcal{I}_{E(\Gamma_{\underline{n}})}) \to H^1(C_a, N_{C_a/T^*C})\}.$$

Again via Serre duality, this map is isomorphic to

$$\mathbb{C}^{E(\Gamma_{\underline{n}})} \simeq \bigoplus_{e \in E(\Gamma_{\underline{n}})} H^0(\mathcal{O}_e)^\vee \to \ker\{H^0(C_a, \nu_{\underline{n},*}\mathcal{O}_{C_{\underline{n},a}})^\vee \to H^0(C_a, \mathcal{O}_{C_a})^\vee\} \simeq \mathbb{C}^{r-1},$$

which in turn can be identified with the boundary map of a quiver underlying the graph $\Gamma_{\underline{n}}$. ■

7. Singularities of universal compactified Jacobians

By the BNR correspondence a Higgs bundle $(\mathcal{E}, \phi)$ corresponds to a rank 1 torsion free sheaf $\mathcal{I}$ on the spectral curve C_a, where $\mathcal{E}$ is isomorphic to $\pi_{a,*}\mathcal{I}$ via the projection $\pi_a \colon C_a \to C$; see [3,42,74]. Suppose that C_a is nodal, and let Σ be the set of nodes where $\mathcal{I}$ fails to be locally free. Define $M^\times_{\underline{n}}(d)$ as the open subset of Higgs bundles of $M^\circ_{\underline{n}}(d)$ with nodal spectral curve.

Proposition 7.1. *Let $\underline{n} = (n_1, \dots, n_r)$ be a partition of n. Then $(\mathcal{E}, \phi)$ lies in $M^\times_{\underline{n}}(d)$ if and only if there exists a partial normalisation $\nu_\Sigma \colon C_\Sigma \to C_a$ obtained by blowing-up the nodes Σ and a line bundle $\mathcal{L}$ on C_Σ such that* (1) *C_Σ has exactly r connected components, and* (2) *$\mathcal{E} \simeq \nu_{\Sigma,*}\mathcal{L}$.*

Proof. By assumption, $(\mathcal{E}, \phi)$ splits as $(\mathcal{E}, \phi) = \bigoplus_{i=1}^r (\mathcal{E}_i, \phi_i)$, where $(\mathcal{E}_i, \phi_i)$ are stable Higgs bundles of rank n_i and degree $d_i = dn_i/n$. The spectral curve C_a of $(\mathcal{E}, \phi)$ is the union of the spectral curves C_{a_i},

$$C_{a_i} \overset{j_i}{\hookrightarrow} C_a = \bigcup_{i=1}^r C_{a_i} \xrightarrow{\pi_a} C.$$

By the BNR correspondence, there exist rank 1 torsion free sheaves $\mathcal{I}$ and $\mathcal{I}_i$ on C_a and C_{a_i} respectively such that $\mathcal{E} = \pi_{a,*}\mathcal{I}$, $\mathcal{E}_i = (\pi_a \circ j_i)_*\mathcal{I}_i$, and

$$\mathcal{I} = \bigoplus_{i=1}^{r} j_{i,*}\mathcal{I}_i. \tag{32}$$

Note that $\mathcal{I}$ fails to be locally free at the node $C_{a_i} \cap C_{a_j}$, $i \neq j$. Therefore, the blowup $\nu_\Sigma\colon C_\Sigma \to C_a$ of C_a at Σ factors through the disjoint union of C_{a_i}

$$\nu_\Sigma\colon C_\Sigma \to \bigsqcup_{i=1}^{r} C_{a_i} \to C_a. \tag{33}$$

Since C_a has only nodal singularities, there exists a line bundle $\mathcal{L}$ on C_Σ such that $\mathcal{I} = \nu_{\Sigma,*}\mathcal{L}$; see [31, Proposition 3.4]. By (33), the curve C_Σ has at least r components, and exactly r by the stability of $(\mathcal{E}_i, \phi_i)$. Otherwise, the decomposition (32) could be further refined, and so (28) could be refined too, which is a contradiction. ∎

Proposition 7.2. *Suppose that $(\mathcal{E}, \phi)$ lies in $M_{\underline{n}}^{\times}(d)$. After removing all the loops, the dual graph $\Gamma_{C_a}(\Sigma)$ of the curve obtained by smoothing the nodes not in Σ is the graph $\Gamma_{\underline{n}}$.*

Proof. As in the proof of Proposition 7.1, we denote the spectral curves of $(\mathcal{E}, \phi)$ and of its stable summands by C_a and C_{a_i} respectively, and we write

$$C_a = \bigcup_{i=1}^{r} C_{a_i} = \bigcup_{j=1}^{r'} C_j, \quad C_{a_i} = \bigcup_{j\in S_i} C_j,$$

where C_j are the irreducible components of C_a, and $[r'] = \bigsqcup_{i=1}^{r} S_i$ is a partition of the set of r' elements. Clearly if C_j has degree n'_j over C, the spectral curves C_{a_i} has degree $n_i = \sum_{j\in S_i} n'_j$.

The graph $\Gamma_{C_a}(\Sigma)$ is obtained from the dual graph Γ_{C_a} of the spectral curve C_a by contracting all the edges not in Σ.[7] Let $q\colon \Gamma_{C_a} \to \Gamma_{C_a}(\Sigma)$ be the contraction map. If some edge between v_j and v_k is not contained in Σ, then $q(v_j) = q(v_k)$, and all edges in Σ are sent to loops in $\Gamma_{C_a}(\Sigma)$. The graph $\Gamma_{C_a}(\Sigma)$ has as many vertices as there are connected components in $\Gamma_{C_a} \setminus \Sigma \simeq \Gamma_{C_\Sigma}$, i.e. r by Proposition 7.1. The number of edges between two distinct vertices w_i and $w_{i'}$ in $\Gamma_{C_a}(\Sigma)$ is exactly

$$\begin{aligned}\sum_{\substack{v_j\in q^{-1}(w_i)\\ v_k\in q^{-1}(w_{i'})}} (\#\text{ edges between } v_j \text{ and } v_k) &= \sum_{\substack{v_j\in q^{-1}(w_i)\\ v_k\in q^{-1}(w_{i'})}} n'_j n'_k(2g-2)\\ &= \Big(\sum_{j\in S_i} n'_j\Big)\cdot\Big(\sum_{k\in S_{i'}} n'_k\Big)(2g-2) = n_i n_{i'}(2g-2).\end{aligned}$$

[7] By abuse of notation, we identify the set Σ of nodes with the corresponding set of edges in the dual graph Γ_{C_a}.

We conclude that the graphs $\Gamma_{C_a}(\Sigma)$ and $\Gamma_{\underline{n}}$ coincide after removing all loops in the former. ∎

Remark 7.3. The dual graph of the spectral curve may depend on the choice of $(\mathcal{E}, \phi)$ in $M_{\underline{n}}^{\times}(d)$. Proposition 7.2 asserts that up to loops the dual graph $\Gamma_{C_a}(\Sigma)$ does not!

The singularities of the universal compactified Jacobian $\overline{J}_{d,\mathcal{B}}$ are affine Lawrence varieties $X(\Gamma_{\underline{n}}^{\pm}, 0)$, as explained in [8, 9]. The corresponding hypertoric quiver variety $Y(\Gamma_{\underline{n}}, 0)$ provides the local model for the singularities of $M(n, d)$.

Theorem 7.4 (Local model of $\overline{J}_{d,\mathcal{B}}$ and $M(n,d)$). *Let $a \in S_{\underline{n}'}^{\times}$, and A_a be the local multisection of $f^{\times}\colon A_{\underline{n}}^{\times} \to f^{\times}(A_{\underline{n}}^{\times}) \subseteq \mathcal{B}$ defined in Proposition 5.5. Let $(\mathcal{E}, \phi)$ be a Higgs bundle in $M_{\underline{n}}^{\times}(d)$, and $\mathcal{I}$ be the corresponding rank 1 torsion free sheaf on the spectral curve C_a. Then there exists an analytic neighbourhood V of an étale chart of $\overline{J}_{d,\mathcal{B}}$ centred at $(C_a, \mathcal{I})$ such that the restriction to V of the fibre product square in Proposition 5.5,*

$$\begin{array}{ccc} M(n,d) \supset U & \hookrightarrow & V \subset \overline{J}_{d,\mathcal{B}} \\ \chi(n,d) \downarrow & & \downarrow \pi \\ \chi(n,d)(U) & \xhookrightarrow{f^{\times}} & \pi(V) \end{array}$$

with $U := V \cap \chi(n,d)^{-1}(A_a)$, is locally isomorphic to

$$\begin{array}{ccc} Y(\Gamma_{\underline{n}}, 0) \times \mathbb{C}^{\dim M_{\underline{n}} - 1 - g} & \xhookrightarrow{(\iota_M, l_1)} & X(\Gamma_{\underline{n}}^{\pm}, 0) \times \mathbb{C}^{c(\underline{n}, g)} \\ (\chi_{\Gamma_{\underline{n}}}, l_2) \downarrow & & \downarrow (\pi_{\Gamma_{\underline{n}}}, l_4) \\ \mathbb{C}^{b_1(\Gamma_{\underline{n}})} \times \mathbb{C}^{d(\underline{n}, g)} & \xhookrightarrow{(\iota_A, l_3)} & \mathbb{C}^{s} \times \mathbb{C}^{3n^2(g-1)-s} \end{array} \tag{34}$$

where the maps χ, π, ι_M and ι_A are defined in (20), the maps l_i are linear, $s := \# E(\Gamma_{\underline{n}})$ and

$$d(\underline{n}, g) := \tfrac{1}{2}(\dim M(n,d) - \dim Y(\Gamma_{\underline{n}}, 0)) - g - 1 = (n^2 - 1)(g - 1) - 1 - b_1(\Gamma_{\underline{n}}),$$
$$c(\underline{n}, g) := \dim \overline{J}_{d,\mathcal{B}} - \dim X(\Gamma_{\underline{n}}^{\pm}, 0) = 4n^2(g-1) + 1 - b_1(\Gamma_{\underline{n}}) - s.$$

Proof. Let Σ be the set of those nodes where $\mathcal{I}$ fails to be locally free, and $\Gamma_{C_a}(\Sigma)$ be the dual graph of the curve obtained by smoothing the nodes not in Σ. By [8, Theorem A] there exists an analytic neighbourhood V of $(C_a, \mathcal{I})$ in $\overline{J}_{d,\mathcal{B}}$ such that

$$V \simeq_{\mathrm{loc}} X(\Gamma_{C_a}(\Sigma)^{\pm}, 0) \times \mathbb{C}^{c(\underline{n}, \Sigma)}$$

with $c(\underline{n}, \Sigma) := 4g(C_a) - 3 - b_1(\Gamma_{C_a}(\Sigma)) - \# E(\Gamma_{C_a}(\Sigma))$. By [9, Lemma 4.3] and Proposition 7.2 we obtain

$$V \simeq_{\mathrm{loc}} X(\Gamma_{C_a}(\Sigma)^{\pm}, 0) \times \mathbb{C}^{c(\underline{n}, \Sigma)} \simeq X(\Gamma_{\underline{n}}^{\pm}, 0) \times \mathbb{C}^{c(\underline{n}, g)}.$$

The fact that the morphism $\pi\colon V \subset \bar{J}_{d,\mathcal{B}} \to \pi(V)$ can be identified with $(\pi_{\Gamma_{\underline{n}}}, l_4)$ is explained in [9, Section 7.2]. The differential of the map $f^{\times}$ is the Kodaira–Spencer map $KS_a\colon T_a A_a \to H^1(C_a, \mathbb{T}^1_{C_a})$, and so the identification of the Hitchin fibration $\chi(n,d)$ with the product of the map $\chi_{\Gamma_{\underline{n}}}\colon Y(\Gamma_{\underline{n}}, 0) \to \mathbb{C}^{b_1(\Gamma_{\underline{n}})}$ with a linear map l_2 follows from Proposition 6.3 and Definition 4.8. ∎

Theorem 7.5. *Let $A_a \subset A_n$ be as in Proposition 5.5. Then*

$$\mathrm{IC}(\bar{J}_{d,\mathcal{B}}, \mathbb{Q})|_{\chi(n,d)^{-1}(A_a)}[\dim M(n,d)]$$

is a semisimple perverse sheaf. In particular,

$$\mathrm{IC}(\bar{J}_{d,\mathcal{B}}, \mathbb{Q})|_{\chi(n,d)^{-1}(A_a)}[\dim M(n,d)] \simeq \bigoplus_{\underline{n} \in \mathcal{P}_d} i_M^* \,\mathrm{IC}(M_{\underline{n}}(d), \mathcal{L}'_{\underline{n}})[\dim M(n,d)]\langle \operatorname{codim} S_{\underline{n}} \rangle,$$

where $\mathcal{L}'_{\underline{n}} := \mathcal{H}^{\operatorname{codim} M_{\underline{n}}(d)}(\mathrm{IC}(\bar{J}_{d,\mathcal{B}}, \mathbb{Q}))$ is a local system on $M_{\underline{n}}(d)$ of rank $h^{b_1(\Gamma_{\underline{n}})}(Y(\Gamma_{\underline{n}}, 0), \mathbb{Q})$.

We anticipate that the local systems $\mathcal{L}'_{\underline{n}}$ are pullbacks of the line bundles $\mathcal{L}_{\underline{n}}$ in [18, Corollary 6.20]; see Proposition 7.6 and the proof of Theorem 1.3 below. In particular, their monodromy is known and induced by the automorphisms of the graph $\Gamma_{\underline{n}}$; see Remark 10.1.

The idea of the proof of Theorem 7.5 is to glue the local Hodge-to-singular correspondence (Theorem 4.19) given the local models in Theorem 7.4.

Proof of Theorem 7.5. Let X be a complex algebraic variety. The category $MH(X, w)$ of polarisable Hodge modules of fixed weight w admits an exact and faithful functor rat: $MH(X, w) \to \mathrm{Perv}(X)_{\mathbb{Q}}$ to the category of perverse sheaves (with coefficient in $\mathbb{Q}$); see [72]. The category $MH(X, w)$ is semisimple by [72, Lemma 5], and the perverse sheaf that one obtains by applying the functor rat is semisimple [72, Lemma 4]. Therefore, the semisimplicity of $\mathrm{IC}(\bar{J}_{d,\mathcal{B}}, \mathbb{Q})|_{\chi(n,d)^{-1}(A_a)}[\dim M(n,d)]$ follows from purity and perversity. Since purity and perversity are local conditions in the analytic topology (see for instance [14, Lemma 2.2]), it is enough to check them on the local model, which is the content of Theorem 4.19. ∎

Proposition 7.6. *Define $M^{\circ\circ}_{\underline{n}}(d) := \chi(n,d)^{-1}(S^{\times}_{\underline{n}}) \cap M_{\underline{n}}(d)$. The local system $\mathcal{L}'_{\underline{n}}$ on $M^{\circ\circ}_{\underline{n}}(d)$ descends to a local system $\mathcal{L}_{\underline{n}}$ on $S^{\times}_{\underline{n}}$,*

$$\mathcal{L}'_{\underline{n}} = \chi(n,d)^* \mathcal{L}_{\underline{n}}.$$

Proof. Denote by F_b the fibre of $\chi(n,d)\colon M^{\circ\circ}_{\underline{n}}(d) \to S_{\underline{n}}$. By Proposition 8.1, the restriction $\mathrm{IC}(\bar{J}_{d,\mathcal{B}}, \mathbb{Q})|_{F_b}$ is isomorphic to the complex

$$\mathbb{Q}_{F_b} \otimes IH^i(X(\Gamma^{\pm}_{\underline{n}}, 0), \mathbb{Q})[-i] \simeq \mathbb{Q}_{F_b} \otimes H^i(Y(\Gamma_{\underline{n}}, 0), \mathbb{Q})[-i]$$

of constant sheaves with trivial differential. Hence,

$$\mathcal{L}'_{\underline{n}}|_{F_b} \simeq \mathbb{Q}_{F_b} \otimes H^{b_1(\Gamma_{\underline{n}})}(Y(\Gamma_{\underline{n}}, 0), \mathbb{Q})$$

is a component of the complex, so a trivial local system. Therefore, $\mathcal{L}'_{\underline{n}}$ descends to a local system on $S^{\times}_{\underline{n}}$. ■

8. Tubular neighbourhood of the Jacobian of the normalisation of a spectral curve

To complete the proof of Proposition 7.6, we must trivialise a tubular neighbourhood of the subvariety F_b in the universal compactified Jacobian. This is a global statement which asserts that the local trivialisations in Theorem 7.4 glue along F_b.

Let $f\colon \mathcal{C} \to B$ be a flat projective versal family of semistable curves of arithmetic genus g', and $\pi\colon \overline{J}_{d,B} \to B$ be the relative compactified Jacobian with respect to the canonical polarisation. For any $b \in B$, we denote by F_b the subvariety in $\pi^{-1}(b)$ parametrising rank 1 torsion free sheaves $\mathcal{I}$ on $C_b := f^{-1}(b)$ which are pushforwards of line bundles from the normalisation of C_b, i.e. $\mathcal{I} = \nu_* M$ where M is a line bundle on the normalisation $\nu\colon C_b^{\nu} = \bigsqcup_{i=1}^{r} C_{b,i} \to C_b$. Denote by Σ the set of nodes of C_b.

Proposition 8.1. *There exists a homeomorphism between a euclidean neighbourhood of F_b in $\overline{J}_{d,B}$ and the product*

$$F_b \times X(\Gamma(C_b)^{\pm}, 0) \times \mathbb{C}^{3g'-3-s}, \tag{35}$$

where $\Gamma(C_b)$ is the dual graph of C_b and s is the number of edges of the graph.

Remark 8.2. Let $B_\Gamma \subseteq B$ the locus of curves with the same dual graph as C_b. The restriction of the family $\mathcal{C}$ to B_Γ can be simultaneously normalised, and let $\mathcal{C}^{\nu} \to \mathcal{C} \times_B B_\Gamma$ be the simultaneous normalisation. Proposition 8.1 says in particular that a euclidean neighbourhood of F_b in the relative Picard scheme $\operatorname{Pic}^{d'}(\mathcal{C}^{\nu}) \subseteq \overline{J}_{d,B}$ is homeomorphic to $F_b \times \mathbb{C}^{3g'-3-s}$, but this is not a formal or analytic isomorphism. Otherwise $T\operatorname{Pic}^{d'}(\mathcal{C}^{\nu})|_{F_b} \simeq \mathcal{O}_{F_b}^{\oplus c'}$ with $c' := \dim F_b + 3g' - 3 - s$, and the boundary map $H^0(N_{F_b/\operatorname{Pic}^{d'}(\mathcal{C}^{\nu})}) \to H^1(TF_b)$ of the short exact sequence

$$0 \to TF_b \to T\operatorname{Pic}^{d'}(\mathcal{C}^{\nu})|_{F_b} \to N_{F_b/\operatorname{Pic}^{d'}(\mathcal{C}^{\nu})} \to 0$$

would be trivial, i.e. $\mathcal{C}^{\nu}$ is an isotrivial family, which is a contradiction. This suggests that Proposition 8.1 is false in the analytic category.

In order to show Proposition 8.1, we need to recall the GIT description of $\overline{J}_{d,B}$ and the deformation theory of the pair $(C_b, \mathcal{I})$ as in [8, 9]. This construction goes back to Simpson [69]. Over the versal curve $f\colon \mathcal{C} \to B$ one considers the relative quot scheme $\operatorname{Quot}(\mathcal{O}^{\oplus R})$, where R is an appropriately high positive integer which can be made explicit, parametrising quotient sheaves $\mathcal{O}^{\oplus R} \to \mathcal{F} \to 0$ with a fixed Hilbert polynomial. The group SL_R acts on $\operatorname{Quot}(\mathcal{O}^{\oplus R})$ by changing the basis of $\mathcal{O}^{\oplus R}$ by linear maps in SL_R.

The GIT stability for the natural linearised action of SL_R on $\mathrm{Quot}(\mathcal{O}^{\oplus R})$ corresponds to slope stability. Then the relative compactified Jacobian $\bar{J}_{d,B}$ can be constructed as the GIT quotient

$$\bar{J}_{d,B} = Q^{\mathrm{ss}} /\!\!/ \mathrm{SL}_R,$$

where $Q^{\mathrm{ss}} \subseteq \mathrm{Quot}(\mathcal{O}^{\oplus R})$ is the locus of semistable rank 1 torsion free quotients. See [78, Section I] for details or [8, Fact 2.7].

Note that Q^{ss} is smooth along the closed SL_R-orbits by [8, Lemma 6.4 (ii)] and by the smoothness of the miniversal deformation of the pair $(C_b, \mathcal{I})$; see for instance [8, Section 3].

Let Z be the locus of closed orbits over $F_b \subset \bar{J}_{d,B} = Q^{\mathrm{ss}} /\!\!/ \mathrm{SL}_R$. Note that F_b is isomorphic to the Jacobian $\mathrm{Pic}^{d'}(C_b^\nu)$ with $d' := d + (n - n^2)(1 - g)$, and Z is a fibre bundle over F_b.

Now let

$$\mathfrak{n}' \colon \mathcal{Q} = \mathrm{Bl}_{Z \times 0}(Q^{\mathrm{ss}} \times \mathbb{C}) \setminus Q^{\mathrm{ss}} \times 0 \to \mathbb{C} \tag{36}$$

be the deformation of Q^{ss} to the normal cone of Z. The central fibre is the total space $\mathrm{Tot}(N_Z)$ of the normal bundle N_Z of Z in Q^{ss}, and the restriction of $\mathfrak{n}'$ to $\mathbb{C}^*$ is the trivial fibration $Q^{\mathrm{ss}} \times \mathbb{C}^* \to \mathbb{C}^*$. Since Z is SL_R-invariant, $\mathcal{Q}$ inherits an SL_R-action preserving the fibres of $\mathfrak{n}'$. The quotient

$$\mathfrak{n} \colon \mathcal{Q} /\!\!/ \mathrm{SL}_R \to \mathbb{C}$$

is a deformation of $\bar{J}_{d,\mathcal{B}}$ to $\mathrm{Tot}(N_Z) /\!\!/ \mathrm{SL}_R$. We can then reduce the proof of Proposition 8.1 to the following statements:

- $\mathrm{Tot}(N_Z) /\!\!/ \mathrm{SL}_R$ is homeomorphic to the product (35);
- $\mathrm{Tot}(N_Z) /\!\!/ \mathrm{SL}_R$ is homeomorphic to a euclidean neighbourhood of F_b in $\bar{J}_{d,B}$.

We prove the statements respectively in Lemma 8.6 and Section 8.2.

Remark 8.3. In general, the deformation $\mathfrak{n}$ is not the deformation of $\bar{J}_{d,B}$ to the normal cone of F_b. Indeed, a neighbourhood U of a point in F_b is locally analytically isomorphic to $X(\Gamma(C_b)^{\pm}, 0) \times \mathbb{C}^{c'}$ with $c' := 4g' - 3 - b_1(\Gamma) - s$ by [8, Theorem A]. Up to a smooth factor, the degeneration to the normal cone deforms U to the normal cone of $X(\Gamma(C_b)^{\pm}, 0)$ at 0. The latter may fail to be isomorphic to $X(\Gamma(C_b)^{\pm}, 0)$: they are isomorphic only if the homogeneous polynomials cutting $X(\Gamma(C_b)^{\pm}, 0)$ have the same degree, equivalently if the number of edges between different vertices of $\Gamma(C_b)^{\pm}$ is constant. On the other hand, $\mathfrak{n}$ is locally trivial, so it deforms U to $X(\Gamma(C_b)^{\pm}, 0) \times \mathbb{C}^{c'}$.

8.1. Deformation theory of rank 1 torsion free sheaves

A slice of Q^{ss} normal to an SL_R-orbit of the pair $(C_b, \mathcal{I})$ is given by a miniversal deformation $\mathrm{Def}(C_b, \mathcal{I})$ of the pair; see [8, Section 6].

The space $\mathrm{Def}(C_b, \mathcal{I})$ is endowed with a natural action of the torus

$$\mathrm{Aut}(\mathcal{I}) \simeq H^0(C_b^\nu, \mathcal{O}^*_{C_b^\nu}) \simeq \bigoplus_{i=1}^{r} H^0(C_{b,i}, \mathcal{O}^*_{C_{b,i}}) \simeq \mathbb{T}^r \ni (\lambda_i)_{i=1}^r$$

(see [8, Definition 3.4]). Since $\mathrm{Aut}(\mathcal{I})$ is reductive, the tangent space $T\mathrm{Def}(C_b, \mathcal{I})$ splits into the product

$$T\mathrm{Def}(C_b, \mathcal{I})_{\mathrm{inv}} \oplus T\mathrm{Def}(C_b, \mathcal{I})_{\mathrm{var}},$$

where the invariant part $T\mathrm{Def}(C_b, \mathcal{I})_{\mathrm{inv}}$ is fixed by the action of $\mathrm{Aut}(\mathcal{I})$, while the variant part $T\mathrm{Def}(C_b, \mathcal{I})_{\mathrm{var}}$ is the unique $\mathrm{Aut}(\mathcal{I})$-invariant complement of $T\mathrm{Def}(C_b, \mathcal{I})_{\mathrm{inv}}$. In fact, these subspaces have a modular interpretation:

$$T\mathrm{Def}(C_b, \mathcal{I})_{\mathrm{inv}} \simeq T\mathrm{Def}^{\mathrm{lt}}(C_b, \mathcal{I}), \quad T\mathrm{Def}(C_b, \mathcal{I})_{\mathrm{var}} \simeq \prod_{e \in \Sigma} T\mathrm{Def}(C_{b,e}, \mathcal{I}_e),$$

where $\mathrm{Def}^{\mathrm{lt}}(C_b, \mathcal{I})$ is the miniversal deformation of locally trivial deformations of $(C_b, \mathcal{I})$, and $\mathrm{Def}(C_{b,e}, \mathcal{I}_e)$ is a miniversal deformation of the localised pair $(C_{b,e}, \mathcal{I}_e)$; see [8, Sections 3, 5]. In particular, it follows from [8, Section 5] that $\mathrm{Aut}(\mathcal{I})$ acts on $T\mathrm{Def}(C_{b,e}, \mathcal{I}_e) \simeq \mathrm{Spec}\, k[z_e, w_e]$ as

$$z_e \mapsto \lambda_{s(e)} z_e \lambda_{t(e)}^{-1}, \quad w_e \mapsto \lambda_{s(e)}^{-1} w_e \lambda_{t(e)}. \tag{37}$$

Now let $q\colon Q^{\mathrm{ss}} \to \overline{J}_{d,B}$ be the quotient map. Choose a universal rank 1 torsion free sheaf $\mathcal{U}$ over $Q^{\mathrm{ss}} \times_B \mathcal{C}$, and let $p\colon Q^{\mathrm{ss}} \times_B \mathcal{C} \to Q^{\mathrm{ss}}$ be the natural projection.

Lemma 8.4. *Let N_Z be the normal bundle of Z in Q^{ss}. There exists a $\mathbb{T}^r$-equivariant splitting*

$$N_Z \simeq N_Z^{\mathbb{T}^r} \oplus R^0 p_* \mathcal{E}xt^1_{C_b}(\mathcal{U}|_Z, \mathcal{U}|_Z),$$

where $N_Z^{\mathbb{T}^r}$ is the $\mathbb{T}^r$-invariant part of N_Z.

Proof. The normal space to Z in Q^{ss} at $z \in Z$ with $q(z) = (C_b, \mathcal{I})$ admits the $\mathbb{T}^r$-equivariant splitting

$$N_{Z,z} \simeq (T_z\mathrm{Def}^{\mathrm{lt}}(C_b, \mathcal{I})/q^* T_{(C_b,\mathcal{I})} F_b) \oplus \bigoplus_{e \in \Sigma} T_z\mathrm{Def}(C_{b,e}, \mathcal{I}_e), \tag{38}$$

and $\mathbb{T}^r$ acts trivially on the first summand $(T_z\mathrm{Def}^{\mathrm{lt}}(C_b, \mathcal{I})/q^* T_{(C_b,\mathcal{I})} F_b)$ and as (37) on the second summand. By [25, Lemma 7.13] and the local-to-global Ext spectral sequence, $\bigoplus_{e \in \Sigma} T_z\mathrm{Def}(C_{b,e}, \mathcal{I}_e)$ can be identified with

$$H^0(C_b, \mathcal{E}xt^1_{C_b}(\mathcal{I}, \mathcal{I})) \simeq \bigoplus_{e \in \Sigma} H^0(C_b, \mathcal{E}xt^1(\mathcal{I}, \mathcal{I})_e) \simeq \mathbb{C}^s,$$

i.e. the fibre of the vector bundle $R^0 p_* \mathcal{E}xt^1_{C_b}(\mathcal{U}|_Z, \mathcal{U}|_Z)$. ■

Lemma 8.5. $R^0 p_* \mathcal{E}xt^1_{C_b}(\mathcal{U}|_Z, \mathcal{U}|_Z)$ *is a trivial vector bundle on Z.*

Proof. The group $F_b \simeq \mathrm{Pic}^{d'}(C_b^\nu)$ acts on Z. Its action lifts to the vector bundle $R^0 p_* \mathcal{E}xt^1_{C_b}(\mathcal{U}|_Z, \mathcal{U}|_Z)$ parallelising it, as we show below.

A point in Z represents a quotient $\mathcal{O}_{C_b}^{\otimes R} \to \nu_* M$, where M is a line bundle on C_b^ν. Note that $\mathcal{E}xt^1_{C_b}(\nu_* M, \nu_* M)$ is independent of the choice of a basis for $\mathcal{O}_X^{\otimes R}$, and its rank is independent of M. Hence, $R^0 p_* \mathcal{E}xt^1_{C_b}(\mathcal{U}|_Z, \mathcal{U}|_Z)$ is the pullback of a vector bundle A on F_b.

The transitive action of $\mathrm{Pic}^{d'}(C_b^\nu)$ by translation induces identification of the fibres of A. Indeed, if $\mathcal{I} := \nu_* M$ and $\mathcal{I}' := \nu_*(M \otimes L)$ with $L \in \mathrm{Pic}^{d'}(C_b^\nu)$, then $\mathcal{I}$ and $\mathcal{I}'$ are locally isomorphic, and so there exists an invertible sheaf L' on X such that $\mathcal{I}' = \mathcal{I} \otimes L'$ with $\nu^* L' = L$; see for instance [11, Lemma 2.2.2 or 3.1.2/3.1.3]. Hence, we have

$$\mathcal{E}xt^1_{C_b}(\nu_*(M \otimes L), \nu_*(M \otimes L)) = \mathcal{E}xt^1_{C_b}(\nu_* M \otimes L', \nu_* M \otimes L') = \mathcal{E}xt^1_{C_b}(\nu_* M, \nu_* M).$$

This gives a global trivialisation of A. ∎

Lemma 8.6. *Let $N_{F_b/\bar{J}^\circ_d(\Gamma)}$ be the normal bundle of F_b in $\mathrm{Pic}^{d'}(\mathcal{C}^\nu) \subseteq \bar{J}_{d,B}$ as constructed in Remark 8.2. Denote by $\mathrm{Tot}(N)$ the total space of a vector bundle N. Then*

$$\mathrm{Tot}(N_Z) /\!\!/ \mathrm{SL}_R \simeq \mathrm{Tot}(N_{F_b/\mathrm{Pic}^{d'}(\mathcal{C}^\nu)}) \times X(\Gamma(C_b)^\pm, 0).$$

Proof. By Lemma 8.4 we have

$$\mathrm{Tot}(N_Z) /\!\!/ \mathrm{SL}_R \simeq (\mathrm{Tot}(N_Z^{\mathbb{T}^r}) \times_Z \mathrm{Tot}(R^0 p_* \mathcal{E}xt^1_{C_b}(\mathcal{U}|_Z, \mathcal{U}|_Z)) /\!\!/ \mathbb{T}^{r-1}) /\!\!/ (\mathrm{SL}_r / \mathbb{T}^{r-1}),$$

and Lemma 8.5 gives

$$\mathrm{Tot}(R^0 p_* \mathcal{E}xt^1_{C_b}(\mathcal{U}|_Z, \mathcal{U}|_Z)) \simeq Z \times \mathbb{C}^s.$$

The algebraic torus $\mathbb{T}^{r-1} = \mathbb{T}^r/\mathbb{T}$ acts on $\mathbb{C}^s$ as prescribed in (37), which coincides with the torus action defining $X(\Gamma(C_b)^\pm, 0)$. Therefore, we obtain

$$\begin{aligned}\mathrm{Tot}(R^0 p_* \mathcal{E}xt^1_{C_b}(\mathcal{U}|_Z, \mathcal{U}|_Z)) /\!\!/ \mathrm{SL}_r &\simeq ((Z \times \mathbb{C}^s) /\!\!/ \mathbb{T}^{r-1}) /\!\!/ (\mathrm{SL}_r / \mathbb{T}^{r-1}) \\ &\simeq Z /\!\!/ (\mathrm{SL}_r / \mathbb{T}^{r-1}) \times X(\Gamma(C_b)^\pm, 0) \simeq F_b \times X(\Gamma(C_b)^\pm, 0).\end{aligned}$$

Finally, the vector bundle $N_Z^{\mathbb{T}^r}$ descends to $N_{F_b/\mathrm{Pic}^{d'}(\mathcal{C}^\nu)}$, because $\mathrm{Pic}^{d'}(\mathcal{C}^\nu)$ parametrises the locally trivial deformations of the pair $(C_b, \mathcal{I})$. ∎

8.2. *Stratified Ehresmann theorem*

We use a stratified version of Ehresmann's theorem to prove that $\pi^{-1}(0)$ is a model of a tubular neighbourhood of F_b in $\bar{J}_{d,B}$.

Lemma 8.7. *The degeneration $\pi\colon \mathcal{Q} /\!\!/ \mathrm{SL}_R \to \mathbb{C}$ is locally analytically trivial along F_b, i.e. for any $x \in F_b \subset \pi^{-1}(0) \subset \mathcal{Q} /\!\!/ \mathrm{SL}_R$ there exists a trivialising neighbourhood $U \subset \mathcal{Q} /\!\!/ \mathrm{SL}_R$ of x, and a local analytic isomorphism*

$$\psi\colon U \to (\mathbb{C}^{c'} \times X(\Gamma(C_b)^\pm, 0)) \times \mathbb{C} \tag{39}$$

with $c' := 4g' - 3 - b_1(\Gamma) - s$ such that $\pi\colon U \to \mathbb{C}$ corresponds to the third projection $(X(\Gamma(C_b)^\pm, 0) \times \mathbb{C}^{c'}) \times \mathbb{C} \to \mathbb{C}$.

Proof. For any $z \in Z$, the degeneration $\mathfrak{n}\colon Z \times \mathbb{C} \subseteq \mathcal{Q} \to \mathbb{C}$ admits a section $\{z\} \times \mathbb{C}$. By (38), an analytic neighbourhood of $\{z\} \times \mathbb{C}$ is locally isomorphic to

$$\Big(T_z O_z \times T_z \mathrm{Def}^{\mathrm{lt}}(C_b, \mathcal{I}) \times \bigoplus_{e \in \Sigma} T_z \mathrm{Def}(C_{b,e}, \mathcal{I}_e)\Big) \times \mathbb{C},$$

where O_z is the SL_R-orbit of z. By Luna's slice theorem, an analytic neighbourhood of $\{[z]\} \times \mathbb{C}$ in $\mathcal{Q} /\!\!/ \mathrm{SL}_R$ is locally isomorphic to

$$\Big(T_z \mathrm{Def}^{\mathrm{lt}}(C_b, \mathcal{I}) \times \bigoplus_{e \in \Sigma} T_z \mathrm{Def}(C_{b,e}, \mathcal{I}_e)\Big) /\!\!/ \mathbb{T}^{r-1} \times \mathbb{C}$$
$$\simeq \big(T_{(C_b,\mathcal{I})} \mathrm{Pic}^{d'}(\mathcal{C}^\nu) \times X(\Gamma(C_b)^{\pm}, 0)\big) \times \mathbb{C} \simeq (\mathbb{C}^{c'} \times X(\Gamma(C_b)^{\pm}, 0)) \times \mathbb{C}. \quad \blacksquare$$

Proof of Proposition 8.1. Let us first show that a relatively compact euclidean neighbourhood of F_b in $\mathfrak{n}^{-1}(0)$ is stratified diffeomorphic to a euclidean neighbourhood of F_b in $\mathfrak{n}^{-1}(0)$ in $\mathfrak{n}^{-1}(t) = \overline{J}_{d,B}$, with $t \neq 0$.

By Lemma 8.7 the degeneration $\mathfrak{n}\colon \mathcal{Q} /\!\!/ \mathrm{SL}_R \to \mathbb{C}$ is locally trivial. This means that we can cover $F_b \subset \mathfrak{n}^{-1}(0)$ by trivialising open sets U_k, with k in the index set K, as in Lemma 8.7. Take now the vector field $v_k := d\psi_k^{-1}(\partial_t)$ on U_k, where ∂_t is a non-vanishing vector field on $\mathbb{C}$ and ψ_k is the stratified diffeomorphism (39). Consider a partition of unity $\{\chi_k\}_{k \in K}$ subordinate to the open cover $\{U_k\}_{k \in K}$. Then the flow of the vector field $v := \sum_{k \in K} \chi_k \cdot v_k$ defines the required stratified diffeomorphism.

We conclude the proof by noticing that the neighbourhood of F_b in $\mathfrak{n}^{-1}(0)$ can be chosen homeomorphic to the product

$$\mathrm{Tot}(N_{F_b/\mathrm{Pic}^{d'}(\mathcal{C}^\nu)}) \times X(\Gamma(C_b)^{\pm}, 0) \underset{\text{homeo}}{\simeq} F_b \times \mathbb{C}^{3g'-3-s} \times X(\Gamma(C_b)^{\pm}, 0)$$

by Lemma 8.6. ■

9. Full support theorem for universal compactified Jacobians

Lemma 9.1 (Weak abelian fibration). *Let $f\colon \mathcal{C} \to B$ be a flat projective versal family of semistable curves of arithmetic genus g', and $\pi^P\colon P := \mathrm{Pic}^0(\mathcal{C}/B) \to B$ be the relative degree* 0 *Picard scheme parametrising line bundles on the fibres of f whose restriction to each irreducible component is of degree* 0*. Let $\pi\colon \overline{J}_{d,B} \to B$ be the relative compactified Jacobian with respect to the canonical polarisation. Then the triple $(\overline{J}_{d,B}, P, B)$ is a weak abelian fibration.*

Proof. We need to check conditions (1)–(3) in Definition 3.1. The relative Picard scheme $\pi^P\colon P \to B$ has fibres of pure dimension g', and $\dim \overline{J}_{d,B} = g' + \dim B$ (see for instance Theorem 7.4). Hence condition (1) holds. The proof of [24, Lemma 3.4.5] gives the affineness of the stabilisers, i.e. (2). The argument of [68, Section 4.12] gives (3); see also [16, Theorem 3.3.1] where one can drop the assumption that $\mathcal{C}$ is a family of curves embedded in a surface. ■

Proposition 9.2 (Whitney stratification of $\overline{J}_{d,B}$). *Under the hypothesis of Lemma 9.1, let $\mathcal{I}$ be a rank 1 torsion free sheaf on $\mathcal{C}_b := f^{-1}(b)$ which fails to be locally free at the nodes Σ of $\mathcal{C}_b$. Set $\Gamma_{\mathcal{C}_b}(\Sigma)$ to be the dual graph of any curve obtained from $\mathcal{C}_b$ by smoothing the nodes not in Σ. Then there exists a complex Whitney stratification*

$$\overline{J}_{d,B} = \bigsqcup_{\Gamma} \overline{J}^{\circ}_{d}(\Gamma),$$

where $\overline{J}^{\circ}_{d}(\Gamma)$ is the locus of rank 1 torsion free sheaves in $\overline{J}_{d,B}$ whose $\Gamma_{\mathcal{C}_b}(\Sigma)$ is isomorphic to Γ, and we have

$$\operatorname{codim}_{\overline{J}_{d,B}} \overline{J}^{\circ}_{d}(\Gamma) \geq 2 \operatorname{codim}_{\pi^{-1}(t)}(\overline{J}^{\circ}_{d}(\Gamma) \cap \pi^{-1}(t)). \tag{40}$$

Proof. The first statement is a reformulation of [8, Theorem A]. For the inequality (40), observe that if Γ has r vertices and s edges, then [8, Theorem A] gives

$$\begin{aligned} \operatorname{codim}_{\overline{J}_{d,B}} \overline{J}^{\circ}_{d}(\Gamma) &= \dim X(\Gamma^{\pm}, 0) = 2s - r + 1, \\ \operatorname{codim}_{\pi^{-1}(t)}(\overline{J}^{\circ}_{d}(\Gamma) \cap \pi^{-1}(t)) &= s - r + 1. \end{aligned}$$ ■

Lemma 9.3. *Under the hypothesis of Lemma 9.1, we have*

$$\tau_{>2c}(R\pi_* \operatorname{IC}(\overline{J}_{d,B}, \mathbb{Q})) = 0, \quad c := \dim \overline{J}_{d,B} - \dim B = g'.$$

Proof. This follows from Propositions 3.3 and 9.2. ■

Lemma 9.4. *Under the hypothesis of Lemma 9.1, any support Z of $R\pi_* \operatorname{IC}(\overline{J}_{d,B}, \mathbb{Q})$ satisfies*

$$\operatorname{codim} Z = \delta_Z.$$

Proof. The inequality $\operatorname{codim} Z \geq \delta_Z$ appears for instance in [66, Fact 2.4]. The reverse inequality follows from Lemmas 9.1 and 9.3, and [58, Theorem 1.8] or Theorem 3.2. ■

Let $\pi\colon \overline{J}_{d,B} \to B$ be the relative compactified Jacobian with respect to the canonical polarisation of a versal deformation of semistable curves of arithmetic genus g'. We define the invariant

$$\Psi_{g'} := \dim \overline{J}_{d,B} - 2 \dim B = 4g' - 3 - 2(3g' - 3) = 3 - 2g'. \tag{41}$$

Theorem 9.5 (Full support for Jacobians of versal families). *Let $f\colon \mathcal{C} \to B$ be a flat projective versal family of semistable curves of arithmetic genus g', and $U \subset B$ the open subscheme over which the morphism f is smooth. Let $\pi\colon \overline{J}_{d,B} \to B$ be the relative compactified Jacobian with respect to the canonical polarisation. Then the complex $R\pi_* \operatorname{IC}(\overline{J}_{d,B}, \mathbb{Q})$ has full support on B, i.e.*

$$\begin{aligned} R\pi_* \operatorname{IC}(\overline{J}_{d,B}, \mathbb{Q}) &\simeq \bigoplus_{l=0}^{2g} \operatorname{IC}(B, \Lambda^l (R^1 \pi_* \mathbb{Q})|_U)[-l] \\ &= \bigoplus_{l=0}^{2g} \operatorname{IC}(B, \Lambda^l (R^1 f_* \mathbb{Q})|_U)[-l]. \end{aligned} \tag{42}$$

In particular, for any $d, d' \in \mathbb{Z}$ there is an isomorphism of complexes of mixed Hodge modules

$$R\pi_* \operatorname{IC}(\bar{J}_{d,B}, \mathbb{Q}) \simeq R\pi_* \operatorname{IC}(\bar{J}_{d',B}, \mathbb{Q}). \tag{43}$$

Proof. The argument of [58, Section 3] can be adapted to work in the present setting. The base B admits a stratification by type

$$B = \bigsqcup B^{\circ}_{\underline{g}'},$$

which extends (4). For any $r \geq 1$, and for any r-tuples $\underline{g}' = (g'_i)$ of positive integers, the closed point $b \in B$ in $B^{\circ}_{\underline{g}'}$ corresponds to the fibre of f of the form $\mathcal{C}_b = \bigcup_i \mathcal{C}_i$, where $\mathcal{C}_i$ is an irreducible curve of genus $g'_i = g(\mathcal{C}_i)$.

Assume that the irreducible subvariety $Z \subseteq B$ is a support of $R\pi_* \operatorname{IC}(\bar{J}_{d,B}, \mathbb{Q})$, and that the generic point of Z is contained in $B^{\circ}_{\underline{g}'}$. As in [58, Proposition 4.4], the inequality $\operatorname{codim} Z \geq \delta_Z$ can be improved to

$$\Psi_{g'} + \operatorname{codim} Z \geq \sum_{i=1}^{r} \Psi_{g'_i} + \delta_Z. \tag{44}$$

Combining (44), (41) and Lemma 9.4, we obtain

$$3(r-1) + 2\Big(g - \sum_{i=1}^{r} g_i\Big) = 0.$$

The only possibility is that $r = 1$. Hence, $Z = B$, which implies (42) and (43). ∎

Remark 9.6. The proof of Theorem 9.5 does not work for the Hitchin fibration. In this case, the invariant $\Psi = \dim M(n,d) - 2\dim A_n$ vanishes, and the inequality (44) is inconclusive.

10. Proofs of the main theorems

Proof of Theorem 1.1 (Hodge-to-singular correspondence). In view of the decomposition (18), it suffices to show that for any partition $\underline{n}$ of n the isomorphism (5) holds on an open euclidean neighbourhood of a general $a \in S^{\times}_{\underline{n}}$ containing a Zariski-dense open set of $S^{\times}_{\underline{n}}$. By Remark 5.6, it is actually enough to show the isomorphism on the subsets A_a constructed in the proof of Proposition 5.5.

By Proposition 5.5, proper base change gives

$$R\chi(n,d)_*(i_M^* \operatorname{IC}(\bar{J}_{d,\mathcal{B}}, \mathbb{Q})) \simeq (f^{\times}|_{A_a})^* R\pi_* \operatorname{IC}(\bar{J}_{d,\mathcal{B}}, \mathbb{Q}).$$

For any étale chart $B \to \mathcal{B}$, Theorem 9.5 says that $R\pi_* \operatorname{IC}(\bar{J}_{e,B}, \mathbb{Q})$ and $R\pi_* \operatorname{IC}(\bar{J}_{d,B}, \mathbb{Q})$ have full support, which implies that

$$(f^{\times}|_{A_a})^* R\pi_* \operatorname{IC}(\bar{J}_{e,\mathcal{B}}, \mathbb{Q}) \simeq (f^{\times}|_{A_a})^* R\pi_* \operatorname{IC}(\bar{J}_{d,\mathcal{B}}, \mathbb{Q}).$$

Since $\chi(n,e)^{-1}(A_a)$ is regularly embedded in $\bar{J}_{e,\mathcal{B}}$, we have

$$i_M^* \operatorname{IC}(\bar{J}_{e,\mathcal{B}}, \mathbb{Q}) \simeq \mathbb{Q}_{\chi(n,e)^{-1}(A_a)}.$$

Together with Theorem 7.5 and Proposition 7.6, we obtain Theorem 1.1. ∎

Proof of Theorem 1.3. By Theorem 7.5, the descent of $\mathcal{L}'_{\underline{n}}$ on $S^\times_{\underline{n}}$ in Proposition 7.6 is a sublocal system of the sheaf $\mathscr{L}_{\underline{n}}(e)$ in the Ngô strings of Theorem 3.13. By [18, Theorem 6.11], Proposition 4.11 (8) and Theorem 7.5, they have the same rank:

$$\operatorname{rank} \mathscr{L}_{\underline{n}}(e) = \dim \widetilde{H}^{2b_1-1}(|\mathscr{C}_Q|, \mathbb{Q}) = \dim H^{2b_1}(Y(Q,\theta), \mathbb{Q}) = \operatorname{rank} \mathcal{L}'_{\underline{n}},$$

so they must coincide. ∎

Proof of Theorem 1.4. Theorem 1.1 for $d = 0$ yields

$$R\chi(n,e)_* \mathbb{Q}_{M(n,e)}|_{A_n^{\mathrm{red}}} \simeq \bigoplus_{\underline{n}\vdash n} R\chi(n,0)_* \operatorname{IC}(M_{\underline{n}}(0), \chi(n,0)^* \mathcal{L}_{\underline{n}})[2(\dim S_n - c(g,n))]|_{A_n^{\mathrm{red}}}. \tag{45}$$

On the open set of $S_{\underline{n}}$ over which the Hitchin map $\chi(n,0)\colon M_{\underline{n}}(0) \to S_{\underline{n}}$ is smooth, there are isomorphisms

$$R\chi(n,0)_* \operatorname{IC}(M_{\underline{n}}(0), \chi(n,0)^* \mathcal{L}_{\underline{n}}) \simeq \bigoplus_{l=0}^{2\dim S_{\underline{n}}} \Lambda^l_{\underline{n}} \otimes \mathcal{L}_{\underline{n}}[-l] \simeq \mathscr{S}_{\underline{n}}.$$

Therefore, by the decomposition theorem [4, 73], the string $\mathscr{S}_{\underline{n}}$ is a direct summand of

$$R\chi(n,0)_* \operatorname{IC}(M_{\underline{n}}(0), \chi(n,0)^* \mathcal{L}_{\underline{n}})$$

on the whole $S_{\underline{n}}$. Now Theorem 1.3 gives

$$\bigoplus_{\underline{n}\vdash n} \mathscr{S}_{\underline{n}}|_{A_n^{\mathrm{red}}} \langle \operatorname{codim} S_{\underline{n}} \rangle \simeq R\chi(n,e)_* \mathbb{Q}_{M(n,e)}|_{A_n^{\mathrm{red}}} \simeq \text{RHS of (45)} \supseteq \bigoplus_{\underline{n}\vdash n} \mathscr{S}_{\underline{n}}|_{A_n^{\mathrm{red}}} \langle \operatorname{codim} S_{\underline{n}} \rangle.$$

This implies that the complexes $R\chi(n,0)_* \operatorname{IC}(M_{\underline{n}}(0), \chi(n,0)^* \mathcal{L}_{\underline{n}})$ have full support on $S_{\underline{n}}$. For $\underline{n} = \{n\}$, this shows that $R\chi(n,0)_* \operatorname{IC}(M(n,0), \mathbb{Q})$ has full support. ∎

Proof of Theorem 1.7. By Theorems 1.3 and 1.4, it suffices to check that no summand of $R\chi(2,1)_* \mathbb{Q}_{M(2,1)}$ and of $R\chi(2,0)_* \operatorname{IC}(M(2,0), \mathbb{Q})$ is supported over $A_2 \setminus A_2^{\mathrm{red}} = \{0\}$. This is indeed the case by Proposition 1.6 (Corollary 3.11). ∎

Remark 10.1. Let $\underline{n} = (n_1, \ldots, n_r) = 1^{\alpha_1} \cdot \ldots \cdot n^{\alpha_n}$ be a partition of n, where α_i is the number of elements in n equal to i. Denote by $\mathfrak{S}_{\underline{n}} := \prod_{i=1}^r \mathfrak{S}_{\alpha_i}$ the subgroup of the symmetric group $\mathfrak{S}_r$ stabilising $\underline{n}$. Then the horizontal arrows of the commutative square

$$\begin{array}{ccc} \prod_{i=1}^r M(n_i,d_i) & \longrightarrow & M_{\underline{n}}(d) \\ {\scriptstyle \prod_{i=1}^r \chi(n_i,d_i)}\downarrow & & \downarrow{\scriptstyle \chi(n,d)} \\ \prod_{i=1}^r A_{n_i} & \xrightarrow{\text{mult}_{\underline{n}}} & S_{\underline{n}} \end{array} \qquad \begin{array}{ccc} (\mathcal{E}_i,\phi_i)_{i=1}^r & \longmapsto & (\bigoplus_{i=1}^r \mathcal{E}_i, \bigoplus_{i=1}^r \phi_i) \\ \downarrow & & \downarrow \\ (\text{char}(\phi_i))_{i=1}^r & \longmapsto & \prod_{i=1}^r \text{char}(\phi_i) \end{array}$$

are quotient maps by the action of $\mathfrak{S}_{\underline{n}}$. By [18, Corollary 6.20], the monodromy of the local systems $\mathcal{L}_{\underline{n}}$ is given by the restriction to the subgroup $\mathfrak{S}_{\underline{n}}$ of the representation of $\mathfrak{S}_r$ induced by a primitive character of a maximal cyclic subgroup. In particular, $\text{mult}_{\underline{n}}^* \mathcal{L}_{\underline{n}}$ is a trivial local system. Hence, $\mathfrak{S}_{\underline{n}}$ acts on the sheaf

$$\boxtimes_{i=1}^r R\chi(n_i,d_i)_* \,\text{IC}(M(n_i,d_i),\mathbb{Q}) \otimes \mathbb{Q}^{\text{rank}\,\mathcal{L}_{\underline{n}}}$$

via the $\mathfrak{S}_{\underline{n}}$-representation $\mathcal{L}_{\underline{n}}$. This means that the summand of the RHS of (5) is the $\mathfrak{S}_{\underline{n}}$-invariant part of

$$\boxtimes_{i=1}^r R\chi(n_i,d_i)_* \,\text{IC}(M(n_i,d_i),\mathbb{Q}) \otimes \mathbb{Q}^{\text{rank}\,\mathcal{L}_{\underline{n}}} \langle \text{codim}\, S_{\underline{n}} \rangle. \tag{46}$$

In particular, the summands of the RHS of (6) are $\mathfrak{S}_{\underline{n}}$-subrepresentations of

$$\bigotimes_{i=1}^r IH^{k-2\,\text{codim}\,S_n}(M(n_i,d_i)^{\text{red}},\mathbb{Q}).$$

Proof of Corollary 1.8. The summation indices and the summands in (5) depend only on $\gcd(n,d)$. ∎

10.1. Dependence of $IH^*(M(n,d),\mathbb{Q})$ on degree

The description of the summands of the decomposition theorem for the Hitchin fibration in the intermediate case, $\gcd(n,d) \neq 0$ or $\neq 1$, is more subtle. A closed formula appears now in [61, Theorems 1.1, 1.6, 1.8], but (5) offers already a recursive approach. Combining Theorems 1.1 and 1.4, one obtains the isomorphism in $D^bMHM_{\text{alg}}(A_n^{\text{red}})$

$$R\chi(n,d)_* \,\text{IC}(M(n,d),\mathbb{Q}) \oplus \bigoplus_{\underline{n}\in\mathcal{P}_d,\,\underline{n}\neq\{n\}} R\chi(n,d)_* \,\text{IC}(M_{\underline{n}}(d), \chi(n,d)^*\mathcal{L}_{\underline{n}})\langle \text{codim}\, S_{\underline{n}} \rangle$$
$$\simeq \bigoplus_{\underline{n}=\{n_i\}\vdash n} \mathscr{S}_{\underline{n}}. \tag{47}$$

Recursively from this identity, one can determine the Ngô strings for the complex $R\chi(n,d)_* \,\text{IC}(M(n,d),\mathbb{Q})$, and the subvarieties $S_{\underline{n}}$ which are actual support of $R\chi(n,d)_* \,\text{IC}(M(n,d),\mathbb{Q})$. We refer to [61] for the explicit computation of the ranks of the Ngô strings. Here we limit ourselves to illustrating the first non-trivial example $n = 4$. More care is needed to determine explicitly the monodromy of the associated local systems.

Example 10.2 ($n = 4$). We determine the rank of the local systems $\mathscr{L}_{\underline{n}}(d)$ in Theorem 3.13 for $n = 4$. To this end, observe that the Hodge-to-singular correspondence gives

$$R\chi(4,2)_* \operatorname{IC}(M(4,2),\mathbb{Q}) \oplus R\chi(4,2)_* \operatorname{IC}(M_{\{2,2\}}(2), \chi(4,2)^*\mathcal{L}_{\{2,2\}})\langle 8g-9\rangle \simeq \bigoplus_{\underline{n}=\{n_i\}\vdash 4} \mathscr{S}_{\underline{n}}. \tag{48}$$

Consider the commutative square

$$\begin{array}{ccc} M(2,1)^2 & \longrightarrow & M_{\{2,2\}}(2) \\ {\scriptstyle \chi(2,1)^2}\big\downarrow & & \big\downarrow{\scriptstyle \chi(4,2)} \\ A_2^2 & \xrightarrow{\operatorname{mult}_{\{2,2\}}} & S_{\{2,2\}} \end{array}$$

The map $\operatorname{mult}_{\{2,2\}}$ factors through the normalisation $\nu_{\{2,2\}}\colon \operatorname{Sym}^2 A_2 \to S_{\{2,2\}}$ of $S_{\{2,2\}}$ as follows:

$$\operatorname{mult}_{\{2,2\}}\colon A_2^2 \xrightarrow{\eta_{\{2,2\}}} \operatorname{Sym}^2 A_2 \xrightarrow{\nu_{\{2,2\}}} S_{\{2,2\}}.$$

Set $\underline{n} = \{2,2\}, \{2,1,1\}$ or $\{1,1,1,1\}$. The degree of $\nu_{\{2,2\}}$ over $S_{\underline{n}}$, denoted by $d_{\underline{n}}$, is

$$d_{\underline{n}} = \begin{cases} 1 & \text{for } \underline{n} = \{2,2\}, \{2,1,1\}, \\ 3 & \text{for } \underline{n} = \{1,1,1,1\}. \end{cases}$$

In fact, the invariant $d_{\underline{n}}$ counts the ways of summing the elements of the partition $\underline{n}$ to get $\{2,2\}$. The map $\eta_{\{2,2\}}$ instead is étale at the general points of $S_{\underline{n}}$. By the behaviour of intersection cohomology under normalisation and étale maps, we find that $R\chi(2,4)_* \operatorname{IC}(M_{\{2,2\}}(2), \chi(4,2)^*\mathcal{L}_{\{2,2\}})$ is locally isomorphic to

$$R\chi(2,1)^2_* \operatorname{IC}(M(2,1)^2,\mathbb{Q}) \otimes \mathbb{Q}^{\operatorname{rank}\mathcal{L}_{\{2,2\}}} \otimes \mathbb{Q}^{d_{\underline{n}}} \tag{49}$$

at the general points of $S_{\underline{n}}$. We omit tensoring by $\mathbb{Q}^{\operatorname{rank}\mathcal{L}_{\{2,2\}}}$ since $\operatorname{rank}\mathcal{L}_{\{2,2\}} = 1$. Indeed,

$$\mathscr{L}_{\underline{n}}(d) \leq \operatorname{rank}\mathcal{L}_{\underline{n}} = (|\underline{n}|-1)! \tag{50}$$

by Theorem 1.1 and [18, Corollary 6.20]. By the Künneth formula and Theorem 1.7, we have

$$R\chi(2,1)^2_* \operatorname{IC}(M(2,1)^2,\mathbb{Q}) \simeq (\mathscr{S}_{\{2\}} \oplus \mathscr{S}_{\{1,1\}}) \boxtimes (\mathscr{S}_{\{2\}} \oplus \mathscr{S}_{\{1,1\}}),$$

so all the Ngô strings of $R\chi(2,1)^2_* \operatorname{IC}(M(2,1)^2,\mathbb{Q})$ start with a local system of rank 1.

$\gcd(n,d)$	$\{4\}$	$\{3,1\}$	$\{2,2\}$	$\{2,1,1\}$	$\{1,1,1,1\}$
0	1	0	0	0	0
1	1	1	1	2	6
2	1	1	0	1	3

Tab. 1. The rank of the local systems $\mathscr{L}_{\underline{n}}(d)$ for $n = 4$.

By (48) and (49), this implies that $\operatorname{rank} \mathscr{L}_{\underline{n}}(2) = \operatorname{rank} \mathcal{L}_{\underline{n}} - d_{\underline{n}}$. In particular, $S_{\{2,2\}}$ is not a support of $R\chi(2,4)_* \operatorname{IC}(M(2,4), \mathbb{Q})$. We summarise the results in Table 1.

Theorem 3.13 says that the complex $R\chi(n,d)_* \operatorname{IC}(M(n,d), \mathbb{Q})$ is a direct sum of Ngô strings

$$\mathscr{S}(\mathscr{L}_{\underline{n}}(d)) := \bigoplus_{l=0}^{2\dim S_{\underline{n}}} \operatorname{IC}(S_{\underline{n}}, \Lambda^l_{\underline{n}} \otimes \mathscr{L}_{\underline{n}}(d))[-l]\langle \operatorname{codim} S_{\underline{n}} \rangle.$$

By Remark 10.1, the summand $R\chi(n,d)_* \operatorname{IC}(M_{\underline{n}}(d), \chi(n,d)^* \mathcal{L}_{\underline{n}})$ of the LHS of (47) is the $\mathfrak{S}_{\underline{n}}$-invariant part of

$$\boxtimes_{i=1}^{r} R\chi(n_i, d_i)_* \operatorname{IC}(M(n_i, d_i), \mathbb{Q}) \otimes \mathbb{Q}^{\operatorname{rank} \mathcal{L}_{\underline{n}}},$$

where $n_i < n$ and $d_i < d$. Therefore, using (47), one could rewrite the local system $\mathscr{L}_{\underline{n}}(d)$ as a combination of direct sums, linear quotients and $\mathfrak{S}_{\underline{n}'}$-quotients of $\mathscr{L}_{\underline{n}'}(d')$, $\mathcal{L}_{\underline{n}}$ and $\mathcal{L}_{\underline{n}'}$, with $n' < n$ and $d' < d$. Recursively, we could remove the dependence of $\mathscr{L}_{\underline{n}}(d)$ on $\mathscr{L}_{\underline{n}'}(d')$, and write $\mathscr{L}_{\underline{n}}(d)$ as depending only on $\mathcal{L}_{\underline{k}}$ with $k \leq n$. For any partition $\underline{n} = (n_1, \ldots, n_r)$, Remark 10.1 says that the Ngô string associated to $\mathcal{L}_{\underline{n}}$, namely

$$\mathscr{S}_{\underline{n}} := \bigoplus_{l=0}^{2\dim S_{\underline{n}}} \operatorname{IC}(S_{\underline{n}}, \Lambda^l_{\underline{n}} \otimes \mathcal{L}_{\underline{n}})[-l]\langle \operatorname{codim} S_{\underline{n}} \rangle,$$

is the $\mathfrak{S}_{\underline{n}}$-invariant part of $(\boxtimes_{i=1}^{r} \mathscr{S}_{n_i})^{\operatorname{rank} \mathcal{L}_{\underline{n}}}$. Thus $R\chi(n,d)_* \operatorname{IC}(M(n,d), \mathbb{Q})$ can be expressed in terms of the Ngô strings $\mathscr{S}_k \simeq R\chi(k,0)_* \operatorname{IC}(M(k,0), \mathbb{Q})$ with $k \leq n$. In other words, we obtain a recursive (though highly non-trivial) formula for $IH^*(M^{\mathrm{red}}(n,d), \mathbb{Q})$ in terms of $IH^*(M^{\mathrm{red}}(k,0), \mathbb{Q}) = H^*(\mathscr{S}_k)$ with $k \leq n$.

10.2. Some applications of the Hodge-to-singular correspondence

Proof of Proposition 1.9. Taking global cohomology in (18), we have

$$IH^{\leq k}(M(n,d)) \simeq IH^{\leq k}(M(n,0))$$

with $k = \min\{2(\dim \mathcal{A}_n - \dim \mathcal{A}_n^{\mathrm{red}}), \dim M_{\underline{n}} : \underline{n} \neq \{n\}\}$. By [5, Lemma 2.2], we have

$$\begin{aligned} k &= 2(n^2(g-1)+1) - 2\max\left\{r + (g-1)\sum_{i=1}^{r} n_i^2 \;\middle|\; m_i, n_i \in \mathbb{N}, \sum_{i=1}^{r} m_i n_i = n\right\} \\ &= 4(g-1)(n-1) - 2 = O(n). \end{aligned} \tag{51}$$

This gives the degree independence of the limits (10). The finiteness follows from the generation of the cohomology of $M(n,e)$ with $\gcd(e,n) = 1$ by finitely many tautological classes [54, Theorem 7]. ∎

Recall that the perverse filtration on $IH^*(M(n,d))$ associated to the Hitchin map $\chi(n,d)$ is given by

$$P_j IH^k(M(n,d),\mathbb{Q}) = \ker\{IH^k(M(n,d),\mathbb{Q}) \to IH^k(\chi(n,d)^{-1}(\Lambda^{k-j-1}),\mathbb{Q})\}, \quad (52)$$

where Λ^s is a general s-dimensional affine hyperplane of A_n; see [23, Theorem 4.1.1]. By (51), Λ^s does not intersect the supports of the RHS of (5) which are properly contained in A_n, provided that $s < 2(g-1)(n-1)-1$. As a result, we obtain the following proposition.

Proposition 10.3. *The graded piece of the perverse filtration*

$$\mathrm{Gr}_j^P\, IH^k(M(n,d),\mathbb{Q}) := P_j IH^k(M(n,d),\mathbb{Q})/P_{j-1}IH^k(M(n,d),\mathbb{Q})$$

is independent of the degree d for $j > k-2(g-1)(n-1)+1$.

Proof of Proposition 1.10. By (52) we have

$$\mathrm{Gr}_k^P\, IH^k(M(n,d),\mathbb{Q}) \simeq \mathrm{Im}\{IH^k(M(n,d),\mathbb{Q}) \to H^k(M_a,\mathbb{Q})\}.$$

Let e be an integer coprime to n. Since

$$\mathrm{Gr}_k^P\, IH^k(M(n,d),\mathbb{Q}) \simeq \mathrm{Gr}_k^P\, H^k(M(n,e),\mathbb{Q})$$

by Proposition 10.3, it suffices to determine the latter. Let $(\mathbb{E},\Phi)$ be the universal family of Higgs bundles on $C \times M(n,e)$ normalised so that

$$\mathrm{ch}_1(\mathbb{E})|_{p\times M(n,e)} = 0 \in H^2(M(n,e),\mathbb{Q}), \quad \mathrm{ch}_1(\mathbb{E})|_{C\times q} = 0 \in H^2(C,\mathbb{Q}).$$

For any $\gamma \in H^j(C,\mathbb{Q})$, the tautological class $c(\gamma,k)$ is the integral

$$\int_\gamma \mathrm{ch}^k(\mathbb{E}) \in H^{j+2k-2}(M(n,d),\mathbb{Q}).$$

By [54, Theorem 7], the cohomology $H^\bullet(M(n,e),\mathbb{Q})$ is generated by the tautological classes as $\mathbb{Q}$-algebra. The proof of [17, Corollary 5.1.3] works in arbitrary rank, and shows that the only tautological classes that do not vanish along M_a are $c(\gamma,1)$ with $\gamma \in H^1(C,\mathbb{Q})$ and the $\chi(n,e)$-relative ample class $\alpha := c([c],2)$, where $[c]$ generates $H^2(C,\mathbb{Q})$. We conclude that

$$\begin{aligned}\mathrm{Gr}_*^P\, H^*(M(n,e),\mathbb{Q}) &= \langle c(\gamma,1)|_{M_a}, \alpha|_{M_a}\rangle \\ &= H^\bullet(C,\mathbb{Q}) \otimes \mathbb{Q}[\alpha|_{M_a}]/(\alpha|_{M_a}^{\dim M(n,d)+1}).\end{aligned} \quad (53)$$

The result for $\check{M}(n,d)$ is an immediate corollary of (53) since

$$\mathrm{Gr}_*^P\, H^*(M(n,d),\mathbb{Q}) \simeq \mathrm{Gr}_*^P\, H^*(\check{M}(n,d),\mathbb{Q}) \otimes H^\bullet(C,\mathbb{Q})$$

(see for instance [17, Section 4.4]). ∎

10.3. Mukai moduli spaces

Let S be a K3 surface. Given an effective Mukai vector[8] $v \in H^*_{\mathrm{alg}}(S,\mathbb{Z})$, we denote by $M(S,v,H)$ the moduli space of Gieseker semistable sheaves on S with Mukai vector v with respect to a polarisation H; see [78, Section 1]. If v is the Mukai vector of a pure one-dimensional sheaf $\mathcal{F}$, then $v=(0,D,\chi(\mathcal{F}))$, where $D \in \mathrm{Pic}(S,\mathbb{Z})$ is the class of the curve supporting $\mathcal{F}$. Taking (Fitting) supports defines a Lagrangian fibration

$$M(S,v,H) \to |D|, \quad \mathcal{F} \mapsto \mathrm{Supp}(\mathcal{F}),$$

whose fibres are compactified Jacobians. The Hodge-to-singular correspondence extends to this context with no change, and so does Proposition 1.9.

Corollary 10.4. *Let S be a K3 surface with $\mathrm{Pic}(S) \simeq \mathbb{Z}$ generated by the class of a smooth curve C of genus $g \geq 2$. Then $\lim_{n\to\infty} \dim IH^{p,q}(M(S,(0,nC,\chi),C))$ is finite and independent of χ.*

Proof. The proof of Proposition 1.9 holds in this context too. The assumption on the Picard rank grants that the codimension of the locus of non-reduced curves in $|nC|$ grows linearly in n. ∎

Definition 10.5 ([70, Definition 1.15]). We say that (S,v,H) is an (m,k)-*triple* if

(1) the polarisation H is primitive and v-generic in the sense of [70, Section 2.1];

(2) $v=mw$, where w is primitive and $w^2=2k$;

(3) if $w=(0,w_1,w_2)$ and $\rho(S)>1$, then $w_2 \neq 0$.

Note that the stabilisation of the cohomology of $M(S,v,H)$ [12, Conjecture 1.1] is known in the literature if v is primitive; see [12, p. 4] and references therein. We can now drop the primitivity assumption.

Corollary 10.6. *Let $v^2=mw^2$ and $w^2=2k$ with $v,w \in H^*_{\mathrm{alg}}(S,\mathbb{Z})$ and $m,k>0$. If H is primitive and v-general, then the intersection Betti and Hodge numbers of $M(S,v,H)$ stabilise to the stable Betti and Hodge numbers of the Hilbert scheme of s-points as s tends to infinity, i.e. the limits $\lim_{m\to\infty} \dim IH^{p,q}(M(S,v,H))$ are finite and independent of k.*

Proof. Given two (m,k)-triples (S_1,v_1,H_1) and (S_2,v_2,H_2), $M(S_1,v_1,H_1)$ and $M(S_2,v_2,H_2)$ are deformation equivalent, and the deformation is locally analytically trivial by [70, Theorem 1.17]. In particular, we have $\dim IH^{p,q}(M(S_1,v_1,H_1)) = \dim IH^{p,q}(M(S_2,v_2,H_2))$.

Now choose a K3 surface S_0 with $\mathrm{Pic}(S_0) \simeq \mathbb{Z}$ generated by the class of a smooth curve C of genus $g=k+1$. Since (S,v,H) and $(S_0,(0,mC,m),C)$ are $(m,g-1)$-triples, we have

$$\dim IH^{p,q}(M(S,v,H)) = \dim IH^{p,q}(M(S_0,(0,mC,m),C)).$$

[8] That is there exists a coherent sheaf $\mathcal{F}$ on S such that $v=(rk(\mathcal{F}),c_1(\mathcal{F}),\chi(\mathcal{F})-rk(\mathcal{F}))$.

By Corollary 10.4, we have

$$\lim_{m\to\infty} \dim IH^{p,q}(M(S_0,(0,mC,m),C)) = \lim_{m\to\infty} \dim IH^{p,q}(M(S_0,(0,mC,1),C)),$$

where $M(S_0,(0,mC,1)$ is deformation equivalent to a Hilbert scheme of $s := 2m^2(g-1)$ points on a K3 surface. ∎

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