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On degenerate blow-up profiles for the subcritical semilinear heat equation

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Abstract. We consider the semilinear heat equation with a superlinear power nonlinearity in the Sobolev subcritical range. We construct a solution which blows up in finite time only at the origin, with a completely new blow-up profile, which is cross-shaped. Our method is general and extends to the construction of other solutions blowing up only at the origin, with a large variety of blow-up profiles, degenerate or not.

Keywords: semilinear heat equation, blow-up behavior, blow-up profile.

1. Introduction

We consider the following subcritical semilinear heat equation:

$$\partial_t u = \Delta u + |u|^{p-1} u, \tag{1.1}$$

where $u: (x,t) \in \mathbb{R}^N \times [0,T) \to \mathbb{R}$, $T > 0$,

$$p > 1 \quad \text{and} \quad (N-2)p < N+2. \tag{1.2}$$

We consider a solution $u(x,t)$ blowing up in finite time $T > 0$, in the sense that

$$\|u(t)\|_{L^\infty} \to \infty \quad \text{as } t \to T,$$

and $a \in \mathbb{R}^N$, a blow-up point of $u(x,t)$, i.e., some a such that $|u(a,t)| \to \infty$ as $t \to T$.

From Giga and Kohn [10] and Giga, Matsui and Sasayama [12], we know that all blow-up solutions are Type 1 in the subcritical case, in the sense that

$$\forall t \in [0,T), \quad \|u(t)\|_{L^\infty} \le C(T-t)^{-\frac{1}{p-1}} \quad \text{for some } C > 0. \tag{1.3}$$

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In order to simplify the exposition, we assume $N = 2$ and focus on the simplest case of the new profiles we handle in this paper (see (1.14) below). Other examples for $N = 2$ and some extensions to the case $N \geq 3$ are given in Section 1.3.

Proceeding as Giga and Kohn did in [9], we introduce the so-called *similarity variables*

$$w_a(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t), \quad \text{where } y = \frac{x-a}{\sqrt{T-t}} \text{ and } s = -\log(T-t). \tag{1.4}$$

From Giga and Kohn [11], we know that up to replacing u by $-u$, we have

$$w_a(y,s) \to \kappa \equiv (p-1)^{-\frac{1}{p-1}} \quad \text{as } s \to \infty, \tag{1.5}$$

uniformly on compact sets. According to Velázquez [26] (see also Filippas and Kohn [6] together with Filippas and Liu [7]), unless $w_a \equiv \kappa$, we may refine that convergence and obtain the following "blow-up profile" $Q(y)$ such that

$$w_a(y,s) - \kappa \sim Q(y,s) \quad \text{as } s \to \infty, \tag{1.6}$$

uniformly on compact sets, with either

$$Q(y,s) = -\frac{\kappa}{4ps}\sum_{i=1}^{l} h_2(y_l), \tag{1.7}$$

where $l = 1$ or 2, after a rotation of coordinates; keeping only the leading terms in the polynomials involved in $Q(y,s)$ and removing the time factor, we obtain the following quadratic form:

$$B(y) = \frac{\kappa}{4p}\sum_{i=1}^{l} y_i^2 \tag{1.8}$$

which is nonzero and nonnegative; or

$$Q(y,s) = -e^{-(\frac{m}{2}-1)s}\sum_{j=0}^{m} C_{m,j} h_{m-j}(y_1) h_j(y_2) \tag{1.9}$$

for some even integer $m = m(a) \geq 4$, where $y = (y_1, y_2)$, $h_j(\xi)$ is the rescaled Hermite polynomial defined by

$$h_j(\xi) = \sum_{i=0}^{[\frac{j}{2}]} \frac{j!}{i!(j-2i)!}(-1)^i \xi^{j-2i}, \tag{1.10}$$

and the homogeneous polynomial (obtained by keeping only the leading terms of the polynomials of $Q(y)$ and removing the time factor)

$$B(y) = \sum_{j=0}^{m} C_{m,j} y_1^{m-j} y_2^j \tag{1.11}$$

is also nonzero and nonnegative. Such a function was first introduced by Velázquez in [26], who called it a "homogeneous multilinear form".

If the origin is the only zero for the homogeneous polynomial $B(y)$ defined in (1.8) and (1.11), we are in the nondegenerate case. If not, we are in the degenerate case. Accordingly, the corresponding blow-up profile $Q(y)$ given in (1.7) or (1.9) will be said to be nondegenerate or degenerate.

With the classification in (1.6), (1.7) and (1.9) at hand, a natural question arises: is a an isolated blow-up point or not?

In the nondegenerate case, we know from [25, p. 1570, Theorem 2] that a is isolated. In particular, for any b in a small ball centered at a with $b \neq a$, $u(b,t)$ has a finite limit denoted by $u(b,T)$, as $t \to T$, with the following equivalences as $b \to a$:

$$u(b,T) \sim \Big[\frac{(p-1)^2}{8p}\frac{|x-a|^2}{|\log|x-a||}\Big]^{-\frac{1}{p-1}} \quad \text{if (1.7) holds with } l = 2,$$

$$u(b,T) \sim \Big[\frac{(p-1)^2}{\kappa}B(x-a)\Big]^{-\frac{1}{p-1}} \quad \text{if (1.9) holds with } B(y) > 0 \text{ for } y \neq 0.$$

In the degenerate case, the situation is less clear. In fact, the only known examples are "artificial", in the sense that the solution depends only on one space variable, say $\omega \cdot x$ where $\omega \in \mathbb{S}^1$. A radial solution blowing up outside the origin in 2 dimensions gives rise to a degenerate situation too. In these two examples, the blow-up point is not isolated. Apart from these trivial examples, no more solutions with a degenerate profile are known.

After this, we wonder whether there exists a solution obeying (1.6) such that (1.9) holds for some $m \geq 4$ with a degenerate homogeneous polynomial in (1.11) and an isolated blow-up point. In this paper, we provide such an example, which is the first ever in the subcritical range (see Theorem 1 below). Note that in the supercritical case, Merle, Raphaël and Szeftel have already provided in [18] an example of a single-point blow-up solution with a degenerate anisotropic profile, strongly relying on the existence of a stationary solution to equation (1.12) given below, which decays to zero at infinity (such a solution does not exist in the subcritical range).

1.1. The existence question: State of the art and difficulties in the degenerate case

We review here the question of the existence of blow-up solutions obeying (1.7) and (1.9). Let us first mention that in the one-dimensional case, the question was positively answered by Bricmont and Kupiainen in [2] (see also Herrero and Velázquez [14] for case (1.9) with $m = 4$).

Let us go back to the two-dimensional case and first focus on the nondegenerate case. The only examples we know concern case (1.7) with $l = 2$, thanks to Bricmont and Kupiainen [2] together with Merle and Zaag [19]. Such a behavior is known to be stable with respect to perturbations in initial data from [19] together with Fermanian, Merle and Zaag [4, 5]. Note that Herrero and Velázquez showed the genericity of such a behavior in [13, 15] dedicated to the one-dimensional case, and in an unpublished preprint in higher space dimensions. We would like to mention also the solutions constructed by Nguyen and Zaag in [24], showing a refinement of (1.7) with $l = 2$. As for case (1.9) with a nondegenerate homogeneous polynomial in (1.11), it has been treated by Amadori [1].

Concerning the degenerate case in (1.9) with a degenerate homogeneous polynomial in (1.11), the only cases we know are the one-dimensional trivial cases we have just mentioned above, showing a non-isolated blow-up point. Apart from these trivial examples, no more solutions with a degenerate profile are known.

As stated earlier, the main goal of the paper is to provide an example of a blow-up solution obeying (1.9) in the degenerate case with an isolated blow-up point.

Let us mention that the question of having nontrivial solutions with degenerate profiles was mentioned by Hiroshi Matano, because of the failure of the formal computation. As a matter of fact, the strategy used in the nondegenerate case is ineffective in the degenerate case, as we will explain below.

Indeed, that strategy consists in working in the similarity variables setting (1.4), where equation (1.1) is transformed into the following equation satisfied by w_a (or w for short): for all $y \in \mathbb{R}^2$ and $s \geq -\log T$,

$$\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1} w. \tag{1.12}$$

For example, the idea used by Bricmont and Kupiainen in [2] to construct their example in one space dimension with (1.9), which holds with $m = 4$, consists in linearizing equation (1.12) around the following profile:

$$(p - 1 + e^{-s}|y|^4)^{-\frac{1}{p-1}}. \tag{1.13}$$

Accordingly, if we intend to construct a solution obeying the following degenerate estimate:

$$w_0(y,s) - \kappa \sim -e^{-s} h_2(y_1) h_2(y_2) \quad \text{as } s \to \infty, \tag{1.14}$$

uniformly on compact sets, a naive idea would be to linearize equation (1.12) around the following profile:

$$\left(p - 1 + \frac{(p-1)^2}{\kappa} e^{-s} y_1^2 y_2^2\right)^{-\frac{1}{p-1}}, \tag{1.15}$$

which already has the same expansion (1.14) as the solution we intend to construct.

Unfortunately, a big problem arises with this profile, since it does not decay to 0 as $|y| \to \infty$, unlike the profile in (1.13). In fact, this smallness of profile (1.13) at infinity combined with the stability of the zero solution of (1.12) is essential in the control of the solution at infinity in space. In other words, with profile (1.15), we can not get such a control, and the naive idea collapses, unless we can manage to get this decaying property. Note that with the naive profile (1.15), the corresponding (approximate) solution is given by

$$u(x,t) = (T-t)^{-\frac{1}{p-1}} \left(p - 1 + \frac{(p-1)^2}{\kappa} \frac{x_1^2 x_2^2}{T-t}\right)^{-\frac{1}{p-1}},$$

which blows up everywhere on the axes $x_1 = 0$ and $x_2 = 0$.

1.2. Main result of the paper

In order to construct a solution obeying (1.14), following the previous subsection, a natural idea would be to refine expansion (1.14) in order to get higher-order terms ensuring some decaying along the axes $y_i = 0$, which are the degenerate directions of the naive profile (1.15). Such a refinement is given below in Lemma 2.1. It shows that the term $C_{6,0}(h_6(y_1) + h_6(y_2))$ may ensure that decaying property, provided that we take $C_{6,0} = -\delta$ with large $\delta > 0$ (note that the parameter $C_{6,0}$ is free in the expansion of Lemma 2.1). Keeping only the leading terms of the polynomials, this leads to the following refinement of profile (1.15):

$$\Phi(y,s) = \left[p - 1 + \frac{(p-1)^2}{\kappa}(e^{-s}y_1^2y_2^2 + \delta e^{-2s}(y_1^6 + y_2^6))\right]^{-\frac{1}{p-1}}, \tag{1.16}$$

as we will explain more in detail in Section 2 below. This refined version clearly has the decaying property at infinity in space, which enables us to rigorously prove the existence of a solution obeying the degenerate profile (1.14) and blowing up only at the origin, answering Matano's question. Note that in our proof, it is no longer possible simply to linearize around (1.16) since it contains higher-order terms in e^{-s} which are crucial near $y_1 = 0$ and $y_2 = 0$, as will be explained in more details in Section 2.

Theorem 1 (A blow-up solution for equation (1.1) with a cross-shaped blow-up profile). *When $N = 2$, there exists $\delta_0 > 0$ such that for any $\delta \geq \delta_0$, there exists a solution $u(x,t)$ to equation (1.1) which blows up in finite time T only at the origin, with*

(i) *Inner profile: We have that*

$$w_0(y,s) - \kappa \sim -e^{-s}h_2(y_1)h_2(y_2) \quad \text{as } s \to \infty, \tag{1.17}$$

in $L^2_\rho(\mathbb{R}^2)$ with

$$\rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{4\pi} \tag{1.18}$$

and uniformly on compact sets.

(ii) *Intermediate profile: For any $K > 0$, it holds that*

$$\sup_{e^{-s}y_1^2y_2^2 + \delta e^{-2s}(y_1^6+y_2^6) < K} |w_0(y,s) - \Phi(y,s)| \to 0 \quad \text{as } t \to T,$$

where $\Phi(y,s)$ is defined in (1.16).

(iii) *Final profile: For any $x \neq 0$, $u(x,t)$ converges to a finite limit $u(x,T)$ uniformly on compact sets of $\mathbb{R}^2 \backslash \{0\}$ as $t \to T$, with*

$$u(x,T) \sim \left[\frac{(p-1)^2}{\kappa}(x_1^2x_2^2 + \delta(x_1^6 + x_2^6))\right]^{-\frac{1}{p-1}} \quad \text{as } x \to 0. \tag{1.19}$$

Remark. One may convince himself that the profiles in (1.16) and (1.19) are cross-shaped, which justifies the title of the theorem.

Remark. This is the first example of a blow-up solution in the subcritical range showing a degenerate homogeneous polynomial in (1.11) with $m \geq 4$ and an isolated blow-up point. Note that in the literature, many examples of blow-up solutions with only one blow-up point are available (see Weissler [27], Bricmont and Kupiainen [2], Amadori [1], Merle and Zaag [19], Nguyen and Zaag [24]). Strikingly enough, almost all the known examples are either radial or asymptotically radial, in the sense that they approach a radially symmetric blow-up profile, except in Amadori [1]. To our knowledge, the only exceptions hold in the Sobolev supercritical case, with the Type 1 solution constructed by Merle, Raphaël and Szeftel [18] (with a $\frac{1}{s}$ rate in one direction and an exponentially decaying rate in the other), and also the Type 2 (i.e., not Type 1, see (1.3)) blow-up solution due to Collot, Merle and Raphaël [3], where the profile is anisotropic for both examples. No example is available in the subcritical range. In Theorem 1, we provide such an example, with a non-radial solution blowing up only at the origin, obeying behavior (1.9) with a degenerate homogeneous polynomial in (1.11), and an anisotropic blow-up profile.

Remark. Note that thanks to our method, we can derive *new* solutions for $N = 2$ and also $N \geq 3$, blowing up only at the origin, in the degenerate case, showing a not necessarily radial profile in (1.9) and (1.11) (see below in Section 1.3).

Remark. Taking $C_{6,0} = -\delta$, where $\delta > 0$ (and large) is crucial in our argument. As a matter of fact, using our analysis and the blow-up criterion that we proved in [20] for solutions of equation (1.12), we show that the construction is impossible in the case $C_{6,0} > 0$. More precisely, if $u(x,t)$ is a symmetric solution (with respect to the axes and the bisectrices) of equation (1.1) blowing up at some time $T > 0$ such that $u(0) \in L^\infty(\mathbb{R}^2)$ and estimate (1.17) holds, then estimate (2.1) holds with $C_{6,0} \leq 0$ (note that a similar statement holds without the symmetry assumption).

1.3. Extensions of the main result

After Theorem 1, we would like to mention that our strategy extends with no difficulty to the construction of solutions to equation (1.1) blowing up only at the origin with other types of profiles (as defined in (1.6), (1.7) and (1.9)) summarized in Tables 1 and 2, dedicated to the nondegenerate and the degenerate cases, respectively, where $C_0 = \frac{(p-1)^2}{\kappa}$ and $\delta_0 > 0$ is large.

Note that in the degenerate case, the idea behind the design of the profiles we are presenting below is simple:

- First, considering the classification given in (1.6) (and its natural extension in higher dimensions), we choose case (1.9) with some $Q(y)$ and a degenerate homogeneous polynomial $B(y)$ in (1.11); in Theorem 1, we choose $Q(y) = e^{-s} h_2(y_1) h_2(y_2)$ and $B(y) = y_1^2 y_2^2$.
- Then, we refine estimate (1.6) by exhibiting higher-order terms, and select among them the polynomials which live on the degenerate directions of $B(y)$; in Theorem 1,

$Q(y,s)$	Equivalent of $u(x,T)$ as $x \to 0$	Conditions
$-e^{(1-\frac{k}{2})s}\sum_{i=0}^{k} C_{k,i}h_{k-i}(y_1)h_i(y_2)$	$[C_0B(x)]^{-\frac{1}{p-1}}$ with $B(x)=\sum_{i=0}^{k} C_{k,i}x_1^{k-i}x_2^i$	$N=2$, $k\geq 4$ is even and $B(x)>0$ for $x\neq 0$
$-e^{-(\frac{k}{2}-1)s}\sum_{j_1+\cdots+j_N=k} C_{k,j_2,\dots,j_N}h_{j_1}(y_1)\cdots h_{j_N}(y_N)$	$[C_0B(x)]^{-\frac{1}{p-1}}$ with $B(x)=\sum_{j_1+\cdots+j_N=k} C_{k,j_2,\dots,j_N}x_1^{j_1}\cdots x_N^{j_N}$	$N\geq 3$, $k\geq 4$ is even and $B(x)>0$ for $x\neq 0$

Tab. 1. Extensions in the nondegenerate case.

$Q(y,s)$	Equivalent of $u(x,T)$ as $x \to 0$	Conditions
$-e^{-s}h_2(y_1)h_2(y_2+a_0y_1)$	$[C_0(x_1^2(x_1+a_0x_2)^2+\delta(x_1^6+x_2^6))]^{-\frac{1}{p-1}}$	$N=2$, $\delta\geq\delta_0$, $a_0\in\mathbb{R}$
$-e^{(1-\frac{k}{2})s}h_{k_1}(y_1)\cdots h_{k_l}(a_ly_1+b_ly_2)$	$[C_0(x_1^{k_1}\cdots(a_lx_1+b_lx_2)^{k_l}+\delta(x_1^{k+2}+x_2^{k+2}))]^{-\frac{1}{p-1}}$	$N=2$, $\delta\geq\delta_0$, $k=k_1+\cdots+k_l$, $k_i\geq 2$ is even, $(a_i,b_i)\neq(0,0)$, the straight lines $\{y_1=0\},\dots,\{a_ly_1+b_ly_2=0\}$ are distinct
$-e^{(1-\frac{k}{2})s}\prod_{i=1}^{m}\Big(\sum_{j\in I_i}h_{2\theta_i}(y_j)\Big)$	$\Big[C_0\Big(\prod_{i=1}^{m}\Big(\sum_{j\in I_i}\lvert x_j\rvert^{2\theta_i}\Big)+\delta(x_1^{k+2}+\cdots+x_N^{k+2})\Big)\Big]^{-\frac{1}{p-1}}$	$N\geq 3$, $\delta\geq\delta_0$, $k=\sum_{i=1}^{m}2\theta_i$, and the set I_i make a partition of $\{1,\dots,N\}$

Tab. 2. Extensions in the degenerate case.

the degenerate directions are $y_1 = 0$ and $y_2 = 0$, the refinement of (1.6) is given below in (2.1) and the polynomials we select in that refinement of (1.6) are $e^{-2s}h_6(y_1)$ and $e^{-2s}h_6(y_2)$.

- Finally, we design a profile similar to (1.16), with two terms, one corresponding to the degenerate homogeneous polynomial $B(y)$ defined in (1.11) (which is $e^{-s}y_1^2y_2^2$ in Theorem 1), and the other to the polynomials we selected on the degenerate directions of $B(y)$ (which is $e^{-2s}(y_1^6 + y_2^6)$ in Theorem 1). The key property of that profile is that it is decaying to zero in all directions, though with different scales according to whether we are in the degenerate or the nondegenerate directions of the homogeneous polynomial $B(y)$.

Concerning these examples, we would like to make some comments:

- The first example in Table 1 was already given by Amadori [1] with a different proof.
- The first example in Table 2 shows that the degeneracy directions of the homogeneous polynomial $B(x)$ need not be orthogonal, unlike one may assume from the constructed example in Theorem 1.

1.4. Main ingredients of the proof

As we have just written before the statement of Theorem 1, our strategy relies on the refinement of goal (1.14) given in Lemma 2.1 below, but not only. Indeed, that refinement allows us to design a good candidate for the profile. Then, linearizing equation (1.12) around that profile, a classical approach based on a (largely adapted) center manifold theory will be used to construct a solution, provided that the following crucial uniform L^∞ bound is proved:

$$\forall s \geq s_0, \quad \|w_0(s)\|_{L^\infty(\mathbb{R}^2)} \leq M$$

for some $M > 0$. Introducing a new method for the proof of this bound is the key estimate in the proof. That method makes our main novelty and contribution. Let us briefly explain how we proceed.

Using the similarity variables' definition (1.4), we remark that

$$w_b(y,s) = w_0(y + be^{\frac{s}{2}}, s). \tag{1.20}$$

This way, we reduce the question to the control of $w_b(0,s)$, for any $b \in \mathbb{R}^2$ and $s \geq s_0$. Thanks to our Liouville theorem recalled in Proposition 5.1 below, we show that the gradient is uniformly small in space and time, hence, we further reduce the question to the control of $\|w_b(s)\|_{L^2_\rho}$, for any $b \in \mathbb{R}^2$ and $s \geq s_0$. This will be done through a careful choice of initial data (at $s = s_0$) for w_0, which completely determines initial data for w_b. Then, starting from these initial data and integrating equation (1.12) for $s \geq s_0$, coordinate by coordinate, we will show that w_b will decrease, providing the control on $\|w_b(s)\|_{L^2_\rho}$, hence on $w_b(0,s) = w_0(be^{\frac{s}{2}}, s)$ (thanks to (1.20)), and finally, collecting all the information, on $\|w_0(s)\|_{L^\infty}$. Note that our integration technique goes beyond the linear level and uses the quadratic term of the equation. Note also that for small values of b, we are in the vicinity of κ from (1.5), which is a stationary solution of (1.12) with both stable and unstable directions. This makes the integration particularly delicate. See Section 5 for the detailed proof, in particular the end of Section 5.2.

To our knowledge, this is the first time such a method is implemented for the control of the L^∞ norm. We strongly believe it is applicable far beyond the case of the semilinear heat equation (1.1).

According to the strategy we presented above, we proceed in several sections to prove Theorem 1:

- first, in Section 2, we formally derive the profile in similarity variables;
- then, in Section 3, we give a setting of the problem;
- in Section 4, we explain the dynamics of the equation and suggest the general form of initial data;
- in Section 5, we prove a crucial L^∞ bound on the solution in similarity variables;
- in Section 6, we conclude the proof of Theorem 1;
- finally, we prove the technical details in Section 7.

2. Formal determination of the profile in similarity variables

In this paper, we ask whether we can have a solution $u(x,t)$ to equation (1.1) which blows up in finite time $T > 0$ at the origin, such that estimate (1.14) holds, namely with

$$w_0(y,s) - \kappa \sim -e^{-s} h_2(y_1) h_2(y_2) \quad \text{as } s \to \infty,$$

uniformly on compact sets. In order to simplify the calculations, we will assume that $u(x,t)$ is symmetric with respect to the axes and the bisectrices, for any $t \in [0,T)$.

As we have already discussed in Section 1.1, applying the strategy of Bricmont and Kupiainen [2] is ineffective, because the naive profile given in (1.15) is not decaying to zero along the axes $y_i = 0$. Proceeding as we wrote in Section 1.2, the idea to refine the naive guess in (1.15) goes through the refinement of the target behavior in (1.14), which we give in the following.

Lemma 2.1 (Second-order Taylor expansion). *Assuming that (1.14) holds and that the solution is symmetric with respect to the axes and bisectrices, it follows that*

$$\begin{aligned} w_0(y,s) = \kappa - e^{-s} h_2 h_2 + e^{-2s} \Big\{ & -\frac{32p}{3\kappa} h_0 h_0 - \frac{16p}{\kappa}(h_2 h_0 + h_0 h_2) \\ & - \frac{4p}{\kappa}(h_4 h_0 + h_0 h_4) - \frac{32p}{\kappa} h_2 h_2 + C_{6,0}(h_6 h_0 + h_0 h_6) \\ & + \Big(\frac{4p}{\kappa} s + C_{6,2}\Big)(h_4 h_2 + h_2 h_4) + \frac{p}{2\kappa} h_4 h_4 \Big\} + O(s^2 e^{-3s}) \end{aligned} \tag{2.1}$$

as $s \to \infty$, uniformly on compact sets, for some constants $C_{6,0}$ and $C_{6,2}$, where the notation $h_i h_j$ stands for $h_i(y_1) h_j(y_2)$.

Proof. The proof is omitted since it is straightforward from the method we explained in [23, Proposition 1]. ■

From this expansion, we remark that the term $C_{6,0}e^{-2s}(h_6(y_1)+h_6(y_2))$ will ensure the good decaying property, if $C_{6,0}<0$, especially on the axes, where the main term $e^{-s}y_1^2y_2^2$ is 0. More precisely, taking $C_{6,0}=-\delta$ for some $\delta>0$ to be fixed large enough later in the proof, then, keeping only the leading term in the polynomials, we naturally suggest the following modified version of (1.15):

$$\varphi(y,s)=\left[\frac{E}{D}\right]^{\frac{1}{p-1}} \tag{2.2}$$

with

$$E=1+e^{-s}P(y)+e^{-2s}Q(y), \tag{2.3}$$

$$D=p-1+\frac{(p-1)^2}{\kappa}(e^{-s}y_1^2y_2^2+\delta e^{-2s}(y_1^6+y_2^6)), \tag{2.4}$$

where $P(y)$ and $Q(y)$ are polynomials which will be chosen so that the numerator in (2.3) is positive (hence, φ is well defined), and

$$\varphi(y,s)=\kappa-e^{-s}h_2h_2+e^{-2s}\left\{-\delta(h_6h_0+h_0h_6)+\gamma(h_4h_2+h_2h_4)+\frac{p}{2\kappa}h_4h_4\right\} + O(e^{-3s}) \tag{2.5}$$

as $s\to\infty$, uniformly on compact sets, for some γ that we fix later. Note that the aimed expansion agrees with the prediction in Lemma 2.1 at the order e^{-s} and most of the order e^{-2s}. Most importantly, we will also require that $\varphi(\cdot,s)\in L^\infty(\mathbb{R}^2)$ and $\varphi(y,s)\to 0$ as $|y|\to\infty$, for any $s\ge 0$, and the choice of γ will be crucial for that.

Let us specify the choices of δ, γ, $P(y)$ and $Q(y)$. Since (2.2) directly implies that

$$\begin{aligned}\varphi(y,s)=\kappa&+e^{-s}\left(\frac{\kappa}{p-1}P(y)-y_1^2y_2^2\right)\\&+e^{-2s}\left(\frac{\kappa}{p-1}Q(y)+\frac{\kappa(2-p)}{2(p-1)^2}P(y)^2-\frac{P(y)}{p-1}y_1^2y_2^2\right.\\&\left.-\delta y_1^6-\delta y_2^6+\frac{p}{2\kappa}y_1^4y_2^4\right)+O(e^{-3s})\end{aligned} \tag{2.6}$$

as $s\to\infty$, uniformly on compact sets, we see by identification with (2.5) that we must have

$$P(y)=\frac{p-1}{\kappa}(y_1^2y_2^2-h_2h_2), \tag{2.7}$$

$$\begin{aligned}Q(y)=\frac{p-1}{\kappa}&\left(\frac{P(y)}{p-1}y_1^2y_2^2+\frac{\kappa(p-2)}{2(p-1)^2}P(y)^2+\delta(y_1^6-h_6(y_1))\right.\\&\left.+\delta(y_2^6-h_6(y_2))+\frac{p}{2\kappa}(h_4h_4-y_1^4y_2^4)+\gamma(h_4h_2+h_2h_4)\right).\end{aligned} \tag{2.8}$$

Now, we claim the following.

Lemma 2.2 (Good definition, boundedness and decaying at infinity of $\varphi(y,s)$ (2.2), (2.7) and (2.8)). *Take $\gamma = \frac{6p-2}{\kappa}$ and consider $\delta \geq \delta_{10}$ and $s \geq s_{10}(\delta)$ for some large enough $\delta_{10} \geq 1$ and $s_{10}(\delta) \geq 0$. Then*

(i) $E \geq \frac{1}{2}$, *hence $\varphi(y,s)$ is well defined and positive.*

(ii) $\|\varphi(\cdot,s)\|_{L^\infty} \leq \kappa + C_0 e^{-\frac{s}{3}}$, *for some $C_0 > 0$.*

(iii) $\|\nabla\varphi(\cdot,s)\|_{L^\infty} \leq C_0 e^{-\frac{s}{6}}$.

(iv) $\varphi(y,s) \to 0$ *as* $|y| \to \infty$.

Proof. Take $\gamma = \frac{6p-2}{\kappa}$ and consider $\delta \geq 1$ and $s \geq 0$ to be taken large enough.

(i) It is enough to show that both $P(y)$ and $Q(y)$ are bounded from below. Since

$$h_2(\xi) = \xi^2 - 2, \quad h_4(\xi) = \xi^4 - 12\xi^2 + 12, \quad h_6(\xi) = \xi^6 - 30\xi^4 + 180\xi^2 - 120$$

from (1.10), it follows from (2.7) and (2.8) that

$$P(y) = \frac{2(p-1)}{\kappa}(y_1^2 + y_2^2 - 2) \geq -\frac{4(p-1)}{\kappa} \tag{2.9}$$

and

$$\begin{aligned}Q(y) &= \frac{p-1}{\kappa}\Big(\frac{2}{\kappa}[y_1^4y_2^2 + y_1^2y_2^4] + 30\delta[y_1^4 + y_2^4] + \frac{6p}{\kappa}[-y_1^4y_2^2 - y_1^2y_2^4] \\ &\qquad + \gamma[y_1^4y_2^2 + y_1^2y_4^2]\Big) + O(\delta[1+|y|^2] + [1+|\gamma|][1+|y|^4]) \\ &= \frac{p-1}{\kappa}\Big(\Big(\frac{2}{\kappa} - \frac{6p}{\kappa} + \gamma\Big)[y_1^4y_2^2 + y_1^2y_2^4] + 30\delta[y_1^4 + y_2^4]\Big) \\ &\quad + O(\delta[1+|y|^2] + [1+|\gamma|][1+|y|^4]).\end{aligned}$$

Choosing $\gamma = \frac{6p-2}{\kappa}$, we see that

$$Q(y) = \frac{30\delta(p-1)}{\kappa}[y_1^4 + y_2^4] + O(\delta[1+|y|^2] + [1+|y|^4]) \quad \text{as } |y| \to \infty. \tag{2.10}$$

Taking δ large enough, then taking $|y|$ large enough, we get that

$$Q(y) \geq \frac{15\delta(p-1)}{\kappa}[y_1^4 + y_2^4], \tag{2.11}$$

which means that $Q(y)$ is bounded from below. Since (2.9) implies that $P(y)$ is bounded from below too, taking s large enough, we see from (2.3) that $E \geq \frac{1}{2}$, hence, φ defined in (2.2) is well defined.

(ii) Using (2.9) and (2.10), we see from (2.2) that

$$|\varphi(y,s)|^{p-1} \leq C\frac{(N_1 + N_2 + N_3)}{D}, \tag{2.12}$$

where

$$N_1 = 1 + e^{-s}, \quad N_2 = \delta e^{-2s}|y|^4, \quad N_3 = e^{-s}|y|^2 \tag{2.13}$$

and D is introduced in (2.4). Since $D \geq p-1$, it follows that

$$\frac{N_1}{D} \leq \frac{1+e^{-s}}{p-1}. \tag{2.14}$$

Introducing $z = e^{-\frac{s}{4}} y$, we write

$$\frac{N_2}{D} \leq \frac{C\delta e^{-s}|z|^4}{1+z_1^2 z_2^2 + e^{-\frac{s}{2}}|z|^6} \leq \frac{C\delta e^{-s}|z|^4}{e^{-\frac{s}{2}} + e^{-\frac{s}{2}}|z|^6} = \frac{C\delta e^{-\frac{s}{2}}|z|^4}{1+|z|^6} \leq C\delta e^{-\frac{s}{2}}. \tag{2.15}$$

Denoting $X = |z|^2$, we get

$$\frac{N_3}{D} \leq \frac{Ce^{-\frac{s}{2}}|z|^2}{1+e^{-\frac{s}{2}}|z|^6} = Cg_\varepsilon(X), \tag{2.16}$$

where

$$\varepsilon = e^{-\frac{s}{2}}, \quad X = |z|^2 \quad \text{and} \quad g_\varepsilon(X) = \frac{\varepsilon X}{1+\varepsilon X^3}. \tag{2.17}$$

Since $g_\varepsilon'(X) = \frac{\varepsilon(1-2\varepsilon X^3)}{(1+\varepsilon X^3)^2}$ which changes from negative to positive at $X = X_\varepsilon \equiv (2\varepsilon)^{-\frac{1}{3}}$, it follows that

$$g_\varepsilon(X) \leq g_\varepsilon(X_\varepsilon) = \frac{2\varepsilon X_\varepsilon}{3} = \frac{(2\varepsilon)^{\frac{2}{3}}}{3}.$$

Using (2.16) and (2.17), we see that

$$\frac{N_3}{D} \leq Ce^{-\frac{s}{3}}. \tag{2.18}$$

Collecting the estimates in (2.12), (2.14), (2.15) and (2.18), then recalling definition (1.5) of κ, we get the desired conclusion.

(iii) By definition (2.2) of $\varphi(y,s)$, we write for all $s \geq 1$ and $y \in \mathbb{R}^2$,

$$(p-1)\log\varphi = \log E - \log D,$$

where E and D are defined in (2.3) and (2.4) (note that $D > 0$ by definition, and that $E > 0$ from item (i) of this lemma). Taking the gradient, we see that

$$\nabla\varphi = \frac{\varphi}{p-1}\Big(\frac{\nabla E}{E} - \frac{\nabla D}{D}\Big).$$

Since

$$|\nabla D| \leq C\delta e^{-2s}|y|^5 + Ce^{-s}|y_1|y_2^2 + Ce^{-s}|y_2|y_1^2 \equiv D_1 + D_2 + D_3 \tag{2.19}$$

from (2.9) and (2.10), we write from item (ii) of this lemma

$$|\nabla\varphi| \leq C\Big[\frac{|\nabla E|}{E} + \frac{D_1}{D} + \frac{D_2}{D} + \frac{D_3}{D}\Big].$$

Noting that

$$E \geq \frac{E_0}{C}, \quad \text{where } E_0 = 1 + e^{-s}(y_1^2 + y_2^2) + e^{-2s}(y_1^4 + y_2^4),$$
$$|\nabla E| \leq Ce^{-s}|y| + C\delta e^{-2s}(1 + |y|^3),$$

from (2.9), (2.11) and (2.10), using definitions (2.4) and (2.19) of D and D_1, then proceeding as for the proof of item (ii) of this lemma, we show that

$$\frac{|\nabla E|}{E} \leq \frac{|\nabla E|}{E_0} \leq C(\delta)e^{-\frac{s}{3}} \quad \text{and} \quad \frac{D_1}{D} \leq C(\delta)e^{-\frac{s}{3}}. \tag{2.20}$$

By symmetry, it remains only to bound the term $\frac{D_2}{D}$, when $y_2 \neq 0$. Since $\delta \geq 1$, using definitions (2.4) and (2.19) of D and D_2, we write

$$\frac{D_2}{D} \leq C \frac{e^{-s}|y_1|y_2^2}{1 + e^{-s}y_1^2y_2^2 + e^{-2s}y_2^6}.$$

As a function of the variable $|y_1|$, the left-hand side realizes its maximum for

$$|y_1| = \sqrt{\frac{1 + e^{-2s}y_2^6}{e^{-s}y_2^2}}.$$

Therefore, it follows that

$$\frac{D_2}{D} \leq C\sqrt{\frac{e^{-s}y_2^2}{1 + e^{-2s}y_2^6}} \leq Ce^{-\frac{s}{6}},$$

in particular

$$\frac{|\nabla D|}{D} \leq Ce^{-\frac{s}{6}}, \tag{2.21}$$

and the estimate on $\nabla\varphi$ in item (iii) follows.

(iv) From (2.9), (2.8) and (2.10), we see that the numerator of the fraction in (2.2) is a polynomial of degree 4, whereas the denominator is of degree 6. Thus, the conclusion follows.

This concludes the proof of Lemma 2.2. ■

3. Setting of the problem

From Section 2, we recall that our goal is to construct $u(x, t)$, a solution of equation (1.1) defined for all $(x, t) \in \mathbb{R}^2 \times [0, T)$ for some small enough $T > 0$, such that

$$w_0(y, s) - \kappa \sim -e^{-s}h_2(y_1)h_2(y_2) \quad \text{as } s \to \infty,$$

uniformly on compact sets, where $w_0(y, s)$ is the similarity variables' version defined in (1.4). We also recall our wish to construct a solution which is symmetric with respect to the axes and the bisectrices.

Consider $\varphi(y,s)$, our candidate for the profile, defined in (2.2), (2.7) and (2.8), with $\gamma = \frac{6p-2}{\kappa}$ and $\delta \geq 1$ fixed large enough, so that Lemma 2.2 holds. From the target expansion of w_0 given in Lemma 2.1 (with $C_{6,0} = -\delta$ and $C_{6,2} = \gamma$) and the expansion of the profile φ given in (2.5), we further specify our goal by requiring that $q(y,s)$ is small in some sense that will be shortly given in (3.20), where

$$q(y,s) = w_0(y,s) - \varphi(y,s). \tag{3.1}$$

In order to achieve this goal, we need to write then understand the dynamics of the equation satisfied by $q(y,s)$ near 0. This will be done in the following subsection.

3.1. *Dynamics for $q(y,s)$ defined in equation* (3.1)

Roughly speaking, our first task is to perform local estimates for bounded y, through a spectral analysis in L^2_ρ near κ from (1.5), the constant solution of equation (1.12), assuming a uniform bound on the solution.

More precisely, from (1.12), we see that $q(y,s)$ satisfies the following equation, for all $(y,s) \in \mathbb{R}^2 \times [s_0, \infty)$, where $s_0 = -\log T$:

$$\partial_s q = (\mathscr{L} + V(y,s))q + B(y,s,q) + R(y,s) \tag{3.2}$$

with

$$\begin{aligned}
\mathscr{L}q &= \Delta q - \frac{1}{2} y \cdot \nabla q + q, \quad V(y,s) = p\varphi(y,s)^{p-1} - \frac{p}{p-1}, \\
B(y,s,q) &= |\varphi(y,s) + q|^{p-1}(\varphi(y,s) + q) - \varphi(y,s)^p - p\varphi(y,s)^{p-1}q, \\
R(y,s) &= -\partial_s\varphi(y,s) + (\mathscr{L} - 1)\varphi(y,s) - \frac{\varphi(y,s)}{p-1} + \varphi(y,s)^p.
\end{aligned} \tag{3.3}$$

Let us give in the following some useful properties of the terms involved in equation (3.2):

- *The linear term:* Note that $\mathscr{L}$ is a self-adjoint operator in L^2_ρ, the L^2 space with respect to the measure ρdy defined in (1.18). The spectrum of $\mathscr{L}$ consists only of eigenvalues

  $$\operatorname{Spec} \mathscr{L} = \left\{1 - \frac{m}{2} \mid m \in \mathbb{N}\right\} \tag{3.4}$$

  with the following set of eigenfunctions:

  $$\{h_i h_j \equiv h_i(y_1)h_j(y_2) \mid i, j \in \mathbb{N}\}, \tag{3.5}$$

  which spans the space L^2_ρ, where we use rescaled Hermite polynomials (1.10). Note also that

  $$\mathscr{L}(h_i h_j) = \left(1 - \frac{i+j}{2}\right) h_i h_j. \tag{3.6}$$

- *The potential:* It is bounded in L^∞ and small in L^r_ρ for any $r \geq 1$, in the sense that

  $$\|V(\cdot,s)\|_{L^\infty} \leq \frac{p}{p-1} \quad \text{and} \quad \|V(\cdot,s)\|_{L^r_\rho} \leq C(r)e^{-s}, \tag{3.7}$$

  for s large enough, thanks to Lemma 2.2 and a small Taylor expansion.

- *The nonlinear term:* Since φ is uniformly bounded in space and time, thanks to item (iii) of Lemma 2.2, assuming the following a priori estimate,

$$\|q\|_{L^\infty(\mathbb{R}^2\times[s_0,\infty))} \le M, \tag{3.8}$$

we easily see that $B(y,s,q)$ is superlinear, in the sense that

$$|B(y,s,q)| \le C(M)|q|^{\bar{p}}, \quad \text{where } \bar{p} = \min(p,2) > 1, \tag{3.9}$$

for all $y \in \mathbb{R}^2$, $q \in \mathbb{R}$ and large s.

- *The remainder term:* By comparing expression (3.3) of $R(y,s)$ with the expression of equation (1.12), we see that $R(y,s)$ measures the quality of $\varphi(y,s)$ as an approximate solution of (1.12). In fact, through a straightforward calculation (see below in Section 7.1), one can show that

$$\forall r \ge 2,\ \forall s \ge 0, \quad \|R(s)\|_{L^r_\rho} \le C(r)e^{-2s}, \tag{3.10}$$

which is consistent with our approach in Section 2 (particularly, Lemma 2.1 and estimate (2.5)), where we constructed φ as an approximate solution to equation (1.12).

Bearing in mind these properties, the construction of a small solution $q(y,s)$ to equation (3.2) seems to be reasonable, except for a serious issue: the control of the nonlinear term $B(y,s,q)$, since the power function is not continuous in L^2_ρ. The control of the potential term Vq is delicate too. In the following subsection, we explain our idea to gain those controls.

3.2. *Control of the potential and the nonlinear terms in equation* (3.2)

In this subsection, we explain how we will achieve the control of the potential Vq and the nonlinear term B in equation (3.2). For simplicity, we present the argument only for the nonlinear term B, and restrict to the case when $p \ge 2$, hence $\bar{p} = 2$ (the actual proof for Vq and B will be done for any $p > 1$).

Assuming the a priori estimate (3.8), we may reduce equation (3.2) to the following linear equation with a source term:

$$\partial_s q = (\mathcal{L} + \bar{V})q + R(y,s), \tag{3.11}$$

where

$$|\bar{V}(y,s)| = \left|V(y,s) + \frac{B(y,s,q)}{q}\right| \le C + C(M)|q|^{\bar{p}-1} \le C + C(M)M^{\bar{p}-1} \equiv \bar{C}(M). \tag{3.12}$$

Thanks to the regularizing effect of the operator $\mathcal{L}$ (see Lemma 7.1 below), we may use equation (3.11) together with (3.12) and (3.10) to control the L^4_ρ norm of the solution with the L^2_ρ norm of the solution, up to some time delay,

$$\|q(s)\|_{L^4_\rho} \le e^{(1+\bar{C})s^*}\|q(s-s^*)\|_{L^2_\rho} + Ce^{-2s}. \tag{3.13}$$

Clearly, this is equivalent to

$$\|q(s)^2\|_{L^2_\rho} \le 2e^{2(1+\bar{C})s^*}\|q(s-s^*)\|^2_{L^2_\rho} + 2Ce^{-4s},$$

which allows us to control the nonlinear term $B(y,s,q)$ in L^2_ρ, thanks to (3.9).

Note that this control is possible thanks to the a priori estimate (3.8), which needs to be checked. We will explain that in the following subsection.

3.3. Proof of the a priori bound (3.8)

This is the key part of the argument, dedicated to the proof of the a priori bound (3.8), under which the local estimates presented in the previous subsections hold. In fact, as we will shortly see from the geometrical transform (3.14), we reduce the uniform bound for w_0 to a local estimate for w_a, where a is arbitrary. Later in Section 5, we will see that the control of w_a follows from the spectral analysis of equation (1.12), in particular the stability of its zero solution and its heteroclinic orbit connecting κ from (1.5) and 0 and given below in (5.8).

More precisely, since φ is uniformly bounded by Lemma 2.2, using definition (3.1) of q, it is enough to control $w_0(a,s)$, for any $a \in \mathbb{R}^2$ and $s \ge s_0$, in order to prove (3.8). Using the similarity variables' definition (1.4), we remark that

$$w_b(y,s) = w_0(y + be^{\frac{s}{2}}, s). \tag{3.14}$$

This way, we reduce the question to the control of $w_b(0,s)$, for any $b \in \mathbb{R}^2$ and $s \ge s_0$. Since w_b satisfies equation (1.12), which is parabolic, we further reduce the question to the control of $\|w_b(s)\|_{L^2_\rho}$, for any $b \in \mathbb{R}^2$ and $s \ge s_0$. This will be done through a careful choice of initial data (at $s = s_0$) for w_0, which completely determines initial data for w_b. Then, starting from these initial data and integrating equation (1.12) for $s \ge s_0$, we will show that w_b will decrease, providing the control on $\|w_b(s)\|_{L^2_\rho}$, hence on $w_b(0,s) = w_0(be^{\frac{s}{2}}, s)$ (thanks to (3.14)), and finally, collecting all the information, on $\|w_0(s)\|_{L^\infty}$.

For details, see Section 5, where we explain our new strategy to achieve this L^∞ uniform control, in particular, Section 5.2.

3.4. Definition of a shrinking set to make $q(s) \to 0$

From our formal analysis in Section 2, in particular the target expansion of w_0 in Lemma 2.1 (with $C_{6,0} = -\delta$ and $C_{6,2} = \gamma$) and the expansion of φ in (2.5), we may write the following target for $q(y,s)$ defined in (3.1):

$$\begin{aligned} q(y,s) = \frac{p}{\kappa}e^{-2s}\{&-32h_0h_0 - 16(h_2h_0 + h_0h_2) - 4(h_4h_0 + h_0h_4) - 32h_2h_2 \\ &+ 4s(h_4h_2 + h_2h_4)\} + O(s^2e^{-3s}) \quad \text{as } s \to \infty. \end{aligned} \tag{3.15}$$

In fact, we will not require such a sharp expansion. We will instead require that the components of $q(y,s)$ are bounded by the same rate as the one in front of the corresponding eigenfunctions shown on the right-hand side of (3.15). More precisely, expanding any function $v \in L^2_\rho$ on the eigenfunctions of $\mathcal{L}$ given in (3.5) as follows:

$$v = \sum_{i=0}^{7}\sum_{j=0}^{i} v_{i,j} h_{i-j} h_j + v_-, \tag{3.16}$$

where

$$v_{i,j} = \int v(y) k_{i-j}(y_1) k_j(y_2) \rho(y) dy \tag{3.17}$$

is the coordinate of v along the eigenfunction $h_{i-j}h_j$ (which corresponds to the eigenvalue $\lambda = 1 - \frac{i}{2}$ by (3.6)), and

$$k_n = \frac{h_n}{\|h_n\|^2_{L^2_\rho}}, \tag{3.18}$$

our target in this paper is to construct a solution to equation (3.2) defined on some interval $[s_0, \infty)$, where $s_0 = -\log T$ such that

$$\forall s \geq s_0, \quad q(s) \in V_A(s), \tag{3.19}$$

where $V_A(s)$ is defined as follows.

Definition 3.1 (A shrinking set to trap the solution). For any $A > 0$ and $s \geq 1$, $V_A(s)$ is the set of all $v \in L^\infty$ such that $\|v + \varphi(\cdot, s)\|_{L^\infty} \leq 2\kappa$, $v_{i,j} = 0$ if i or j is odd, $|v_{i,j}| \leq Ae^{-2s}$ if $i \leq 4$, $|v_{6,0}| = |v_{6,6}| \leq As^2 e^{-3s}$, $|v_{6,2}| = |v_{6,4}| \leq Ase^{-2s}$ and $\|v_-\|_{L^2_\rho} \leq A^2 s^2 e^{-3s}$, where v is decomposed as in (3.16), and the profile φ is defined in (2.2) and Lemma 2.2.

Remark. The L^∞ bound in this definition is crucial to control the nonlinear term (see (3.8) and Section 3.2).

Clearly, we see that

$$\text{if } v \in V_A(s), \quad \text{then } \|v\|_{L^2_\rho} \leq CAse^{-2s} \tag{3.20}$$

for s large enough, which shows that our goal in (3.19) implies indeed that $q(s) \to 0$ as $s \to \infty$ in $L^2_\rho(\mathbb{R}^2)$. The next part of the paper is devoted to the rigorous proof of our goal in (3.19), and to the fact that it implies our main result stated in Theorem 1.

4. Dynamics of the equation and general form of initial data

In this section, we study the dynamics of equation (3.2) and suggest a general form of initial data, depending on some parameters, which will be fine-tuned in some further steps to provide a suitable solution so that Theorem 1 holds.

We proceed in three steps, each presented in a subsection, starting by the projections of equation (3.2) on the different components of decomposition (3.16), then we study the behavior of the flow on the boundary of the shrinking set $V_A(s)$ defined in (3.1). Finally, we suggest a general form for initial data. Note that technical details will be postponed to Section 7 below, hoping to make our exposition clearer.

4.1. Dynamics of equation (3.2) in the shrinking set $V_A(s)$

Aiming at proving (3.19), we assume $q(s) \in V_A(s)$ for all $s \in [s_0, s_1]$ for some $s_1 \geq s_0$ and derive differential equations satisfied by $q_{i,j}$ and q_- in this subsection. In fact, we will first derive the size of the components of the remainder term $R(y,s)$ from (3.3) in the following.

Lemma 4.1 (Expansion of $R(y,s)$ from (3.3)). *For all $r \geq 2$, it follows that*

$$R(y,s) = \frac{pe^{-2s}}{\kappa}\big\{32h_0h_0 + 32(h_2h_0 + h_0h_2) + 4(h_4h_0 + h_0h_4) + 32h_2h_2 + 4(h_4h_2 + h_2h_4)\big\} + O(e^{-3s})$$

in $L^r_\rho(\mathbb{R}^2)$, as $s \to \infty$.

Proof. See Section 7.1 below. ∎

Now, we project equation (3.2) in the following.

Proposition 4.2 (Dynamics of equation (3.2) in $V_A(s)$). *For any $A \geq 1$, there exists $s_{11}(A) \geq 1$ such that the following holds. Assume that*

$$q(s_0) \in L^\infty, \quad \nabla q(s_0) \in L^\infty \quad \text{and} \quad \forall r \geq 2, \quad \|q(s_0)\|_{L^r_\rho} \leq C(r)As_0e^{-2s_0}, \tag{4.1}$$

and that $q(s) \in V_A(s)$ satisfies equation (3.2), for all $s \in [s_0, s_1]$, for some $s_1 \geq s_0 \geq s_{11}(A)$. Then for all $s \in [s_0, s_1]$,

(i) *for all $i \in \mathbb{N}$ and $0 \leq j \leq i$, $|q'_{i,j}(s) - (1 - \frac{i}{2})q_{i,j}(s)| \leq C_iAse^{-3s} + R_{i,j}(s)$,*

(ii) $\frac{d}{ds}\|q_-(s)\|_{L^2_\rho} \leq -3\|q_-(s)\|_{L^2_\rho} + CAse^{-3s} + \|R_-(s)\|_{L^2_\rho}$, *where $q(\cdot,s)$ and $R(\cdot,s)$ are decomposed as in (3.16).*

Proof. The proof is straightforward, except for the control of the potential term Vq and the superlinear term B in equation (3.2), which both need a regularizing delay estimate, already mentioned informally in (3.13). For that reason, we only state that delay estimate here, and postpone the proof to Section 7.2 below. ∎

Remark. The fact that q satisfies PDE (3.2) on the interval $[s_0, s]$ and not just at a particular time s is important for the delay estimate (3.13).

As we have just mentioned, let us precisely state that delay estimate, which is crucial for the proof of Proposition 4.2.

Proposition 4.3 (A delay regularizing estimate for equation (3.2)). *Under the hypothesis of Proposition 4.2 and for any $r \geq 2$ and $s \in [s_0, s_1]$, it holds that*

$$\|q(s)\|_{L^r_\rho} \leq C(r)Ase^{-2s}.$$

Proof. See below in Section 7.3. ∎

4.2. Behavior of the flow on the boundary of $V_A(s)$

Recall that our aim is to suitably choose $q(s_0)$ so that $q(s) \in V_A(s)$ for all $s \geq s_0$.

As in the previous subsection, we assume that $q(s) \in V_A(s)$ for all $s \in [s_0, s_1]$ for some $s_1 \geq s_0$. This time, we will assume in addition that at $s = s_1$, one component of $q(s_1)$ (as defined in (3.16)) "touches" the corresponding part of the boundary of its bound defined in Definition 3.1. We will then derive the position of the flow of that component with respect to the boundary, and see whether it is inward (which leads to a contradiction) or outward. This way, only the components with an outward flow may touch. Hopefully, those components will be in a finite-dimensional space, leading the way to the application of a consequence of Brouwer's lemma linked to some fine-tuning of the initial data, in order to guarantee that the solution will stay in $V_A(s)$ for all $s \geq s_0$. More precisely, this is our statement.

Proposition 4.4 (Position of the flow on the boundary of $V_A(s)$). *There exists $A_{12} \geq 1$ such that for any $A \geq A_{12}$, there exists $s_{12}(A) \geq 1$ such that if the hypotheses of Proposition 4.2 hold with $s_0 \geq s_{12}$, then*

(i) *If $q_{i,j}(s_1) = \theta A e^{-2s_1}$ with $i \leq 4$, i and j even, and $\theta = \pm 1$, then*

$$\theta q'_{i,j}(s_1) > \frac{d}{ds} A e^{-2s}\Big|_{s=s_1}.$$

(ii) *If $q_{6,j}(s_1) = \theta A s^2 e^{-3s}$ with $j = 0$ or $j = 6$, and $\theta = \pm 1$, then*

$$\theta q'_{i,j}(s_1) > \frac{d}{ds} A s^2 e^{-3s}\Big|_{s=s_1}.$$

(iii) *If $q_{6,j(s_1)} = \theta A s e^{-2s}$ with $j = 2$ or $j = 4$, and $\theta = \pm 1$, then*

$$\theta q'_{i,j}(s_1) < \frac{d}{ds} A s e^{-2s}\Big|_{s=s_1}.$$

(iv) *If $\|q_-(s_1)\|_{L^2_\rho} = A^2 s_1^2 e^{-3s_1}$, then*

$$\frac{d}{ds}\|q_-(s_1)\|_{L^2_\rho} < \frac{d}{ds} A^2 s^2 e^{-3s}\Big|_{s=s_1}.$$

Remark. The flow in items (i) and (ii) is outward, while in items (iii) and (iv), it is inward.

Proof of Proposition 4.4. See Section 7.4 below. ∎

Thanks to this statement, we immediately see that the constraints on $q_{6,0} = q_{6,6}$ and q_- in Definition 3.1 of $V_A(s)$ can never achieve equality.

Corollary 4.5 ($q_{6,2} = q_{6,4}$ and q_- never quit). *Under the settings of Proposition 4.4, assuming that $s_1 > s_0$, it holds that $|q_{6,2}(s_1)| = |q_{6,4}(s_1)| < A s_1 e^{-2s_1}$ and $\|q_-(s_1)\|_{L^2_\rho} < A^2 s_1^2 e^{-3s_1}$.*

Proof. We will only prove the estimate on q_-, since the other follows in the same way. Proceeding by contradiction, we assume that $\|q_-(s_1)\|_{L^2_\rho} \geq A^2 s_1^2 e^{-3s_1}$. Since $q(s) \in V_A(s)$ for all $s \in [s_0, s_1]$ by hypothesis, it follows that

$$\forall s \in (s_0, s_1], \quad \|q_-(s)\|_{L^2_\rho} \leq A^2 s^2 e^{-3s} \quad \text{and} \quad \|q_-(s_1)\|_{L^2_\rho} = A^2 s_1^2 e^{-3s_1}.$$

In particular, this translates into the following estimate between the derivatives of both curves:

$$\frac{d}{ds}\|q_-(s_1)\|_{L^2_\rho} \geq \frac{d}{ds} A^2 s^2 e^{-3s},$$

which is a contradiction by item (iv) of Proposition 4.4. ■

4.3. *Choice of initial data*

Thanks to Corollary 4.5, we see that the control of $q(s)$ in $V_A(s)$ reduces to the control of five components: $q_{0,0}$, $q_{2,0}$, $q_{4,0}$, $q_{4,2}$ and $q_{6,0}$ (together with the control of $\|q(s) + \varphi(s)\|_{L^\infty}$, which is a major novelty of our paper). It happens that item (i) in Proposition 4.4 indicates that the flow of those components is "transverse outgoing" on the boundary of the constraint introduced in Definition 3.1. As in our various papers where we construct solutions to PDEs with a prescribed behavior (see, for example, [19]), the control of this finite-dimensional part will be done through a Brouwer-type lemma, involving some fine-tuning of parameters in initial data. Namely, the idea would be to introduce the following family of initial data (for $w_0(y, s_0)$ linked to $q(y, s_0)$ by definition (3.1)) depending on parameters $d_{0,0}$, $d_{2,0}$, $d_{4,0}$, $d_{4,2}$ and $d_{6,0}$:

$$w_0(y, s_0) = \varphi(y, s_0) + [Ae^{-2s_0} S(y) + As^2 e^{-3s} \bar{S}(y)]\chi(y),$$

where

$$\begin{aligned} S(y) &= d_{0,0} + d_{2,0}[h_2(y_1) + h_2(y_2)] + d_{4,0}[h_4(y_1) + h_4(y_2)] \\ &\qquad + d_{4,2} h_2(y_1) h_2(y_2), \\ \bar{S}(y) &= d_{6,0}(h_6(y_1) + h_6(y_2)) \end{aligned} \tag{4.2}$$

for some sufficiently decaying function χ. In fact, in order to stick to the shape of the profile φ from (2.2), we will take the following initial data for equation (1.12):

$$w_0(y, s_0) = \left[\frac{E}{D} + \frac{p-1}{\kappa D^2}(Ae^{-2s_0} S(y) + As^2 e^{-3s} \bar{S}(y))\right]^{\frac{1}{p-1}}, \tag{4.3}$$

where E, D, S and $\bar{S}$ are introduced in (2.3), (2.4) and (4.2), which yields the following initial data for equation (3.2):

$$q(y, s_0) = w_0(y, s_0) - \varphi(y, s_0). \tag{4.4}$$

In the following, we exhibit a set of the parameters such that $q(s_0)$ is well defined, $q(s_0) \in V_A(s_0)$ with other smallness and decay properties, inherited from the profile $\varphi(y, s_0)$ given in (2.2). More precisely, this is our statement.

Proposition 4.6 (Initialization). *For any $A \ge 1$, there exists $s_{13}(A) \ge 1$ such that for all $s_0 \ge s_{13}(A)$, there exists a set $\mathcal{D}(A, s_0) \subset [-2,2]^5$ such that for all parameters $d \equiv (d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}$, we have the following two properties:*

(i) $E + \frac{p-1}{\kappa D}(Ae^{-2s_0}S(y) + As^2e^{-3s}\overline{S}(y)) \ge \frac{1}{4}$*, hence $w_0(y,s_0)$ and $q(y,s_0)$ from* (4.3) *and* (4.4)*, respectively, are well defined.*

(ii) *the function $q(s_0) \in V_A(s_0)$ introduced in Definition* 3.1*, estimate* (4.1) *holds true, $|q_{6,2}(s_0)| = |q_{6,4}(s_0)| \le CAe^{-3s_0} < As_0e^{-2s_0}$, $\|q_-(s_0)\|_{L^2_\rho} \le CAe^{-3s_0} < A^2s_0^2e^{-3s_0}$, $\|w_0(s_0)\|_{L^\infty} \le \kappa + Ce^{-\frac{s_0}{3}}$ and $\|\nabla w_0(s_0)\|_{L^\infty} \le Ce^{-\frac{s_0}{6}}$.*
Moreover, the function

$$\begin{aligned} \mathcal{D} &\to [-Ae^{-2s_0}, Ae^{-2s_0}]^4 \times [-As_0^2e^{-3s_0}, As_0^2e^{-3s_0}], \\ d &\mapsto (q_{0,0}(s_0), q_{2,0}(s_0), q_{4,0}(s_0), q_{4,2}(s_0), q_{6,0}(s_0)) \end{aligned} \tag{4.5}$$

is one-to-one.

Proof. See Section 7.5. ∎

5. Control of $w_a(y,s)$ for any a

Starting from this section, our goal is to show that the L^∞ bound on $q(y,s) + \varphi(y,s)$ in Definition 3.1 of $V_A(s)$ never satisfies an equality case. In other words, if $q(s_0)$ is given by (4.4) for some parameters $(d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}$ defined in Proposition 4.6 and $q(y,s)$ satisfies equation (3.2) with $q(s) \in V_A(s)$ given in Definition 3.1, for all $s \in [s_0, s_1]$, for some $s_1 \ge s_0$, with s_0 large enough, then

$$\forall s \in [s_0, s_1],\ \forall y \in \mathbb{R}^2, \quad |q(y,s) + \varphi(y,s)| = |w_0(y,s)| < 2\kappa, \tag{5.1}$$

where we have used definition (3.1) of $q(y,s)$. Using relation (3.14), this is equivalent to showing that

$$\forall a \in \mathbb{R}^2,\ \forall s \in [s_0, s_1], \quad |w_a(0,s)| < 2\kappa. \tag{5.2}$$

We will proceed in several steps in order to control $w_a(y,s)$ for any $a \in \mathbb{R}^2$. We first give a uniform control of the gradient, then we explain our strategy, depending on the region where a belongs to. The next three subsections are dedicated to the proof of the estimate in each region. Finally, we give in the last subsection a concluding statement for the whole section.

5.1. *Control of the gradient*

In this subsection, we use the Liouville theorem we proved in [20, 22] for equation (1.1) in order to show that $\|\nabla q(s)\|_{L^\infty}$ is small, provided that s_0 is large enough. Let us first recall our version of the Liouville theorem, stated for equation (1.12).

Proposition 5.1 (A Liouville theorem for equation (1.12)). *Under condition (1.2), consider a solution $W(y,s)$ of equation (1.12) defined and uniformly bounded for all $(y,s) \in \mathbb{R}^N \times (-\infty, \bar{s})$ for some $\bar{s} \le +\infty$. Then either $W \equiv 0$, or $W \equiv \pm\kappa$ or $W(y,s) = \pm\kappa(1 \pm e^{s-s^*})^{-\frac{1}{p-1}}$ for all $(y,s) \in \mathbb{R}^N \times (-\infty, \bar{s}]$ and for some $s^* \in \mathbb{R}$. In all cases, it holds that $\nabla W \equiv 0$.*

Remark. If $\bar{s} = +\infty$, then the unbounded solution $W(y,s) = \pm\kappa(1 - e^{s-s^*})^{-\frac{1}{p-1}}$ never occurs. If $\bar{s} < +\infty$, then that solution may occur with some s^* satisfying $e^{\bar{s}-s^*} < 1$.

Proof of Proposition 5.1. If $\bar{s} = +\infty$, see [20, p. 143, Theorem 1.4] for the nonnegative case and [22, p. 106, Theorem 1] for the unsigned case.

If $\bar{s} < +\infty$, then the statement follows from a small adaptation of the previous case. See [20, p. 144, Corollary 1.5], where a similar adaptation is carried out. ■

Let us now state our result for the gradient.

Proposition 5.2 (Smallness of the gradient). *For all $A \ge 1$ and $\delta_0 > 0$, there exists $s_{14}(A, \delta_0) \ge 1$ such that for all $s_0 \ge s_{14}(A, \delta_0)$, if $q(s_0)$ is given by (4.4) for some $(d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}$ defined in Proposition 4.6, and $q(s) \in V_A(s)$ for all $s \in [s_0, s_1]$ for some $s_1 \ge s_0$ satisfies equation (3.2), then for all $s \in [s_0, s_1]$,*

$$\|\nabla q(s) + \nabla \varphi(s)\|_{L^\infty} \le \delta_0.$$

Proof. See Section 7.6. ■

5.2. *Strategy for the control of $w_a(y,s)$*

With this estimate, we can make a reduction of our goal (5.2) in the following.

Claim 5.3 (Reduction). *Under the hypotheses of Proposition 5.2, assuming that*

$$\delta_0 \le \frac{\kappa}{4} \quad \text{and} \quad s_0 \ge s_{14}(A, \delta_0), \tag{5.3}$$

we see that estimate (5.2) follows from the following:

$$\forall a \in \mathbb{R}^2,\ \forall s \in [s_0, s_1], \quad \|w_a(s)\|_{L^2_\rho} \le \frac{3}{2}\kappa. \tag{5.4}$$

Proof. Under the hypotheses of Proposition 5.2, assume that (5.4) holds. Noting that $\|\nabla w_a(s)\|_{L^\infty} \le \delta_0$ from Proposition 5.2 together with relations (3.1) and (3.14), we use a Taylor expansion to write

$$|w_a(y,s) - w_a(0,s)| \le |y| \cdot \|\nabla w_a(s)\|_{L^\infty} \le \delta_0 |y|. \tag{5.5}$$

Therefore, since

$$\int \rho(y)dy = 1 \quad \text{and} \quad \int |y|\rho(y)dy = 1 \tag{5.6}$$

by definition (1.18) of ρ, it follows that

$$|w_a(0,s)| \le \int |w_a(y,s)|\rho(y)dy + \delta_0 \le \|w_a(s)\|_{L^2_\rho} + \delta_0 \le \frac{3}{2}\kappa + \frac{\kappa}{4} < 2\kappa,$$

and the claim follows. ∎

In the following subsections, assuming that (5.3) holds, we will prove either (5.1), (5.2) or (5.4), according to the context. Using the sharper gradient estimate at initial time s_0 given in item (ii) of Proposition 4.6, let us remark that at $s = s_0$, $w_a(y, s_0)$ is "flat" in L^2_ρ, in the sense that it is close to some constant independent from space, as we prove in the following.

Lemma 5.4 (Flatness of $w_a(y,s_0)$). *For any $A \ge 1$, $s_0 \ge s_{13}(A)$ and parameters $d = (d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}(A, s_0)$, for any $a \in \mathbb{R}^2$,*

$$\|w_a(\cdot, s_0) - w_0(ae^{\frac{s_0}{2}}, s_0)\|_{L^2_\rho} \le Ce^{-\frac{s_0}{6}}, \tag{5.7}$$

where $s_{13}(A)$ and $\mathcal{D}(A, s_0)$ are defined in Proposition 4.6.

Proof. The proof of the lemma follows by the same argument as in Claim 5.3, in particular, estimate (5.5). ∎

From Lemma 5.4, let us remark that equation (1.12) satisfied by $w_a(y,s)$ has three bounded and nonnegative explicit "flat" solutions: 0, κ, and

$$\psi(s) = \kappa(1+e^s)^{-\frac{1}{p-1}} \tag{5.8}$$

(note that ψ is a heteroclinic orbit connecting κ to 0, and that all its time shifts are also solutions). In fact, it happens that $w_a(0,s_0) = w_0(ae^{\frac{s_0}{2}}, s_0)$ belongs to the interval $[0, \kappa + Ce^{-\frac{s_0}{3}}]$, from (3.14) and Proposition 4.6. In other words, from Lemma 5.4, we are in the vicinity of one of those three explicit solutions, whose stability properties are known! Indeed, if 0 and ψ are stable, as we will show below (see Propositions 5.6 and 5.8), κ has both stable and unstable directions, as one may see from the linearization of equation (1.12) around κ (see below in (7.25)), where the linearized operator appears to be $\mathcal{L}$ introduced in (3.3), whose spectrum is given in (3.4). In particular, the existence of the heteroclinic orbit ψ from (5.8) shows the instability of κ. In other words, if $w_0(ae^{\frac{s_0}{2}}, s_0)$ is close to κ, we need to refine estimate (5.7) in order to better understand the dynamics of $w_a(y,s)$ for $s \ge s_0$.

Accordingly, we will decompose the space into three regions, where $w_a(y,s_0)$ will be in the vicinity of one of the three above-mentioned explicit solutions, leading to three different scenarios for the behavior of $w_a(y,s)$ for $s \ge s_0$. More precisely, given $m < M$, we introduce three regions $\mathcal{R}_i(m, M, s_0)$ for $i = 1, 2, 3$ as follows:

$$\begin{aligned}
\mathcal{R}_1 &= \{a \in \mathbb{R}^2 \mid Me^{-s_0} \le G_0(a)\},\\
\mathcal{R}_2 &= \{a \in \mathbb{R}^2 \mid me^{-s_0} \le G_0(a) \le Me^{-s_0}\},\\
\mathcal{R}_3 &= \{a \in \mathbb{R}^2 \mid G_0(a) \le me^{-s_0}\},
\end{aligned} \tag{5.9}$$

where

$$G_0(a) = \frac{p-1}{\kappa}[a_1^2 a_2^2 + \delta(a_1^6 + a_2^6)]. \tag{5.10}$$

In the following lemma, we will see that $G_0(a)$ is a kind of norm which measures the size of $w_a(0, s_0) = w_0(ae^{-\frac{s_0}{2}}, s_0)$.

Lemma 5.5 (Size of initial data in the three regions). *For any $M \geq 1$, there exists such $C_{15}(M) > 0$ that for any $A \geq 1$, there exists $s_{15}(A, M)$ such that for any $s_0 \geq s_{15}(A, M)$ and $d \in \mathcal{D}(A, s_0)$ defined in Proposition 4.6, for any $m \in (0, 1)$, the following holds:*

- *If $a \in \mathcal{R}_1$, then $0 \leq w_0(ae^{\frac{s_0}{2}}, s_0) \leq \kappa(1+M)^{-\frac{1}{p-1}} + C_{15}(M)e^{-\frac{s_0}{3}}$.*
- *If $a \in \mathcal{R}_2$, then $\kappa(1+M)^{-\frac{1}{p-1}} - C_{15}(M)e^{-\frac{s_0}{3}} \leq w_0(ae^{\frac{s_0}{2}}, s_0) \leq \kappa(1+m)^{-\frac{1}{p-1}} + Ce^{-\frac{s_0}{3}}$.*
- *If $a \in \mathcal{R}_3$, then $\kappa(1+m)^{-\frac{1}{p-1}} - Ce^{-\frac{s_0}{3}} \leq w_0(ae^{\frac{s_0}{2}}, s_0) \leq \kappa + Ce^{-\frac{s_0}{3}}$.*

Proof. See Section 7.7. ∎

Thanks to the two previous lemmas and what we have mentioned concerning the stability of the three explicit solutions of equation (1.12) mentioned in (5.8) and the line before, three scenarios become clear for the proof of estimate (5.4).

Scenario 1: If $a \in \mathcal{R}_1$, provided that M is large enough, we are in the vicinity of the zero solution. Thanks to its stability, $w_a(s)$ will remain small and (5.4) will follow.

Scenario 2: If $a \in \mathcal{R}_2$, we are in the vicinity of $\psi(s)$ from (5.8). Thanks to its stability, $w_a(s)$ will remain close to $\psi(s)$. Since $\psi(s) \leq \kappa$, estimate (5.4) holds.

Scenario 3: If $a \in \mathcal{R}_3$, we are in the vicinity of κ, which has both stable and unstable directions as mentioned earlier. For that reason, the bounds on $w_a(0, s_0) = w_0(ae^{\frac{s_0}{2}}, s_0)$ given in Lemma 5.5 are not enough, and we need a more refined expansion of $w_a(y, s_0)$, followed by an integration of PDE (1.12) satisfied by w_a. This step is the key point of our argument. It is inspired by our techniques in [23] for the control of w_a, where a is a blow-up point located near some given non-isolated blow-up point. In fact, from our careful design of initial data in (4.3), the integration of the PDE will show that for a not "very small" (in a sense that will naturally appear in the proof), $w_a(s)$ will be attracted to the vicinity of the heteroclinic orbit ψ from (5.8), leading us to Scenario 2, where the stability of ψ will imply estimate (5.4). If a is "very small", then, thanks to the gradient estimate of Proposition 5.2, the boundedness of $\|w_a(s)\|_{L^2_\rho}$ will follow from the boundedness of $\|w_0(s)\|_{L^2_\rho}$, a consequence of the fact that $q(s) \in V_A(s)$ given in Definition 3.1 (see (3.20)).

The integration of the PDE when a is not "very small" is in fact the most original part of our paper. In our proof, we make the integration coordinate by coordinate and do not limit ourselves to the linear approximation of the PDE around κ (talking about a *nonlinear* integration is perhaps more appropriate). Indeed, we do take into account the quadratic terms. As we have pointed out in the paragraph after (5.8), the linearized operator around κ has both stable and unstable directions, which makes the integration even more delicate. For details, see Section 5.5 below, in particular Lemma 5.11 and its proof given in Section 7.8.

In the following, we give details for those three scenarios in three subsections.

5.3. *Control of $w_a(y,s)$ in region $\mathcal{R}_1$*

The stability of the zero solution to equation (1.12) (under some L^∞ a priori bound) is crucial for the argument, as we wrote above in Scenario 1. Let us state it in the following.

Proposition 5.6 (Stability of the zero solution to equation (1.12) under an L^∞ a priori bound). *There exist $\varepsilon_0 > 0$ and $M_0 \geq 1$ such that if w solves equation (1.12) with $|w(y,s)| \leq 2\kappa$ for all $(y,s) \in \mathbb{R}^2 \times [0,\sigma_1]$ for some $\sigma_1 \geq 0$, with $\|w(0)\|_{L^2_\rho} \leq \varepsilon_0$ and $\nabla w(0)(1+|y|)^{-k} \in L^\infty$ for some $k \in \mathbb{N}$, then*

$$\forall s \in [0,\sigma_1], \quad \|w(s)\|_{L^2_\rho} \leq M_0 \|w(0)\|_{L^2_\rho} e^{-\frac{s}{p-1}}.$$

Proof. The proof is somehow classical, apart from a delay estimate to control the nonlinear estimate we have already presented in Section 3.2. In order to focus only on the main arguments, we postpone the proof to Appendix B. ■

Fixing $M \geq 1$ such that

$$\kappa(1+M)^{-\frac{1}{p-1}} \leq \max\left(\frac{\varepsilon_0}{2}, \frac{\kappa}{2M_0}\right), \tag{5.11}$$

where M_0 and ε_0 are given in Proposition 5.6, then taking s_0 large enough, we see from Proposition 4.6 that $\nabla w(s_0) \in L^\infty$, and from Lemmas 5.5 and 5.4 that the smallness condition required in Proposition 5.6 holds in region $\mathcal{R}_1$, leading to the trapping of w_a near 0, proving bound (5.4). More precisely, this is our statement.

Corollary 5.7 (Exponential decay of $w_a(s)$ in region $\mathcal{R}_1$). *For all $A \geq 1$, there exists $s_{16}(A) \geq 1$ such that if $s_0 \geq s_{16}(A)$, $d \in \mathcal{D}(A,s_0)$ and $a \in \mathcal{R}_1$ defined in (5.9), then $\nabla w(s_0) \in L^\infty$ and $\|w_a(s_0)\|_{L^2_\rho} \leq 2\kappa(1+M)^{-\frac{1}{p-1}} \leq \varepsilon_0$ introduced in Proposition 5.6. If in addition we have $|w_a(y,s)| \leq 2\kappa$, for all $(y,s) \in \mathbb{R}^2 \times [s_0,s_2]$, for some $s_2 \geq s_0$, then*

$$\forall s \in [s_0,s_2], \quad \|w_a(s)\|_{L^2_\rho} \leq M_0 \|w_a(s_0)\|_{L^2_\rho} e^{-\frac{s-s_0}{p-1}} \leq 2\kappa(1+M)^{-\frac{1}{p-1}} M_0 e^{-\frac{s-s_0}{p-1}} \leq \kappa,$$

where M_0 is also introduced in Proposition 5.6. In particular, (5.4) *holds.*

5.4. *Control of $w_a(y,s)$ in region $\mathcal{R}_2$*

As we explained above in Scenario 2, the stability of the solution ψ from (5.8) is the key argument. Let us first state that stability result.

Proposition 5.8 (Stability of the heteroclinic orbit for equation (1.12) under an L^∞ a priori bound). *There exists $M_1 \ge 1$ such that if w solves equation* (1.12) *with $|w(y,s)| \le 2\kappa$ for all $(y,s) \in \mathbb{R}^2 \times [0,\sigma_1]$ for some $\sigma_1 \ge 0$, and*

$$\nabla w(0)(1+|y|)^{-k} \in L^\infty, \quad \|w(0) - \psi(\sigma^*)\|_{L^2_\rho} \le \frac{|\psi'(\sigma^*)|}{M_1} \tag{5.12}$$

for some $k \in \mathbb{N}$ and $\sigma^* \in \mathbb{R}$, where ψ is defined in (5.8)*, then*

$$\forall s \in [0,\sigma_1], \quad \|w(s) - \psi(s+\sigma^*)\|_{L^2_\rho} \le M_1 \|w(0) - \psi(\sigma^*)\|_{L^2_\rho} \frac{|\psi'(s+\sigma^*)|}{|\psi'(\sigma^*)|}. \tag{5.13}$$

Remark. By definition (5.8) of ψ, we have exponential decay in (5.13). Moreover, since κ and $\psi(s)$ are both solutions to equation (1.12), κ should never satisfy condition (5.12). This is clear, except when $\sigma^* \to -\infty$, since in that case $\psi(\sigma^*) \to \kappa$. More precisely, we see from (5.8) that $\kappa - \psi(\sigma^*) \sim \frac{\kappa e^{\sigma^*}}{p-1} \sim |\psi'(\sigma^*)|$, which shows that (5.12) is indeed sharp, up to a multiplying (small) factor, $\frac{1}{M_1}$.

Proof of Proposition 5.8. The proof is much more involved than the proof of Proposition 5.6, since by definition (5.8), $\psi(\sigma^*)$ may be close to κ, the unstable equilibrium of (1.12). As for the previous proposition, we postpone the proof to Appendix B. ∎

In the following corollary, using Proposition 4.6, Lemmas 5.4 and 5.5, we show that $w_a(s)$ is trapped near the heteroclinic orbit ψ from (5.8) whenever a is in region $\mathcal{R}_2$ from (5.9). More precisely, this is our statement.

Corollary 5.9 (Trapping of $w_a(s)$ near ψ in region $\mathcal{R}_2$). *There exists $\bar\sigma \in \mathbb{R}$ such that for all $m \in (0,1)$, there exists $\underline{\sigma}(m) \le \bar\sigma$ such that for all $A \ge 1$, there exists $s_{17}(A,m) \ge 1$ such that for all $s_0 \ge s_{17}(A,m)$, $d \in \mathcal{D}(A,s_0)$ and $a \in \mathcal{R}_2$ defined in* (5.9) *(with M defined in* (5.11)*), $\nabla w_a(s_0) \in L^\infty$ and $\|w_a(s_0) - \psi(\sigma^*)\|_{L^2_\rho} \le Ce^{-\frac{s_0}{6}} \le \frac{|\psi'(\sigma^*)|}{M_1}$ for some $\sigma^* \in [\underline{\sigma}(m), \bar\sigma]$, where M_1 was introduced in Proposition* 5.8*. If in addition we have $|w_a(y,s)| \le 2\kappa$, for all $(y,s) \in \mathbb{R}^2 \times [s_0,s_2]$, for some $s_2 \ge s_0$, then*

$$\forall s \in [s_0,s_2], \quad \|w_a(s)\|_{L^2_\rho} \le \psi(s+\sigma^*-s_0) + \frac{CM_1}{C(m)} e^{-\frac{s_0}{6}} \le \kappa + \frac{\kappa}{4} \le \frac{3}{2}\kappa.$$

In particular, (5.4) *holds.*

Proof. The proof is omitted, since it is a direct consequence of Proposition 5.8, thanks to Proposition 4.6, Lemmas 5.4 and 5.5. ∎

5.5. Control of $w_a(y,s)$ in region $\mathcal{R}_3$

Given an initial time s_0, some $m \in (0,1)$ and $a \in \mathcal{R}_3$ defined in (5.9), we write a as follows:

$$a = (Ke^{-\frac{s_0}{2}}, Le^{-\frac{s_0}{2}}) \tag{5.14}$$

for some real numbers K and L.

Since $w_0(\cdot, s_0)$ is symmetric with respect to the axes and the bisectrices (see (4.3)), so is $w_0(\cdot, s)$ for any later time s. For that reason, we only consider the case where

$$0 \leq K \leq L.$$

Since

$$w_a(y, s_0) = w_0(y + ae^{\frac{s_0}{2}}, s_0) = w_0(y_1 + K, y_2 + L, s_0) \tag{5.15}$$

from (3.14), we will use the explicit expression (4.3) of w_0 in order to estimate the components of $w_a(s_0)$ in L^2_ρ. Taking this as initial data, we will integrate equation (1.12) (which is satisfied by w_a) in order to estimate $w_a(s)$ for later times $s \geq s_0$.

It happens that the outcome of the integration depends on the size of a, which can be measured in terms of the position of $K + L$ and A. For that reason, we distinguish two cases in the following.

5.5.1. Case where $K + L \geq A$. From decomposition (5.14) of $a = (a_1, a_2)$, this is the case of "large" a, where $a_1 + a_2 \geq Ae^{-\frac{s_0}{2}}$. Let us first estimate $w_a(s_0)$. In order to be consistent with definition (2.2) of our profile φ and the decomposition in regions we suggest in (5.9), we will give the expansion of $w_a(s_0)$ in L^2_ρ, uniformly with respect to the small variable

$$\iota = e^{-s_0} K^2 L^2 + \delta e^{-2s_0}[K^6 + L^6] \tag{5.16}$$

and the large parameter A. This is our statement.

Lemma 5.10 (Initial value of $w_a(y, s)$ for large a). *For any $A \geq 1$, there exists $s_{18}(A) \geq 1$ such that for any $s_0 \geq s_{18}(A)$ and any parameters $(d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}$ defined in Proposition 4.6, if $w_0(y, s_0)$ is given by (4.3), then for any $m \in (0, 1)$ and $a \in \mathcal{R}_3$ defined in (5.9) with a decomposed as in (5.14) for some $L \geq K \geq 0$, with $K + L \geq A$, the following expansion holds in $L^r_\rho(\mathbb{R}^2)$ for any $r \geq 2$:*

$$\begin{aligned} w_a(y, s_0) = \kappa - \iota - e^{-s_0}\big\{&2KL^2 h_1 h_0 + 2K^2 L h_1 h_0 + L^2 h_2 h_0 + 4KL h_1 h_1 + K^2 h_0 h_2 \\ &+ 2L h_2 h_1 + 2K h_1 h_2 + h_2 h_2\big\} + O\Big(\frac{\iota}{A}\Big) + O(\iota^2), \end{aligned}$$

where ι is defined in (5.16).

Proof. See Section 7.8. ∎

Noting that

$$\iota \leq \frac{m\kappa}{p-1} \leq \frac{\kappa}{p-1} \tag{5.17}$$

from (5.16) and (5.9), it follows from this lemma that $\|w_a(s_0) - \kappa\|_{L^2_\rho} \leq C\iota + CJ$, where

$$\begin{aligned} J &\equiv e^{-s_0}(K^2 + L^2) \leq 2e^{-s_0}(K^6 + L^6)^{\frac{1}{3}} \leq 2e^{-s_0}\Big(\frac{e^{2s_0}\iota}{\delta}\Big)^{\frac{1}{3}} \\ &= 2e^{-\frac{s_0}{3}}\Big(\frac{\iota}{\delta}\Big)^{\frac{1}{3}} \leq 2\iota^{\frac{1}{3}} \end{aligned} \tag{5.18}$$

from (5.16) and the fact that $s_0 \geq 0$ and $\delta \geq 1$ (see the beginning of Section 3). Using again (5.17), it follows that $w_a - \kappa$ is small at $s = s_0$, whenever m is small. It is natural then to make a linear approximation for equation (1.12) around κ in order to obtain the expansion of $w_a(y,s)$ for later times, as long as $w_a(s) - \kappa$ remains small. In fact, we will see that the projection on $h_0 h_0 = 1$ will dominate in $w_a(y,s)$. For that reason, given some small η^* such that

$$\eta^* \geq \frac{m\kappa}{p-1}, \tag{5.19}$$

where $m \in (0,1)$ is the constant appearing in definition (5.9) of $\mathcal{R}_3$, our integration will be valid only on the interval $[s_0, s^*]$, where $s^* = s^*(s_0, a, \eta^*)$ is defined by

$$e^{s^*-s_0}\iota = \eta^* \tag{5.20}$$

and ι is given in (5.16). Note that s^* is well defined since $L + K \geq A > 0$, hence $\iota > 0$. Note also that $s^* \geq s_0$ thanks to condition (5.19), as we will see in item (i) of Lemma 5.11 below. More precisely, this is our statement.

Lemma 5.11 (Decreasing from κ to $\kappa - \eta^*$ for "large" a under some a priori L^∞ bound). *There exist $M_{19} > 0$, $A_{19} \geq 1$ and $\eta_{19} > 0$ such that for all $A \geq A_{19}$, there exists $s_{19}(A)$ such that for any $s_0 \geq s_{19}(A)$ and any parameters $(d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}$ defined in Proposition 4.6, if $w_0(y, s_0)$ is given by (4.3), then for any $\eta^* \in (0, \eta_{19}]$ and $m \in (0, \min(1, \frac{\eta^*(p-1)}{\kappa}))$, for any $a \in \mathcal{R}_3$ defined in (5.9), where a is given by (5.14) for some $L \geq K \geq 0$, with $K + L \geq A$, the following holds:*

(i) *$s^* \geq s_0$, where s^* is introduced in (5.20).*

(ii) *If we assume in addition that*

$$\|w_a(s)\|_{L^\infty} \leq 2\kappa$$

for all $s \in [s_0, s_1]$ for some $s_1 \geq s_0$, then for all $s \in [s_0, \min(s^*, s_1)]$,

$$\|w_a(\cdot, s) - (\kappa - e^{s-s_0}\iota)\|_{L^2_\rho} \leq M_{19}(\eta^* + A^{-1})e^{s-s_0}\iota + M_{19}e^{-\frac{s_0}{3}}.$$

Proof. See Section 7.8. ∎

With this result, we are able to prove estimate (5.4).

Corollary 5.12 (Proof of the L^2_ρ bound when $a \in \mathcal{R}_3$ is "large" under some L^∞ a priori bound). *There exist $A_{20} > 0$, $\eta_{20} > 0$ and $s_{20}(\eta^*) \geq 1$, such that under the hypotheses of Lemma 5.11, if in addition $A \geq A_{20}$, $\eta^* \leq \eta_{20}$ and $s_0 \geq s_{20}(\eta^*)$, then for all $s \in [s_0, s_1]$,*

$$\|w_a(s)\|_{L^2_\rho} \leq \frac{3}{2}\kappa.$$

Proof. Using Lemma 5.11, we further assume that $s_0 \geq s_{14}(A, 1)$ defined in Proposition 5.2, so we can apply that proposition. Consider then $s \in [s_0, s_1]$. We distinguish two cases.

Case 1: $s \le s^*$. We write from Lemma 5.11 and (5.6), together with definitions (5.16) and (5.20) of ι and s^*,

$$\begin{aligned}\|w_a(s)\|_{L^2_\rho} &\le \kappa + e^{s^*-s_0}\iota + M_{19}(\eta^* + A^{-1})e^{s^*-s_0}\iota + M_{19}e^{-\frac{s_0}{3}}\\ &\le \kappa + \eta^* + M_{19}(\eta^* + A^{-1})\eta^* + M_{19}e^{-\frac{s_0}{3}}.\end{aligned}$$

Since $A \ge 1$, by taking η^* small enough and s_0 large enough, we get the result.

Case 2: $s \ge s^*$. In this case, we have $s_0 \le s^* \le s \le s_1$. We will show that starting at time s^*, w will be trapped near the heteroclinic orbit ψ from (5.8), thanks to Proposition 5.8. In particular, its L^2_ρ norm will remain bounded by $\frac{3}{2}\kappa$. More precisely, from item (ii) of Lemma 5.11 and (5.20), we see that

$$\|w_a(s^*) - (\kappa - \eta^*)\|_{L^2_\rho} \le M_{19}(\eta^* + A^{-1})\eta^* + M_{19}e^{-\frac{s_0}{3}}.$$

Assuming that $\eta^* < \kappa$, we may introduce $\sigma^* \in \mathbb{R}$ such that $\psi(\sigma^*) = \kappa - \eta^*$, where ψ is defined in (5.8). Noting that

$$|\psi'(\sigma^*)| \sim \frac{\kappa e^{\sigma^*}}{p-1} \sim \kappa - \psi(\sigma^*) = \eta^* \quad \text{as } \eta^* \to 0, \tag{5.21}$$

we see that taking η^* small enough, then A and s_0 large enough, we have

$$\|w_a(s^*) - \psi(\sigma^*)\|_{L^2_\rho} \le \frac{|\psi'(\sigma^*)|}{M_1[1 + \frac{2}{\kappa}\|\psi'\|_{L^\infty}]},$$

where M_1 is introduced in Proposition 5.8. Since $\nabla w_a(s^*) \in L^\infty$, thanks to Proposition 5.2, together with definition (3.1) of q and transformation (3.14), Proposition 5.8 applies, and we see that at time s, we have

$$\begin{aligned}\|w_a(s) - \psi(s + \sigma^* - s^*)\|_{L^2_\rho} &\le M_1\|w_a(s^*) - \psi(\sigma^*)\|_{L^2_\rho}\frac{|\psi'(s+\sigma^*)|}{|\psi'(\sigma^*)|}\\ &\le \frac{|\psi'(s + \sigma^* - s^*)|}{1 + \frac{2}{\kappa}\|\psi'\|_{L^\infty}} \le \frac{\kappa}{2}.\end{aligned}$$

Since $\psi \le \kappa$ by definition (5.8), using (5.6) we obtain

$$\|w_a(s)\|_{L^2_\rho} \le \psi(s + \sigma^* - s^*) + \frac{\kappa}{2} \le \frac{3}{2}\kappa.$$

This concludes the proof of Corollary 5.12. ∎

5.5.2. *Case where* $K + L \le A$. Given $m \in (0, 1)$ and $a \in \mathcal{R}_3$ from (5.9), we aim in this section to handle the case where a is "small", namely when it belongs to the triangle $\mathcal{T}_0$ defined by

$$\mathcal{T}_0 = \{(Ke^{-\frac{s_0}{2}}, Le^{-\frac{s_0}{2}}) \mid 0 \le K \le L \text{ and } K + L \le A\}.$$

Let us recall from the beginning of Section 5 that $q(s) \in V_A(s)$ given in Definition 3.1, for all $s \in [s_0, s_1]$ for some $s_1 \geq s_0$, hence,

$$\forall s \in [s_0, s_1], \quad \|w_0(s)\|_{L^\infty} \leq 2\kappa.$$

Introducing the segment

$$\mathcal{G}_\sigma = \{(K'e^{-\frac{\sigma}{2}}, L'e^{-\frac{\sigma}{2}}) \mid 0 \leq K' \leq L' \text{ and } K' + L' = A\} \tag{5.22}$$

and the triangle

$$\mathcal{T}_1 = \{(K'e^{-\frac{s_1}{2}}, L'e^{-\frac{s_1}{2}}) \mid 0 \leq K' \leq L' \text{ and } K' + L' \leq A\},$$

we see that

$$\mathcal{T}_0 = \bigcup_{s_0 \leq \sigma < s_1} \mathcal{G}_\sigma \cup \mathcal{T}_1.$$

We then proceed in two steps:

- In Step 1, we handle the case of "small" s, namely when $a \in \mathcal{G}_\sigma$ with $s_0 \leq s \leq \sigma \leq s_1$, and also the case where $a \in \mathcal{T}_1$ with $s_0 \leq s \leq s_1$.
- In Step 2, we handle the case of "large" s, namely when $a \in \mathcal{G}_\sigma$ with $s_0 \leq \sigma \leq s \leq s_1$.

Step 1: *Case of "small" a and "small" s.* Consider $a \in \mathcal{G}_\sigma$ with $s_0 \leq s \leq \sigma < s_1$, or $a \in \mathcal{T}_1$ with $s_0 \leq s \leq s_1$. The conclusion will follow from the L^2_ρ estimate on w_0 together with the gradient estimate of Proposition 5.2. Indeed, using a Taylor expansion together with (5.15) and (5.6), we write

$$w_a(0,s) = w_0(ae^{\frac{s}{2}}, s) = w_0(0,s) + O(ae^{\frac{s}{2}}\|\nabla w_0(s)\|_{L^\infty}),$$
$$\int w_0(z,s)\rho(z)dz = w_0(0,s) + O(\|\nabla w_0(s)\|_{L^\infty})$$

on the one hand. On the other hand, since $w_0 = \varphi + q$ by definition (3.1), recalling that $q(s) \in V_A(s)$, we write by definitions (2.2) and (3.17) of φ and $q_{0,0}(s)$, together with Definition 3.1 of $V_A(s)$,

$$\begin{aligned} \int w_0(z,s)\rho(z)dz &= \int \varphi(z,s)\rho(z)dz + \int q(z,s)\rho(z)dz \\ &= \int \varphi(z,s)\rho(z)dz + q_{0,0}(s) \\ &= \kappa + O(e^{-s}) + O(Ae^{-2s}). \end{aligned}$$

Introducing (K', L') such that

$$a = (K'e^{-\frac{\sigma}{2}}, L'e^{-\frac{\sigma}{2}}) \tag{5.23}$$

with $\sigma = s_1$ if $a \in \mathcal{T}_1$, we see that

$$0 \leq K' \leq L' \quad \text{and} \quad K' + L' \leq A.$$

Recalling that $s \le \sigma$, we write

$$|ae^{\frac{s}{2}}| = \sqrt{K'^2 + L'^2} e^{\frac{s-\sigma}{2}} \le \sqrt{2}A.$$

Taking $A \ge 1$ and recalling that $\|\nabla w_0(s)\|_{L^\infty} \le \delta_0$ from Proposition 5.2 provided that $s_0 \ge s_{14}(A, \delta_0)$, we write from the previous estimates that

$$|w_a(0,s)| \le \kappa + C\delta_0 + CA\delta_0 + Ce^{-s} + CAe^{-2s} \le \frac{3}{2}\kappa,$$

whenever $\delta_0 \le \delta_{21}(A)$ and $s_0 \ge s_{21}(A, \delta_0)$ for some $s_{21}(A, \delta_0) \ge 1$ and $\delta_{21}(A) > 0$. In particular, estimate (5.2) holds.

Step 2: *Case of "small" a and "large" s.* Now, we consider $a \in \mathcal{G}_\sigma$ with $s_0 \le \sigma \le s \le s_1$. As we will shortly see, the conclusion follows here from the case $K + L \ge A$ treated in Section 5.5.1, if one replaces there K, L and s_0 by K', L' and σ, respectively. Indeed, by definition (5.22) of $\mathcal{G}_\sigma$, we have

$$K' + L' = A, \tag{5.24}$$

where K' and L' are defined in (5.23). Proceeding as in Section 5.5.1, we first start by expanding $w_a(y, \sigma)$ as in Lemma 5.10.

Lemma 5.13 (Expansion of $w_a(y,\sigma)$). *For any $A \ge 1$, there exists $s_{22}(A) \ge 1$ such that for any $s_0 \ge s_{22}(A)$, the following holds: Assume that $q(s) \in V_A(s)$ satisfies equation (3.2) for any $s \in [s_0, \sigma]$ for some $\sigma \ge s_0$, such that (4.1) holds and $\nabla q(s_0) \in L^\infty(\mathbb{R}^2)$. Assume in addition that*

$$a = (K', L')e^{-\frac{\sigma}{2}}$$

such that (5.24) holds. Then

$$\begin{aligned}
w_a(y,\sigma) = \kappa - \iota' + O\Big(\frac{\iota'}{A}\Big) + O(\iota'^2) \\
- e^{-\sigma}\{2K'L'^2 h_1 h_0 + 2K'^2 L' h_1 h_0 + L'^2 h_2 h_0 + 4K'L' h_1 h_1 \\
+ K'^2 h_0 h_2 + 2L' h_2 h_1 + 2K' h_1 h_2 + h_2 h_2\} \\
+ q_{6,2}(\sigma)\{L'^4 h_2 h_0 + K'^4 h_0 h_2 + 4L'^3 h_2 h_1 + 4K'^3 h_1 h_2 \\
+ 6(K'^2 + L'^2) h_2 h_2 + 4K' h_3 h_2 + 4L' h_2 h_3 \\
+ h_4 h_2 + h_2 h_4\}
\end{aligned} \tag{5.25}$$

in L^r_ρ for any $r \ge 2$, with $|q_{6,2}(\sigma)| \le A\sigma e^{-2\sigma}$, where

$$\iota' = e^{-\sigma} K'^2 L'^2 + \delta e^{-2\sigma}[K'^6 + L'^6]. \tag{5.26}$$

Proof. See Section 7.8. ■

Now, arguing as for Lemma 5.11, we see $w_a(\sigma)$ as initial data, then integrate equation (1.12) to get an expansion of $w_a(s)$ for later times.

Lemma 5.14 (Decreasing $w_a(\sigma)$ from κ to $\kappa - \eta^*$). *There exist $M_{23} > 0$, $A_{23} \geq 1$ and $\eta_{23} > 0$ such that for all $A \geq A_{23}$ and $\eta^* \in (0, \eta_{23}]$, there exists $s_{23}(A, \eta^*)$ such that for any $\sigma \geq s_{23}$ and $s_1 \geq \sigma$, if $q(s) \in V_A(s)$ given in Definition 3.1 satisfies equation (3.2) on $[\sigma, s_1]$ and $\nabla q(\sigma) \in L^\infty$, if $a = (K'e^{-\frac{\sigma}{2}}, L'e^{-\frac{\sigma}{2}})$ with $K' + L' = A$, then*

(i) *$s^* \geq \sigma$, where s^* is such that $e^{s^*-\sigma}\iota' = \eta^*$, where ι' is defined in (5.26).*

(ii) *For all $s \in [\sigma, \min(s^*, s_1)]$,*

$$\|w_a(s) - (\kappa - e^{s-\sigma}\iota')\|_{L^2_\rho} \leq M_{23}(\eta^* + A^{-1})e^{s-\sigma}\iota' + M_{23}e^{-\frac{\sigma}{3}}.$$

Proof. The proof follows from a straightforward adaptation of the proof of Lemma 5.11, given in Section 7.8. For that reason, the proof is omitted. ■

As in the case $K + L \geq A$, we derive from this result the following corollary, where we prove estimate (5.4).

Corollary 5.15 (Proof of the L^2_ρ bound when $a \in \mathcal{R}_3$ is "small" and s is "large"). *There exist $A_{24} > 0$, $\eta_{24} > 0$ and $s_{24}(\eta^*) \geq 1$, such that under the hypotheses of Lemma 5.14, if in addition $A \geq A_{24}$, $\eta^* \leq \eta_{24}$ and $s_0 \geq s_{24}(\eta^*)$, then for all $s \in [\sigma, s_1]$, $\|w_a(s)\|_{L^2_\rho} \leq \frac{3}{2}\kappa$.*

Proof. The proof is omitted since this statement follows from Lemma 5.14 exactly in the same way Corollary 5.12 follows from Lemma 5.11. ■

5.5.3. Concluding statement for the control of $w_a(s)$ when $a \in \mathcal{R}_3$. Combining the previous statements given when $a \in \mathcal{R}_3$ defined in (5.9), we obtain the following lemma.

Lemma 5.16 (Control of $w_a(s)$ when $a \in \mathcal{R}_3$ if $w(s_0)$ is given by (4.3) and $q(s) \in V_A(s)$). *There exist $A_{25} \geq 1$ and $m_{25} \in (0, 1)$ such that for all $A \geq A_{25}$, there exists $\delta_{25}(A) > 0$ such that for all $\delta_0 \in (0, \delta_{25}(A))$, there exists $s_{25}(A, \delta_0) \geq 1$ such that for all $s_0 \geq s_{25}(A, \delta_0)$, $d \in \mathcal{D}(A, s_0)$ defined in Proposition 4.6 and $s_1 \geq s_0$, the following holds: Assume that w is the solution of equation (1.12) with initial data $w_0(y, s_0)$ defined in (4.3), such that for all $s \in [s_0, s_1]$, $q(s) \in V_A(s)$ given in Definition 3.1, where $q(s)$ is defined in (3.1). Then for all $s \in [s_0, s_1]$,*

(i) $\|\nabla w(s)\|_{L^\infty} \leq \delta_0$.

(ii) *For all $m \in (0, m_{25}]$ and $a \in \mathcal{R}_3$ defined in (5.9), $|w_a(0, s)| < 2\kappa$.*

Proof. The proof is omitted since it is straightforward from Claim 5.3, Corollary 5.12, Step 1 given in Section 5.5.2 and Corollary 5.15. ■

5.6. Concluding statement for the control of $w_a(s)$ for any $a \in \mathbb{R}^2$

Fixing M as in (5.11) and taking $m = m_{25}$ as introduced in Lemma 5.16, we see that the three regions in (3.3) are properly defined. Then, combining the previous statements given for the different regions (namely, Corollaries 5.7 and 5.9, together with Lemma 5.16), we derive the following.

Proposition 5.17 (Control of $\|w(s)\|_{L^\infty}$ if $w(s_0)$ is given by (4.3) and $q(s) \in V_A(s)$). *There exists $A_{26} \geq 1$ such that for all $A \geq A_{26}$, there exists $s_{26}(A) \geq 1$ such that for all $s_0 \geq s_{26}(A)$, $d \in \mathcal{D}(A, s_0)$ defined in Proposition 4.6 and $s_1 \geq s_0$, the following holds: Assume that w_0 is the solution of equation (1.12) with initial data $w_0(y, s_0)$ defined in (4.3), such that for all $s \in [s_0, s_1]$, $q(s) \in V_A(s)$ given in Definition 3.1, where $q(s)$ is defined in (3.1). Then*

(i) *For all $s \in [s_0, s_1]$, $\|w_0(s)\|_{L^\infty} < 2\kappa$.*

(ii) *For all $a \in \mathbb{R}^2$ such that*

$$|a_1| + |a_2| \geq A e^{-\frac{s_1}{2}}, \tag{5.27}$$

there exist $\bar{s}(a) \geq s_0$ and $\bar{M}(a) \geq 0$ such that if $\bar{s}(a) \geq s_1$, then for all $s \in [\bar{s}(a), s_1]$, $\|w_a(s)\|_{L^2_\rho} \leq \bar{M}(a) e^{-\frac{s}{p-1}}$.

Proof. (i) The proof is omitted since it is straightforward from the above-mentioned statements.

(ii) Take $a \in \mathbb{R}^2$ such that (5.27) holds. The conclusion follows according to the position of a in the three regions $\mathcal{R}_i$ defined in (5.9), with the constants M and m fixed in the beginning of the current subsection.

If $a \in \mathcal{R}_1$, then the conclusion follows from Corollary 5.7.

If $a \in \mathcal{R}_2$, then we see from Corollary 5.9 and its proof that Proposition 5.8 applies. In particular, $w_a(s)$ is trapped near the heteroclinic orbit ψ from (5.8), and the exponential bound follows from (5.13) and (5.8).

Finally, if $a \in \mathcal{R}_3$, then, following Section 5.5, we write

$$a = (K, L) e^{-\frac{s_0}{2}} \tag{5.28}$$

as in (5.14) and reduce to the case where $0 \leq K \leq L$ thanks to the symmetries of initial data $w_0(\cdot, s_0)$ defined in (4.3).

If $K + L \geq A$, then we see from Corollary 5.12 and its proof that $w_a(s)$ is trapped near the heteroclinic orbit $\psi(s)$ from (5.8), and the exponential bound follows from estimate (5.13) given in Proposition 5.8.

If $K + L \leq A$, using (5.27) and (5.28), we may write $a = (K', L') e^{-\frac{\sigma}{2}}$ for some $\sigma \in [s_0, s_1]$ with $K' + L' = A$, as we did in (5.23) and (5.24). Using Corollary 5.15, we see that $w_a(s)$ is trapped near the heteroclinic orbit $\psi(s)$, and the conclusion follows from estimate (5.13) again. This concludes the proof of Proposition 5.17. ■

6. Proof of the main result

In this section, we collect all the arguments to derive Theorem 1. We proceed in three subsections:

- We first prove the existence of a solution of equation (3.2) trapped in the set $V_A(s)$ given in Definition 3.1.

- Then, we derive a better description on larger sets, yielding the so-called *intermediate profile*.
- Finally, we prove that the origin is the only blow-up point and derive the *final profile* given in (1.19).

6.1. *Existence of a solution to equation* (3.2) *trapped in the set* $V_A(s)$

This is our statement in this subsection.

Proposition 6.1 (Existence of a solution $q(y,s)$ in the shrinking set $V_A(s)$).

(i) *There exist* $C_{27} > 0$, $A \geq 1$, $s_0 \geq 1$ *and a parameter* $d = (d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}(A, s_0)$ *defined in Proposition* 4.6 *such that the solution* $w_0(y,s)$ *of equation* (1.12) *with initial data at* $s = s_0$ *given by* (4.3) *satisfies* $q(s) \in V_A(s)$ *and* $\|\nabla q(s)\|_{L^\infty} \leq C_{27}$ *for any* $s \geq s_0$, *where* q *is defined in* (3.1) *and the set* $V_A(s)$ *is given in Definition* 3.1.

(ii) *If* $u(x,t) = (T-t)^{-\frac{1}{p-1}} w_0(\frac{x}{\sqrt{T-t}}, -\log(T-t))$ *with* $T = e^{-s_0}$, *then* u *is a solution of equation* (1.1) *which blows up only at the origin.*

Remark. Since initial data in (4.3) is symmetric with respect to the axes and the bisectrices, the same holds for $w_0(y,s)$.

Before proving Proposition 6.1, let us recall the following consequence of the Liouville theorem stated in Proposition 5.1.

Corollary 6.2 (Behavior of the gradient). *In addition to Proposition* 6.1, *it holds that* $\|\nabla w_0(s)\|_{L^\infty} \to 0$ *as* $s \to \infty$.

Proof. This is a classical consequence of the Liouville theorem stated in Proposition 5.1, which follows similarly to Proposition 5.2. For a statement and a proof, see Merle and Zaag [20, p. 141, Theorem 1.1]. ∎

Let us now prove Proposition 6.1.

Proof of Proposition 6.1. (i) This is a classical combination of our previous analysis, namely Lemma 2.2 and Propositions 4.4, 4.6, 5.2 and 5.17. The argument relies on a reduction of the problem in finite dimensions, then the proof of the reduced problem thanks to the degree theory. We give the argument only for the sake of completeness. Expert readers may skip this proof.

In order to apply the above-mentioned statements, let us fix $A = \max(A_{12}, 1, A_{26})$ and $s_0 = \max(s_{10}(\delta), s_{12}(A), s_{13}(A), s_{14}(A,1), s_{26}(A))$, where δ is the constant appearing in the definition of the profile $\varphi(y,s)$ and fixed at the beginning of Section 3.

We proceed by contradiction and assume that for all $d = (d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}(A, s_0)$ defined in Proposition 4.6, $q(d,y,s)$ does not remain in the set $V_A(s)$ for all $s \geq s_0$, where $q(d,y,s) = w_0(d,y,s) - \varphi(y,s)$, $\varphi(y,s)$ is defined in (2.2), where the constants γ and δ are fixed in Lemma 2.2 and the beginning of Section 3, and $w_0(d,y,s)$ is the solution of equation (1.12) with initial data at $s = s_0$ given by (4.3).

Since the function given in (4.5) is one-to-one from item (ii) in Proposition 4.6, we may take $\bar{d} = (\bar{d}_{0,0}, \bar{d}_{2,0}, \bar{d}_{4,0}, \bar{d}_{4,2}, \bar{d}_{6,0}) \in [-1,1]^5$ as a new parameter, where

$$\bar{d}_{i,j} = \frac{e^{2s_0}}{A} q_{i,j}(d, s_0) \quad \text{if } (i,j) \in I_0, \quad \bar{d}_{6,0} = \frac{e^{3s_0}}{A s_0^2} q_{6,0}(d, s_0) \tag{6.1}$$

and $I_0 \equiv \{(0,0),(2,0),(4,0),(4,2)\}$. Accordingly, we introduce the notation

$$\bar{q}(\bar{d}, y, s) = q(d, y, s), \quad \bar{w}_0(\bar{d}, y, s) = w_0(d, y, s)$$

and so on. Since $q(d, y, s_0) \in V_A(s_0)$ by Proposition 4.6, from continuity, we may introduce $s^*(d) = \bar{s}^*(\bar{d})$ as the minimal time such that $q(d, y, s) \in V_A(s)$ for all $s \in [s_0, s^*]$ and an equality case occurs at time $s = s^*$ in one of the $\le$ defining $V_A(s^*)$ in Definition 3.1.

According to Proposition 5.17, no equality case occurs for $\|w_0(d, s^*)\|_{L^\infty}$. We claim that no equality case occurs for $|q_{6,2}(d,s^*)| = |q_{6,4}(d,s^*)|$ neither $\|q_-(d,s_0)\|_{L^2_\rho}$. Indeed, if $s^*(d) = s_0$, this follows from Proposition 4.6. If $s^*(d) > s_0$, then this follows from Corollary 4.5, which can be applied thanks to Proposition 4.6. In other words, it holds that

$$\text{either} \quad |q_{i,j}(d, s^*)| = A e^{-2s^*} \text{ for some } (i,j) \in I_0 \quad \text{or} \quad |q_{6,0}(d, s^*)| = A {s^*}^2 e^{-3s^*}.$$

This way, the following rescaled flow is well defined:

$$\Phi\colon [-1,1]^5 \to \partial[-1,1]^5, \quad \bar{d} \mapsto \frac{e^{2s^*}}{A}\Big(q_{0,0}, q_{2,0}, q_{4,0}, q_{4,2}, \frac{e^{s^*}}{{s^*}^2} q_{6,0}\Big)(d, s^*(d)).$$

We claim that

(a) Φ is continuous;

(b) $\Phi|_{[-1,1]^5} = \mathrm{Id}$, which will imply a contradiction, by the degree theory and finish the proof.

Let us prove (a) and (b).

(a) Using items (i) and (ii) in Proposition 4.4, we see that the flow of the five components $|q_{i,j}|$ with $(i,j) \in I_0 \cup \{(6,0)\}$ is transverse outgoing on the corresponding boundary (i.e., Ae^{-2s} if $(i,j) \in I_0$ or $As^2 e^{-3s}$ if $(i,j) = (6,0)$). This implies the continuity of Φ.

(b) If $\bar{d} \in \partial[-1,1]^5$ and d is the corresponding original parameter in $\mathcal{D}$, then we see by definition (6.1) that

$$\text{either} \quad |q_{i,j}(d, s_0)| = A e^{-2s_0} \text{ for some } (i,j) \in I_0 \quad \text{or} \quad |q_{6,0}(d, s_0)| = A {s_0}^2 e^{-3s_0}. \tag{6.2}$$

Since $q(d, s_0) \in V_A(s_0)$ by Definition 3.1, this implies that $\bar{s}^*(\bar{d}) = s^*(d) = s_0$. From (6.2) and (6.1), we see that $\Phi(\bar{d}) = \bar{d}$ and (b) holds.

Since we know from the degree theory that there is no continuous function from $[-1,1]^5$ to its boundary which is equal to the identity on the boundary, a contradiction follows. Thus, there exists a parameter $d = (d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}(A, s_0)$ defined in Proposition 4.6 such that $q(d, y, s) \in V_A(s)$ for all $s \ge s_0$. Applying Proposition 5.2 and Lemma 2.2, we see that $\|\nabla q(d, s)\|_{L^\infty} \le 1 + C_0$, for all $s \ge s_0$.

(ii) From the similarity variables definition (1.4), u is indeed a solution of equation (1.1) defined for all $(x,t) \in \mathbb{R}^2 \times [0,T)$. From the classical theory by Giga and Kohn [11], we know that when $s \to \infty$, $w_b(s) \to \kappa$ when b is a blow-up point, and $w_b(s) \to 0$ if not, in L^2_ρ. Therefore, it is enough to prove that $w_0(s) \to \kappa$ and $w_a(s) \to 0$ as $s \to \infty$ in order to conclude.

First, since $q(s) \in V_A(s)$ for all $s \geq s_0$ from item (i), we see from (3.20), relation (3.1) and definition (2.2) of φ that $w_0(s) \to \kappa$ as $s \to \infty$.

Second, if $a \neq 0$, we see that for any $s_1 \geq 2\log \frac{A}{|a_1|+|a_2|}$, (5.27) holds. Since Proposition 5.17 holds here, applying its item (ii), we see that for some $\overline{s}(a) \geq s_0$ and $\bar{M}(a) \geq 0$, for any $s_1 \geq \max(2\log \frac{A}{|a_1|+|a_2|}, \overline{s}(a))$, for any $s \in [\overline{s}(a), s_1]$, $\|w_a(s)\|_{L^2_\rho} \leq \bar{M}(a)e^{-\frac{s}{p-1}}$. Making $s_1 \to \infty$, we see that $w_a(s) \to 0$.

This concludes the proof of Proposition 6.1. ∎

6.2. *Intermediate profile*

From the construction given in Proposition 6.1 together with (3.20) and expansion (2.5) of the profile φ defined in (2.2), we have a solution $w_0(y,s)$ of equation (1.12) such that our goal in (1.14) holds, namely

$$w_0(y,s) = \kappa - e^{-s}h_2(y_1)h_2(y_2) + o(e^{-s}) \quad \text{as } s \to \infty,$$

in L^2_ρ and uniformly on compact sets, by parabolic regularity.

For large s, the second term of this expansion will be small with respect to the constant κ, which means that we only see a flat shape for w_0. Extending the convergence beyond compact sets, say, for $|y| \sim e^{\frac{s}{4}}$, would allow us to see w escapes that constant. This is indeed what Velázquez did in [25, p. 1570, Theorem 1], proving that w_0 has the degenerate profile (1.15), in the sense that

$$\sup_{|y|<Ke^{\frac{s}{4}}} \left| w_0(y,s) - \left(p-1+\frac{(p-1)^2}{\kappa}e^{-s}y_1^2y_2^2\right)^{-\frac{1}{p-1}} \right| \to 0 \quad \text{as } s \to \infty$$

for any $K > 0$.

If we see in this expansion that w_0 departs from the constant κ from (1.5) for $y_1 = K_1e^{\frac{s}{4}}$ and $y_2 = K_2e^{\frac{s}{4}}$ for some $K_1 > 0$ and $K_2 > 0$, this is not the case on the axes, namely when $y_1 = 0$ or $y_2 = 0$. In accordance with our idea in deriving the profile φ in (2.2), our techniques in this paper provide us with the following sharper profile, valid on a larger region.

Proposition 6.3 (Intermediate profile). *Consider the solution $w_0(y,s)$ of (1.12) constructed in Proposition 6.1. For any $K > 0$, it holds that*

$$\sup_{e^{-s}y_1^2y_2^2+\delta e^{-2s}(y_1^6+y_2^6)<K} |w_0(y,s) - \Phi(y,s)| \to 0 \quad \textit{as } t \to T,$$

where $\Phi(y,s)$ is defined in (1.16).

Remark. $\Phi(y,s)$ is referred to as the "intermediate profile", since it holds in the region $\{s \ge s_0 = -\log T\}$, which corresponds to the region $\{0 \le t < T\}$ in the original variables defined by (1.4). Later in Proposition 6.5, we will define the "final profile" as a limit of $u(a,t)$ as $t \to T$ when $a \ne 0$, which holds in some sense for $t = T$, justifying its character as "final".

Proof of Proposition 6.3. Consider w_0 the solution of equation (1.12) constructed in Proposition 6.1 and defined for all $(y,s) \in \mathbb{R}^2 \times [s_0, \infty)$. Given $K_0 > 0$ and $\varepsilon > 0$, we look for $S_0 = S_0(K_0, \varepsilon)$ such that for any $S \ge S_0$ and $Y \in \mathbb{R}^2$ such that

$$e^{-S} Y_1^2 Y_2^2 + \delta e^{-2S}(Y_1^6 + Y_2^6) < K_0, \tag{6.3}$$

it holds that

$$|w_0(Y,s) - \Phi(Y,s)| \le C\varepsilon \tag{6.4}$$

for some universal $C > 0$, where $\Phi(y,s)$ is defined in (1.16). Consider then $S \ge s_0$ and $Y \in \mathbb{R}^2$ such that (6.3) holds. Since $w_0(y,s)$ is symmetric with respect to the axes and the bisectrices (see the remark following Proposition 6.1), we may assume that

$$0 \le Y_1 \le Y_2.$$

Proceeding as in Section 5.5.1, we introduce $a = a(Y,S)$ such that

$$a e^{\frac{S}{2}} = Y. \tag{6.5}$$

Since

$$w_0(Y,S) = w_a(0,S) \tag{6.6}$$

by (3.14), our conclusion will follow from the study of $w_a(y,s)$ for $s \in [s_0, S]$. From (6.3) and (6.5), we see that

$$G_0(a) \le \frac{K_0(p-1)}{\kappa} e^{-S},$$

where G_0 is defined in (5.10). In particular,

$$0 \le a_1 \le a_2 \le \left(\frac{K_0(p-1)}{\kappa\delta}\right)^{\frac{1}{6}} e^{-\frac{S}{6}}. \tag{6.7}$$

Consider $\bar{A} > 0$ to be fixed large enough later. Taking $S \ge S_1$ for some $S_1(K_0, \bar{A}) \ge s_0$, we see that

$$a_1 + a_2 \le \bar{A} e^{-\frac{s_0}{2}}.$$

Therefore, we may introduce $\sigma \ge s_0$, K' and L' such that

$$a = (K', L') e^{-\frac{\sigma}{2}} \quad \text{with } 0 \le K' \le L' \text{ and } K' + L' = \bar{A}. \tag{6.8}$$

The conclusion will follow from applying Lemma 5.14 and Proposition 5.8, exactly as we did in Corollaries 5.15 and 5.12. In order to apply those two statements, let us assume that $\bar{A} \ge \max(A, A_{23})$ and consider $\eta^* \in (0, \eta_{23}]$, where A is fixed in Proposition 6.1 and A_{23} together with η_{23} are introduced in Lemma 5.14.

From (6.7) and (6.8), we see that whenever $S \geq S_2$ for some $S_2(K_0, \bar{A}, \bar{\eta}^*) \geq s_0$, it follows that $\sigma \geq s_{23}(\bar{A}, \eta^*)$ defined in Lemma 5.14. Let us then assume that $S \geq S_2$.

Since $\bar{A} \geq A$, we see from Proposition 6.1 and Definition 3.1 that $q(s) \in V_A(s) \subset V_{\bar{A}}(s)$ and $\nabla q(s) \in L^\infty$ for all $s \geq s_0$.

Using (6.8), we see that Lemma 5.14 applies with any $s_1 \geq \sigma$. In particular, if we introduce s^* such that

$$e^{s^*-\sigma}\iota' = \eta^*, \tag{6.9}$$

where ι' is defined in (5.26), then we see that $s^* \geq \sigma$ and for all $s \in [\sigma, s^*]$,

$$\|w_a(s) - (\kappa - e^{s-\sigma}\iota')\|_{L^2_\rho} \leq M_{23}\Big(\eta^* + \frac{1}{\bar{A}}\Big)\eta^* + M_{23}e^{-\frac{\sigma}{3}}, \tag{6.10}$$

where the constant M_{23} is defined in Lemma 5.14.

Assuming that $\eta^* < \kappa$, we may introduce $\sigma^* \in \mathbb{R}$ such that

$$\psi(\sigma^*) = \kappa - \eta^*, \tag{6.11}$$

where ψ is defined in (5.8). Using (5.21), (6.8) and (6.7), we see that whenever $\bar{A} \geq A_3$, $\eta^* \leq \eta_3$ and $S \geq S_3$ for some $A_3 \geq 1$, $\eta_3 > 0$ and $S_3(K_0, \bar{A}, \eta^*)$, we have

$$\|w_a(s^*) - \psi(\sigma^*)\|_{L^2_\rho} \leq M_{23}\Big(\eta^* + \frac{1}{\bar{A}}\Big)\eta^* + M_{23}e^{-\frac{\sigma}{3}} \leq \frac{|\psi'(\sigma^*)|}{M_1}, \tag{6.12}$$

where M_1 is introduced in Proposition 5.8. Since $\nabla w_a(s^*) \in L^\infty$ and $\|w_a(s)\|_{L^\infty} \leq 2\kappa$ for any $s \geq s^*$, thanks to Proposition 6.1, together with definition (3.1) of q and transformation (3.14), Proposition 5.8 applies to the shifted function $w_a(y, s + s^*)$ (with any $\sigma_1 \geq 0$), and we see that for all $s \geq s^*$, we have

$$\|w_a(s) - \psi(s + \sigma^* - s^*)\|_{L^2_\rho} \leq M_1\|w_a(s^*) - \psi(\sigma^*)\|_{L^2_\rho}\frac{|\psi'(s+\sigma^*)|}{|\psi'(\sigma^*)|}.$$

Since $\psi' \in L^\infty$ by definition (5.8) of ψ, using again (6.12) and (5.21), we see that for all $s \geq s^*$,

$$\|w_a(s) - \psi(s + \sigma^* - s^*)\|_{L^2_\rho} \leq 2M_1M_{23}\|\psi'\|_{L^\infty}\Big(\eta^* + \frac{1}{\bar{A}} + \frac{e^{-\frac{\sigma}{3}}}{\eta^*}\Big), \tag{6.13}$$

whenever $\eta^* \leq \eta_4$, for some $\eta_4 > 0$.

Now, if $s \in [\sigma, s^*]$, using (5.6), we write

$$\begin{aligned}\|w_a(s) - \psi(s + \sigma^* - s^*)\|_{L^2_\rho} \leq{}& \|w_a(s) - (\kappa - e^{s-\sigma}\iota')\|_{L^2_\rho}\\ &+ |\psi(s + \sigma^* - s^*) - \kappa| + e^{s-\sigma}\iota'.\end{aligned} \tag{6.14}$$

Since $s \leq s^*$, hence $s + \sigma^* - s^* \leq \sigma^*$ and ψ is decreasing by definition (5.8), we obtain from (6.11) and (6.9) that

$$|\psi(s + \sigma^* - s^*) - \kappa| \leq |\psi(\sigma^*) - \kappa| = \eta^* \quad \text{and} \quad e^{s-\sigma}\iota' \leq e^{s^*-\sigma}\iota' = \eta^*. \tag{6.15}$$

Therefore, using (6.14) together with (6.10), we see that for all $s \in [\sigma, s^*]$,

$$\|w_a(s) - \psi(s + \sigma^* - s^*)\|_{L^2_\rho} \le M_{23}\Big(\eta^* + \frac{1}{\bar{A}}\Big)\eta^* + M_{23}e^{-\frac{\sigma}{3}} + 2\eta^*. \tag{6.16}$$

Finally, if $s \in [s_0, \sigma]$, noting again that $s + \sigma^* - s^* \le \sigma + \sigma^* - s^* \le \sigma^*$, we may use (6.15) and write similarly

$$\|w_a(s) - \psi(s + \sigma^* - s^*)\|_{L^2_\rho} \le \|w_a(s) - \kappa\|_{L^2_\rho} + \eta^*.$$

Recalling that $\nabla w(s) \in L^\infty$ from the construction in Proposition 6.1, we may use a Taylor expansion as in the proof of Claim 5.3 and get

$$\|w_a(s) - \kappa\|_{L^2_\rho} \le C|w_a(0, s) - \kappa| + C\|\nabla w_0(s)\|_{L^\infty}. \tag{6.17}$$

Since $w_a(0, s) = w_0(ae^{\frac{s}{2}}, s)$ by (3.14) and

$$|ae^{\frac{s}{2}}| = |(K', L')|e^{\frac{s-\sigma}{2}} \le \sqrt{K'^2 + L'^2}e^{\frac{s-\sigma}{2}} \le \bar{A}\sqrt{2}$$

by (6.8) and the fact that $s \le \sigma$, we may use again a Taylor expansion for w_0 and write

$$|w_a(0, s) - \kappa| = |w_0(ae^{\frac{s}{2}}, s) - \kappa| \le C\|w_0(s) - \kappa\|_{L^2_\rho} + C(1 + \bar{A})\|\nabla w_0(s)\|_{L^\infty}.$$

Furthermore, using the construction in Proposition 6.1, together with definitions (3.1) and (2.2) of $q(y, s)$ and $\varphi(y, s)$, and estimate (3.20), we have

$$\|w_0(s) - \kappa\|_{L^2_\rho} \le \|q(s)\|_{L^2_\rho} + \|\varphi(s) - \kappa\|_{L^2_\rho} \le CAse^{-2s} + Ce^{-s},$$

where A is given in Proposition 6.1. Collecting the previous estimates, we write for all $s \in [s_0, \sigma]$,

$$\begin{aligned} \|w_a(s) - \psi(s + \sigma^* - s^*)\|_{L^2_\rho} &\le CAse^{-2s} + Ce^{-s} \\ &\quad + C(1 + \bar{A})\|\nabla w(s)\|_{L^\infty} + \eta^*. \end{aligned} \tag{6.18}$$

Using (6.8), (6.7) and the gradient estimate in Corollary 6.2, we see from (6.13), (6.16) and (6.18) that taking $\bar{A} \ge A_5$, $\eta^* \le \eta_5$ and $S \ge S_5$ for some $A_5(\varepsilon) \ge 1$, $\eta_5(\varepsilon) > 0$ and $S_5(\varepsilon, \eta^*, K_0, \bar{A}) \ge s_0$, we get for all $s \ge s_0$,

$$\|w_a(s) - \psi(s + \sigma^* - s^*)\|_{L^2_\rho} \le 4\varepsilon.$$

Taking $s = S$, using (6.5) and (6.6), then proceeding as with (6.17), we write

$$\begin{aligned} |w_0(Y, S) - \psi(S + \sigma^* - s^*)| &= |w_a(0, S) - \psi(S + \sigma^* - s^*)| \\ &\le C\|w_a(S) - \psi(S + \sigma^* - s^*)\|_{L^2_\rho} + C\|\nabla w_0(S)\|_{L^\infty} \\ &\le C\varepsilon + C\|\nabla w_0(S)\|_{L^\infty}. \end{aligned}$$

Taking S larger and using again the gradient estimate of Corollary 6.2, we see that

$$|w_0(Y, S) - \psi(S + \sigma^* - s^*)| \le 2C\varepsilon. \tag{6.19}$$

Since $e^{s^*-\sigma}\iota' = \eta^*$ from (6.9), $e^{\sigma^*} \sim \eta^* \frac{(p-1)}{\kappa}$ as $\eta^* \to 0$ from (5.21), and $e^{S-\sigma}\iota' = e^{-S}Y_1^2Y_2^2 + \delta e^{-2S}(Y_1^6 + Y_2^6)$ by definitions (6.5), (6.8) and (5.26) of a, (K', L') and ι', we write

$$\begin{aligned} e^{S+\sigma^*-s^*} &= e^{\sigma^*}e^{-(s^*-\sigma)}e^{S-\sigma} = \Big(\frac{p-1}{\kappa} + \overline{\eta}\Big)\eta^* e^{-(s^*-\sigma)}e^{S-\sigma} \\ &= \Big(\frac{p-1}{\kappa} + \overline{\eta}\Big)\iota' e^{S-\sigma} = \Big(\frac{p-1}{\kappa} + \overline{\eta}\Big)[e^{-S}Y_1^2Y_2^2 + \delta e^{-2S}(Y_1^6 + Y_2^6)], \end{aligned}$$

where $\overline{\eta} \to 0$ as $\eta^* \to 0$. Recalling condition (6.3), then making an expansion of ψ defined in (5.8), we see that for $\eta^* \leq \eta_6$ for some $\eta_6(K_0, \varepsilon) > 0$, we have

$$\begin{aligned} \Big|\psi(S + \sigma^* - s^*) \\ - \Big[p - 1 + \frac{(p-1)^2}{\kappa}(e^{-s}Y_1^2Y_2^2 + \delta e^{-2s}(Y_1^6 + Y_2^6))\Big]^{-\frac{1}{p-1}}\Big| \leq \varepsilon. \end{aligned} \tag{6.20}$$

Combining (6.19) and (6.20) yields (6.4). This concludes the proof of Proposition 6.3. ∎

6.3. *Derivation of the final profile*

Throughout this subsection, we consider the solution $u(x, t)$ of equation (1.1) constructed in Proposition 6.1 which blows up at time $T > 0$ only at the origin.

Using Propositions 6.1 and 6.3, together with the similarity variables' transformation (1.4), we derive the following.

Corollary 6.4 (Intermediate profile for $u(x, t)$). *For any $K > 0$, it holds that*

$$\sup_{\substack{x_1^2x_2^2+\delta(x_1^6+x_2^6) \\ \leq K(T-t)}} \Big|(T-t)^{\frac{1}{p-1}}u(x,t) - \Big[p - 1 + \frac{(p-1)^2}{\kappa}\frac{[x_1^2x_2^2 + \delta(x_1^6 + x_2^6)]}{T-t}\Big]^{-\frac{1}{p-1}}\Big|$$

goes to 0 *as* $t \to T$.

Using this result, we derive the following estimate for the final profile.

Proposition 6.5 (Final profile). *For any $a \neq 0$, $u(a, t)$ has a limit as $t \to T$, denoted by $u(a, T)$. Moreover,*

$$u(a, T) \sim u^*(a) \quad \text{as } a \to 0, \tag{6.21}$$

where

$$u^*(a) = \Big[\frac{(p-1)^2}{\kappa}(a_1^2a_2^2 + \delta(a_1^6 + a_2^6))\Big]^{-\frac{1}{p-1}}.$$

Remark. Similarly to our remark given after Proposition 6.3, we justify the name of u^* as "final profile".

This result is in fact a consequence of the following ODE localization property of the PDE, proved in our earlier paper [20], and which is a direct consequence of the Liouville theorem stated in Proposition 5.1.

Proposition 6.6 (ODE localization for $u(x,t)$). *For any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all $(x,t) \in \mathbb{R}^2 \times [0,T)$,*

$$(1-\varepsilon)u(x,t)^p - C_\varepsilon \le \partial_t u(x,t) \le (1+\varepsilon)u(x,t)^p + C_\varepsilon.$$

Remark. Since initial data given in (4.3) are nonnegative, the same holds for $w_0(y,s)$, solution of equation (1.12), and $u(x,t)$, solution of equation (1.1), both constructed in Proposition 6.1. This justifies the notation $u(x,t)^p$ without absolute value.

Proof of Proposition 6.6. See [20, p. 144, Theorem 1.7]. The fact that initial data given in (4.3) are in $W^{2,\infty}(\mathbb{R}^2)$ is necessary to have the estimate in $\mathbb{R}^2 \times [0,T)$, otherwise, if initial data are only in $L^\infty(\mathbb{R}^2)$, we will have $u(t_0) \in W^{2,\infty}(\mathbb{R}^2)$ for any $t_0 > 0$ by parabolic regularity, and the statement will hold uniformly in $\mathbb{R}^2 \times [t_0, T)$ from any $t_0 \in (0,T)$. ∎

Now, we are ready to prove Proposition 6.5, thanks to Corollary 6.4 and Proposition 6.6.

Proof of Proposition 6.5. The existence of the limiting profile $u(a,T)$ follows by compactness, exactly as in [17, p. 269, Proposition 2.2]. It remains to prove (6.21).

Consider any $K_0 > 0$. Given any small enough $a \neq 0$, we introduce the time $t^*(a) \in [0,T)$ such that

$$a_1^2 a_2^2 + \delta(a_1^6 + a_2^6) = K_0 \frac{\kappa}{p-1}(T - t^*(a)). \tag{6.22}$$

Note that $t^*(a) \to T$ as $a \to 0$. The conclusion will follow from the study of $u(a,t)$ on the time interval $[t^*(a), T)$. Consider first some arbitrary $\varepsilon > 0$.

Step 1: *Initialization*. From Corollary 6.4, we get that

$$|(T - t^*(a))^{\frac{1}{p-1}} u(a, t^*(a)) - \kappa(1+K_0)^{-\frac{1}{p-1}}| \le \varepsilon\kappa(1+K_0)^{-\frac{1}{p-1}}, \tag{6.23}$$

provided that a is small enough.

Step 2: *Dynamics for $t \in [t^*(a), T)$*. From Proposition 6.6, we see that

$$\forall t \in [t^*(a), T), \quad (1-\varepsilon)u(a,t)^p - C_\varepsilon \le \partial_t u(x,t) \le (1+\varepsilon)u(a,t)^p + C_\varepsilon \tag{6.24}$$

for some $C_\varepsilon > 0$. Using (6.23) and (6.24), one easily shows that for a small enough,

$$\forall t \in [t^*(a), T), \quad C_\varepsilon \le \varepsilon u(a,t)^p.$$

Therefore, we see from (6.24) that

$$\forall t \in [t^*(a), T), \quad (1-2\varepsilon)u(a,t)^p \le \partial_t u(x,t) \le (1+2\varepsilon)u(a,t)^p.$$

Using again (6.23), we may explicitly integrate the two differential inequalities we have just derived and write for all $t \in [t^*(a), T)$,

$$\begin{aligned}\kappa[(T - t^*(a))(1+K_0)(1-\varepsilon)^{1-p} - (1-2\varepsilon)(t - t^*(a))]^{-\frac{1}{p-1}} &\le u(a,t)\\ \le \kappa[(T - t^*(a))(1+K_0)(1+\varepsilon)^{1-p} - (1+2\varepsilon)(t - t^*(a))]^{-\frac{1}{p-1}}&.\end{aligned}$$

Making $t \to T$, we see that

$$\kappa(T - t^*(a))^{-\frac{1}{p-1}}[(1 + K_0)(1 - \varepsilon)^{1-p} - (1 - 2\varepsilon)]^{-\frac{1}{p-1}} \le u(a, T)$$
$$\le \kappa(T - t^*(a))^{-\frac{1}{p-1}}[(1 + K_0)(1 + \varepsilon)^{1-p} - (1 + 2\varepsilon)]^{-\frac{1}{p-1}}.$$

Taking ε small enough, we obtain that

$$|u(a, T) - \kappa(T - t^*(a))^{-\frac{1}{p-1}} K_0^{-\frac{1}{p-1}}| \le C(K_0)\varepsilon(T - t^*(a))^{-\frac{1}{p-1}}.$$

Using definitions (6.22) and (6.21) of $t^*(a)$ and $u^*(a)$, we see that

$$|u(a, T) - u^*(a)| \le C'(K_0)\varepsilon u^*(a).$$

Since $\varepsilon > 0$ was arbitrarily chosen, this concludes the proof of Proposition 6.5. ■

6.4. *Conclusion of the proof of the main statement*

In this subsection, we gather the previous statements to derive Theorem 1.

Proof of Theorem 1. Let us first note that $\delta > 0$ was chosen large enough in the beginning of Section 3. In fact, as one can easily see from the proof, our analysis holds for any $\delta \ge \delta_0$, for some δ_0 large enough.

Using Proposition 6.1, we have the existence of a solution $u(x, t)$ of equation (1.1) blowing up at some time $T > 0$ only at the origin, such that $q(s) \in V_A(s)$ given in Definition 3.1 for any $s \ge s_0 = -\log T$, where q is defined in (3.1). In particular, $\|q(s)\|_{L^2_\rho} \le Ase^{-2s}$ for all $s \ge s_0$, by (3.20). From expansion (2.5) of the profile φ defined in (2.2), we see that item (i) of Theorem 1 holds in $L^2_\rho(\mathbb{R}^2)$. Using parabolic regularity, the convergence holds also uniformly in any compact set of $\mathbb{R}^2$.

As for items (ii) and (iii) of Theorem 1, they directly follow from Propositions 6.3 and 6.5. This concludes the proof of Theorem 1. ■

7. Proof of the technical details

In this section, we proceed in several subsections to justify all the technical ingredients, which were stated without proofs in the previous sections.

7.1. *Estimates of the remainder term $R(y, s)$ defined in formula* (3.3)

In this subsection, we prove estimate (3.10) and Lemma 4.1. Since the former is a direct consequence of the latter, thanks to decomposition (3.16), we only prove the lemma.

Proof of Lemma 4.1. Take $r \ge 2$. In the following, all the expansions are valid in L^r_ρ for $s \to \infty$. This is the case in particular for (2.5). As one may convince himself by differentiating expression (2.2), then making an expansion as $s \to \infty$, we will find the same result as if we have directly differentiated expression (2.5). Using (3.6), it follows that

$$\begin{aligned}\partial_s\varphi(y,s) &= e^{-s}h_2h_2 - 2e^{-2s}\Big\{-\delta(h_6h_0+h_0h_6)+\gamma(h_4h_2+h_2h_4)+\frac{p}{2\kappa}h_4h_4\Big\}\\ &\quad + O(e^{-3s}),\\ (\mathcal{L}-1)\varphi(y,s) &= 2e^{-s}h_2h_2 + e^{-2s}\Big\{3\delta(h_6h_0+h_0h_6)-3\gamma(h_4h_2+h_2h_4)-\frac{2p}{\kappa}h_4h_4\Big\}\\ &\quad + O(e^{-3s}).\end{aligned}$$

Since $\|\varphi(\cdot,s)\|_{L^\infty}\le\kappa+C_0$ for all $s\ge 0$ (see item (ii) of Lemma 2.2), using a Taylor expansion, we write

$$\varphi^p = \kappa^p + p\kappa^{p-1}(\varphi-\kappa)+\frac{p(p-1)}{2}\kappa^{p-2}(\varphi-\kappa)^2+O((\varphi-\kappa)^3).$$

Since

$$\varphi-\kappa = O(e^{-s}) \quad\text{and}\quad (\varphi-\kappa)^2 = e^{-2s}h_2(y_1)^2h_2(y_2)^2+O(e^{-3s})$$

from (2.5), noting that

$$h_2(\xi)^2 = h_4(\xi)+8h_2(\xi)+8h_0(\xi)$$

by definition (1.10), the result follows from the above estimates, thanks to definitions (3.3) and (1.5) of $R(y,s)$ and κ. This concludes the proof of Lemma 4.1. ∎

7.2. *Dynamics of equation* (3.2)

We prove Proposition 4.2 in this subsection.

Proof of Proposition 4.2. Consider a solution $q(s)\in V_A(s)$ of equation (3.2) on some time interval $[s_0,s_1]$ for some $A>0$. Note that $\|q(s)\|_{L^\infty}$ is uniformly bounded, hence, (3.9) holds. Assume that the initial condition (4.1) holds. Consider then $s\in[s_0,s_1]$.

(i) Consider $i\in\mathbb{N}$ and $0\le j\le i$. If we multiply equation (3.2) by $k_{i-j}(y_1)k_j(y_2)$, where the polynomial k_n is introduced in (3.18), we get the conclusion, by definition (3.17) of $q_{i,j}$ together with (3.6), (3.7), (3.20), (3.9) and Hölder's inequality.

(ii) If P_- is the L^2_ρ projector on

$$E_- \equiv \operatorname{span}\{h_{i-j}h_j \mid i\ge 8 \text{ and } 0\le j\le i\}, \tag{7.1}$$

then $q_-(s)=P_-(q(s))$ and $R_-(s)=P_-(R(s))$. Applying this projector to equation (3.2), we get

$$\partial_s q_- = \mathcal{L}q_- + P_-(Vq+B)+R_-.$$

Multiplying by $q_-\rho$ and then integrating in space, we write

$$\frac{1}{2}\frac{d}{ds}\|q_-\|^2_{L^2_\rho} = \int_{\mathbb{R}^2} q_-\mathcal{L}q_-\rho dy + \int_{\mathbb{R}^2} q_-[P_-(Vq+B)+R_-]\rho dy.$$

Since the highest eigenvalue of $\mathcal{L}$ on E_- is $\lambda = -3$ (see (3.6)), it follows that

$$\int_{\mathbb{R}^2} q_- \mathcal{L} q_- \rho dy \le -3\|q_-\|^2_{L^2_\rho}.$$

Recalling that $|B| \le C|q|^{\bar{p}}$, where $\bar{p} = \min(p, 2) > 1$ thanks to (3.9), then using the Cauchy–Schwarz inequality, we write

$$\begin{aligned}\left|\int_{\mathbb{R}^2} q_-[P_-(Vq + B) + R_-]\rho dy\right| &\le \|q_-\|_{L^2_\rho}[\|P_-(Vq + B)\|_{L^2_\rho} + \|R_-\|_{L^2_\rho}]\\ &\le \|q_-\|_{L^2_\rho}[\|Vq\|_{L^2_\rho} + C\|B\|_{L^2_\rho} + \|R_-\|_{L^2_\rho}]\\ &\le \|q_-\|_{L^2_\rho}[\|V\|_{L^4_\rho}\|q\|_{L^4_\rho} + C\|q\|^{\bar{p}}_{L^{2\bar{p}}_\rho} + \|R_-\|_{L^2_\rho}].\end{aligned}$$

Using the delay regularizing estimate given in Proposition 4.3, together with estimate (3.7) satisfied by V, we conclude the proof of Proposition 4.2. ∎

7.3. *Parabolic regularity*

In this subsection, we prove Proposition 4.3, which gives a parabolic regularity estimate for equation (3.2). Through a Duhamel formulation for equation (3.2), we reduce the question to the linear level, where the estimate was already proved by Herrero and Velázquez [16], as we recall in the following, along with another classical regularity estimate.

Lemma 7.1 (Regularizing effect of the operator $\mathcal{L}$). *We have*

(i) (Herrero and Velázquez [16]) *For any* $r > 1$, $\bar{r} > 1$, $s > \max(0, -\log(\frac{r-1}{\bar{r}-1}))$ *and* $v_0 \in L^r_\rho(\mathbb{R}^N)$ *it holds that*

$$\|e^{s\mathcal{L}} v_0\|_{L^{\bar{r}}_\rho(\mathbb{R}^N)} \le \frac{C(r, \bar{r})e^s}{(1 - e^{-s})^{\frac{N}{2r}}(r - 1 - e^{-s}(\bar{r} - 1))^{\frac{N}{2\bar{r}}}}\|v_0\|_{L^r_\rho(\mathbb{R}^N)}.$$

(ii) *Consider* $r \ge 2$ *and* $v_0 \in L^r_\rho(\mathbb{R}^N)$ *such that* $|v_0(y)| + |\nabla v_0(y)| \le C(1 + |y|^k)$ *for some* $k \in \mathbb{N}$. *Then for all* $s \ge 0$, *we have* $\|e^{s\mathcal{L}} v_0\|_{L^r_\rho(\mathbb{R}^N)} \le Ce^s\|v_0\|_{L^r_\rho(\mathbb{R}^N)}$.

(iii) *For any* $v_0 \in L^\infty(\mathbb{R}^N)$ *and* $s \ge 0$, *it holds that* $e^{s\mathcal{L}} v_0 \in L^\infty$ *with* $\|e^{s\mathcal{L}} v_0\|_{L^\infty} \le Ce^s\|v_0\|_{L^\infty}$.

(iv) *For any* $v_0 \in W^{1,\infty}(\mathbb{R}^N)$ *and* $s > 0$, *it holds that* $e^{s\mathcal{L}}\nabla v_0 \in L^\infty$ *with* $\|e^{s\mathcal{L}}\nabla v_0\|_{L^\infty} \le \frac{Ce^s}{\sqrt{1-e^{-s}}}\|v_0\|_{L^\infty}$.

Remark. When $r = \bar{r}$ in item (i), then constant in the right-hand side blows up as $s \to 0$, which may seem surprising, since one expects some continuity of the norm in L^r_ρ. In fact, such a continuity is obtained in item (ii), thanks to an additional growth control of the gradient.

Proof of Lemma 7.1. (i) See [16, Section 2, p. 139] where the one-dimensional case is proved. The adaptation to higher dimensions is straightforward.

(ii) See [23, Lemma 2.1].

(iii) and (iv) These estimates are straightforward from the definition of the kernel,

$$e^{s\mathcal{L}}(y,x) = \frac{e^s}{[4\pi(1-e^{-s})]^{N/2}} \exp\Big[-\frac{|ye^{-\frac{s}{2}}-x|^2}{4(1-e^{-s})}\Big]. \tag{7.2}$$

We conclude the proof of Lemma 7.1. ∎

Let us now give the proof of Proposition 4.3.

Proof of Proposition 4.3. Let us consider $A > 0$ and $s_1 \geq s_0 \geq 0$, and assume that $q(y,s)$ satisfies equation (3.2) for all $(y,s) \in \mathbb{R}^2 \times [s_0, s_1]$, with $q(s) \in V_A(s)$ defined in (3.1). Assume also that the initial condition (4.1) holds. Note in particular that $\|q(s)\|_{L^\infty}$ is uniformly bounded. Using (3.11) together with (3.12), we see that for some universal constant $C^* > 0$ and for almost every $(y,s) \in \mathbb{R}^2 \times [s_0, s_1]$, we have

$$\partial_s |q| \leq (\mathcal{L} + C^*)|q| + |R|. \tag{7.3}$$

Consider now some $r \geq 2$ and

$$s^* = -\log\Big(\frac{2-1}{r-1}\Big) \geq 0,$$

which is involved in the statement of Lemma 7.1. Consider then some $s \in [s_0, s_1]$ and introduce

$$s' = \max[s_0, s - s^*]. \tag{7.4}$$

We then introduce the following Duhamel formulation of (7.3) on the interval $[s', s]$:

$$|q(s)| \leq e^{(\mathcal{L}+C^*)(s-s')}|q(s')| + \int_{s'}^{s} e^{(\mathcal{L}+C^*)(s-\tau)}|R(\tau)|d\tau,$$

which implies that

$$\|q(s)\|_{L^r_\rho(\mathbb{R}^2)} \leq I + J, \tag{7.5}$$

where

$$I = \|e^{(\mathcal{L}+C^*)(s-s')}|q(s')|\|_{L^r_\rho(\mathbb{R}^2)} \quad \text{and} \quad J = \int_{s'}^{s} \|e^{(\mathcal{L}+C^*)(s-\tau)}|R(\tau)|\|_{L^r_\rho(\mathbb{R}^2)} d\tau.$$

Let us first bound J, and then I.

Since $R(y,s)$ and $\nabla R(y,s)$ are clearly bounded by a polynomial in y by definition (3.3), we write from item (ii) of Lemma 7.1 and bound (3.10) on R,

$$\begin{aligned} J &\leq C \int_{s'}^{s} e^{(1+C^*)(s-\tau)} \|R(\tau)\|_{L^r_\rho(\mathbb{R}^2)} d\tau \\ &\leq C \int_{s'}^{s} e^{(1+C^*)(s-\tau)} e^{-2\tau} d\tau \leq e^{(1+C^*)(s-s')}(s-s')e^{-2s'} d\tau. \end{aligned}$$

Since $s' \geq s - s^*$ from (7.4), it follows that $s - s' \leq s^*$ and $e^{-2s'} \leq e^{2s^*-2s}$, hence

$$J \leq C s^* e^{(3+C^*)s^*} e^{-2s}. \tag{7.6}$$

In order to bound I, we consider two cases:

- If $s - s^* \geq s_0$, then $s' = s - s^*$ and $s - s' = s^*$. Using item (i) in Lemma 7.1, we write

$$I \leq C(r)\|q(s-s^*)\|_{L^2_\rho}.$$

Since $q(s-s^*) \in V_A(s-s^*)$ by hypothesis, it follows from (3.20) that

$$\|q(s-s^*)\|_{L^2_\rho} \leq CA(s-s^*)e^{-2(s-s^*)} \leq Ce^{2s^*}Ase^{-2s}.$$

Therefore, we conclude that

$$I \leq C(r)Ase^{-2s}. \tag{7.7}$$

- If $s - s^* < s_0$, then $s' = s_0$. This time, from hypothesis (4.1), we see that we can apply item (ii) of Lemma 7.1 to write

$$I \leq Ce^{(1+C^*)(s-s_0)}\|q(s_0)\|_{L^r_\rho} \leq C(r)e^{(1+C^*)s^*}As_0e^{-2s_0}.$$

Since $s_0 \leq s < s_0 + s^*$, it follows that

$$I \leq C(r)Ase^{-2s}. \tag{7.8}$$

Combining (7.5), (7.6), (7.7) and (7.8) concludes the proof of Proposition 4.3. ∎

7.4. *Position of the flow of equation* (3.2) *on the boundary of* $V_A(s)$

We prove Proposition 4.4 in this subsection.

Proof of Proposition 4.4. Consider a solution $q(s) \in V_A(s)$ of equation (3.2) on some time interval $[s_0, s_1]$ for some $A > 0$, where $s_0 \geq s_{11}(A)$ is large enough so that both Proposition 4.2 and Lemma 4.1 apply. Assume that the initial condition (4.1) holds. We only prove items (i) and (iv), since items (ii) and (iii) follow in the same way.

(i) Consider $i \leq 4$, with i and j both even, and assume that

$$q_{i,j}(s_1) = \theta Ae^{-2s_1},$$

for some $\theta = \pm 1$. Since Lemma 4.1 implies that $|R_{i,j}(s)| \leq Ce^{-2s}$, by definition (3.17), we write from item (i) of Proposition 4.2,

$$\theta q'_{i,j}(s_1) \geq \left(1 - \frac{i}{2}\right)Ae^{-2s_1} - C_i As_1e^{-3s_1} - Ce^{-2s_1} \geq \frac{1-i}{2}Ae^{-2s_1},$$

on the one hand, taking A large enough, then s_0 large enough. On the other hand, we have

$$\frac{d}{ds}Ae^{-2s}\Big|_{s=s_1} = -2Ae^{-2s_1}.$$

Since $\frac{1-i}{2} \geq \frac{1-4}{2} = -\frac{3}{2} \geq -2$, the conclusion follows.

(iv) Let us assume that $\|q_-(s_1)\|_{L^2_\rho} = A^2 s_1^2 e^{-3s_1}$. Using Lemma 4.1, we see that

$$\|R_-(s)\|_{L^2_\rho} \leq Ce^{-3s},$$

by definition (3.16). Using item (ii) in Proposition 4.2, it follows that

$$\frac{d}{ds}\|q_-(s_1)\|_{L^2_\rho} \leq -3A^2 s_1^2 e^{-3s_1} + CAs_1 e^{-3s_1} + Ce^{-3s_1} \leq -3A^2 s_1^2 e^{-3s_1} + A^2 s_1 e^{-3s_1}$$

on the one hand, taking A large enough, then s_0 large enough. On the other hand, we compute

$$\frac{d}{ds} A^2 s^2 e^{-3s}\Big|_{s=s_1} = A^2 e^{-3s_1}(2s_1 - 3s_1^2),$$

and the conclusion follows. This concludes the proof of Proposition 4.4. ∎

7.5. *Details for the initialization*

We prove Proposition 4.6 here.

Proof of Proposition 4.6. We will be using the notation $d = (d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0})$ for simplicity. Let us consider $A \geq 1$ and $s_0 \geq 1$. The first item (i) will be proved for any $d \in [-2,2]^5$. The set $\mathcal{D}$ will be introduced while proving item (ii). Since the set $\mathcal{D}$ we intend to construct will be in $[-2,2]^5$, there will be no need to revisit the proof of item (i) afterward. Note that all the expansions given below are valid in L^r_ρ for any $r \geq 2$.

(i) By definition (4.2) of $S(y)$ and $\bar{S}(y)$ and definition (2.4) of D, together with (2.13) and (2.15), we write for s_0 large enough,

$$\begin{aligned} \frac{Ae^{-2s_0}|S(y)|}{D} &\leq \frac{CAe^{-2s_0}(1+|y|^4)}{D} \leq CAe^{-\frac{s_0}{2}}, \\ \frac{As_0^2 e^{-3s_0}|\bar{S}(y)|}{D} &\leq CAs_0^2 e^{-3s_0}\frac{1+|y|^6}{e^{-2s_0}+e^{-2s_0}|y|^6} \leq CAs_0^2 e^{-s_0}. \end{aligned} \tag{7.9}$$

Using item (i) of Lemma 2.2, the conclusion follows for s_0 large enough.

(ii) Since $D \geq p-1 > 0$ by definition (2.4) of D, using expressions (4.3) and (2.2) of $w_0(y,s_0)$ and φ, together with item (ii) of Lemma 2.2 and (7.9), we write

$$\begin{aligned} |w_0(y,s_0)|^{p-1} &\leq |\varphi(y,s_0)|^{p-1} + \frac{p-1}{\kappa D}(Ae^{-2s_0}S(y) + As_0^2 e^{-3s_0}\bar{S}(y)) \\ &\leq \frac{1}{p-1} + Ce^{-\frac{s_0}{3}}, \end{aligned} \tag{7.10}$$

for s_0 large enough, and the bound on $\|w_0(s_0)\|_{L^\infty}$ follows.

Taking the logarithm and then the gradient of $w_0(y,s_0)$ in (4.3), we write

$$|\nabla w_0(y,s_0)| \leq \frac{|w_0(y,s_0)|}{p-1}\Big[\frac{|\nabla \bar{E}|}{\bar{E}} + \frac{|\nabla D|}{D}\Big], \tag{7.11}$$

where

$$\bar{E} = E + \frac{p-1}{\kappa D}(Ae^{-2s_0}S(y) + As_0^2 e^{-3s_0}\bar{S}(y)).$$

Using (7.9), (2.3), (2.9) and (2.11), we write

$$\bar{E} \geq \frac{E_0}{C}, \quad \text{where } E_0 = 1 + e^{-s_0}(y_1^2 + y_2^2) + e^{-2s_0}(y_1^4 + y_2^4). \tag{7.12}$$

Then, we write

$$\nabla\Big[\frac{S(y)}{D}\Big] = \frac{\nabla S(y)}{D} - \frac{S(y)\nabla D}{D^2},$$

hence, by definition (4.2) of $S(y)$, we have

$$Ae^{-2s_0}\Big|\nabla\Big[\frac{S(y)}{D}\Big]\Big| \leq CAe^{-2s_0}\frac{(1+|y|^3)}{D} + Ae^{-2s_0}\frac{|S(y)|}{D}\frac{|\nabla D|}{D} \leq CAe^{-\frac{2}{3}s_0},$$

thanks to (7.9) and (2.21), together with the technique we used for (2.15). Similarly, we derive that

$$As_0^2 e^{-3s_0}\Big|\nabla\Big[\frac{\bar{S}(y)}{D}\Big]\Big| \leq As_0^2 e^{-\frac{7}{6}s_0}.$$

Therefore, using (7.11), (7.12) together with (2.20), we write

$$\frac{|\nabla \bar{E}|}{\bar{E}} \leq C\frac{|\nabla E|}{E_0} + CAe^{-\frac{2}{3}s_0} + CAs_0^2 e^{-\frac{7}{6}s_0} \leq Ce^{-\frac{s_0}{3}}. \tag{7.13}$$

Using (7.11), (7.10), (7.13) together with (2.21), we obtain the following bound:

$$|\nabla w_0(y, s_0)| \leq Ce^{-\frac{s_0}{6}}.$$

Arguing as for (2.6), we show the following:

$$\begin{aligned}
w_0(y, s_0) = \kappa &+ e^{-s_0}\Big(\frac{\kappa}{p-1}P(y) - y_1^2 y_2^2\Big) + e^{-2s_0}\Big(\frac{\kappa}{p-1}Q(y) + AS(y)\\
&+ \frac{\kappa(2-p)}{2(p-1)^2}P(y)^2 - \frac{P(y)}{p-1}y_1^2 y_2^2 - \delta y_1^6 - \delta y_2^6 + \frac{p}{2\kappa}y_1^4 y_2^4\Big)\\
&+ As_0^2 e^{-3s_0}\bar{S}(y) + O(Ae^{-3s_0}),
\end{aligned}$$

uniformly for $d \in [-2,2]^5$. Using again expansion (2.6) of φ together with definition (4.4) of $q(y, s_0)$, we derive that

$$q(y, s_0) = Ae^{-2s_0}S(y) + As_0^2 e^{-3s_0}\bar{S}(y) + O(Ae^{-3s_0}) \quad \text{as } s_0 \to \infty.$$

By definition (4.2) of $S(y)$ and $\bar{S}(y)$, together with definition (3.17) of the projections, we clearly see that for all $d \in [-2,2]^4$ and $(i, j) \in I_0 \equiv \{(0,0), (2,0), (4,0), (4,2)\}$,

$$q_{i,j}(s_0) = Ae^{-2s_0}d_{i,j} + O(Ae^{-3s_0}) \quad \text{and} \quad q_{6,0}(s_0) = As_0^2 e^{-3s_0}d_{6,0} + O(Ae^{-3s_0})$$

(please note that this identity holds after differentiation in d). We also have $\|q_-(s_0)\|_{L^2_\rho} = O(Ae^{-3s_0})$ and

$$q_{i,j}(s_0) = O(Ae^{-3s_0}) \quad \text{whenever } (i, j) \text{ and } (i, i-j) \text{ are not in } I_0 \cup \{(6,0)\}.$$

Recalling estimate (7.10) and definition (4.4) of $q(y, s_0)$, we see that this clearly gives the existence of $\mathcal{D} \subset [-2, 2]^5$ such that $q(s_0) \in V_A(s_0)$, (4.1) holds and $|q_{6,2}(s_0)| + \|q_-(s_0)\|_{L^2_\rho} \le CAe^{-3s_0}$, whenever $d \in \mathcal{D}$, with the function defined in (4.5) being one-to-one, provided that s_0 is large enough. This concludes the proof of Proposition 4.6. ∎

7.6. Gradient estimate in the shrinking set

This subsection is devoted to the proof of Proposition 5.2, thanks to the Liouville theorem recalled in Proposition 5.1.

Proof of Proposition 5.2. Consider $A \ge 1$ and $\delta_0 > 0$. Proceeding by contradiction, we may exhibit a sequence q_n of solutions to equation (3.2) defined for all $(y, s) \in \mathbb{R}^2 \times [s_{0,n}, s_{1,n}]$ for some $s_{1,n} \ge s_{0,n} \ge n$ such that $q_n(s_{0,n})$ is given by (4.4) for some parameters $(d_{0,0,n}, d_{2,0,n}, d_{4,0,n}, d_{4,2,n}, d_{6,0,n}) \in \mathcal{D}_n$, where $\mathcal{D}_n = \mathcal{D}(A, s_{0,n})$ is defined in Proposition 4.6, with $q_n(s) \in V_A(s)$ for all $s \in [s_{0,n}, s_{1,n}]$, and

$$\|\nabla q_n(s_{2,n}) + \nabla\varphi(s_{2,n})\|_{L^\infty} > \delta_0 \quad \text{for some } s_{2,n} \in [s_{0,n}, s_{1,n}]. \tag{7.14}$$

Note that $s_{2,n} \to \infty$ as $n \to \infty$. We claim that it is enough to prove that

$$s_{2,n} - s_{0,n} \to \infty \quad \text{as } n \to \infty \tag{7.15}$$

in order to conclude. Indeed, if (7.15) holds, then, introducing

$$w_n(y, s) = q_n(y, s + s_{2,n}) + \varphi(y, s + s_{2,n}), \tag{7.16}$$

we see from (3.1) that w_n is a solution of equation (1.12) defined for all $(y, s) \in \mathbb{R}^2 \times [s_{0,n} - s_{2,n}, 0]$. Since $q_n(s) \in V_A(s)$ for all $s \in [s_{0,n}, s_{2,n}]$, it follows by Definition 3.1 that $\|q_n(s) + \varphi(s)\|_{L^\infty} \le 2\kappa$. Using this together with (7.14), we see that for n large enough, we have

$$\forall s \in [s_{0,n} - s_{2,n}, 0], \quad \|w_n(s)\|_{L^\infty} \le 2\kappa, \tag{7.17}$$

$$\|\nabla w_n(0)\|_{L^\infty} \ge \delta_0. \tag{7.18}$$

Applying a classical parabolic regularity technique to equation (1.12), we obtain the following estimate.

Lemma 7.2 (Parabolic regularity for equation (1.12)). *For all $n \in \mathbb{N}$, it holds that*

$$\|w_n\|_{C^{2,1,\alpha}(\mathbb{R}^2 \times (s_{0,n} - s_{2,n} + 1, 0))} \le C_0 \tag{7.19}$$

for some $C_0 > 0$, where $C^{2,1,\alpha}$ stands for the set of functions of space and time, with two space derivatives and one time derivative which are α-Hölder continuous in both variables.

Proof. Since this estimate is classical, we leave its justification to Appendix A. ∎

Recalling that $s_{0,n} - s_{2,n} \to -\infty$ from (7.15), we may use the compactness provided by this lemma, combined to a diagonal process, in order to extract a subsequence (still denoted by w_n) such that $w_n \to w$ in $C^{2,1}$ of any compact subset of $\mathbb{R}^2 \times (-\infty, 0]$. Since w_n is a solution of equation (1.12), the same holds for w. From properties (7.17) and (7.18), it follows that

$$\forall s \le 0, \quad \|w(s)\|_{L^\infty} \le 2\kappa,$$
$$\|\nabla w(0)\|_{L^\infty} \ge \delta_0.$$

Using Proposition 5.1, we see that this is a contradiction. Thus, it remains to prove (7.15) in order to conclude.

Now we prove (7.15). Since w_n satisfies equation (1.12), by differentiating this equation in space, we obtain the following vector-valued equation on ∇w_n, for all $(y, s) \in \mathbb{R}^2 \times [s_{0,n} - s_{2,n}, 0]$:

$$\partial_s \nabla w_n = \Big(\mathcal{L} - \frac{3}{2} - \frac{1}{p-1}\Big)\nabla w_n + p|w_n|^{p-1}\nabla w_n,$$

where the operator $\mathcal{L}$ is defined in (3.3).

Using a Duhamel formulation based on the kernel (7.2), together with the L^∞ bound in (7.17), we see that for all $s \in [s_{0,n} - s_{2,n}, 0]$,

$$\begin{aligned}\|\nabla w_n(s)\|_{L^\infty} \le\ & e^{-(\frac{1}{2}+\frac{1}{p-1})(s+s_{2,n}-s_{0,n})}\|\nabla w_n(s_{0,n} - s_{2,n})\|_{L^\infty} \\ & + p(2\kappa)^{p-1}\int_{s_{0,n}-s_{2,n}}^{0} \|\nabla w_n(s')\|_{L^\infty} ds'.\end{aligned}$$

Since $\kappa^{p-1} = \frac{1}{p-1}$ by definition (1.5) using Gronwall's lemma, we see that

$$\|\nabla w_n(0)\|_{L^\infty} \le e^{(\frac{p2^{p-1}-1}{p-1}-\frac{1}{2})(s_{2,n}-s_{0,n})}\|\nabla w_n(s_{0,n} - s_{2,n})\|_{L^\infty} \qquad (7.20)$$

on the one hand. On the other hand, recalling that by hypothesis, $q_n(s_{0,n})$ is given by (4.4) for some parameters $(d_{0,0,n}, d_{2,0,n}, d_{4,0,n}, d_{4,2,n}, d_{6,0,n}) \in \mathcal{D}_n$, where $\mathcal{D}_n = \mathcal{D}(A, s_{0,n})$ is defined in Proposition 4.6, we see from that proposition and definition (7.16) of w_n that

$$\|\nabla w_n(s_{0,n} - s_{2,n})\|_{L^\infty} = \|\nabla q_n(s_{0,n}) + \nabla\varphi(s_{0,n})\|_{L^\infty} \le Ce^{-\frac{s_{0,n}}{6}}.$$

Using this together with (7.18) and (7.20), we write

$$\begin{aligned}0 < \delta_0 \le \|\nabla w_n(0)\|_{L^\infty} &\le e^{(\frac{p2^{p-1}-1}{p-1}-\frac{1}{2})(s_{2,n}-s_{0,n})}\|\nabla w_n(s_{0,n} - s_{2,n})\|_{L^\infty} \\ &\le Ce^{(\frac{p2^{p-1}-1}{p-1}-\frac{1}{2})(s_{2,n}-s_{0,n})}e^{-\frac{s_{0,n}}{6}}.\end{aligned}$$

Since $s_{0,n} \ge n$ and $\frac{p2^{p-1}-1}{p-1} - \frac{1}{2} > \frac{1}{2} > 0$, it follows that $s_{2,n} - s_{0,n} \to \infty$, and (7.15) holds. Since we have already shown that (7.15) yields a contradiction, this concludes the proof of Proposition 5.2. ∎

7.7. Size of the solution in the three regions

In this subsection, we prove Lemma 5.5.

Proof of Lemma 5.5. Consider $A \geq 1$, $s_0 \geq s_{13}(A)$ and $d = (d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}(A, s_0)$, where $s_{13}(A)$ and $\mathcal{D}(A, s_0)$ are defined in Proposition 4.6. Consider then $a \in \mathbb{R}^2$ and introduce $y = ae^{\frac{s_0}{2}}$. By definitions (4.3) and (2.2) of initial data $w_0(y, s_0)$ and the profile φ, arguing as for (2.12), we may improve that estimate and write

$$\begin{aligned}\left|w_0(y,s_0)^{p-1} - \frac{1}{D}\right| &\leq C\left\{e^{-s_0} + \frac{N_2 + N_3}{D} + \frac{Ae^{-2s_0}S(y)}{D^2} + As_0^2 e^{-3s_0}\bar{S}(y)D^2\right\}\\ &\leq Ce^{-\frac{s_0}{3}}, \end{aligned} \tag{7.21}$$

where D, N_2, N_3, $S(y)$ and $\bar{S}(y)$ are defined in (2.4), (2.13) and (4.2), and where we have used the bounds (2.15), (2.18) and (7.9).

By definitions (2.4) and (5.10) of D and $G_0(a)$, we see that

$$D = p - 1 + \frac{(p-1)^2}{\kappa}(e^{-s_0}y_1^2y_2^2 + \delta e^{-2s_0}(y_1^6 + y_2^6)) = (p-1)[1 + e^{s_0}G_0(a)].$$

Consider now two nonnegative numbers m and M such that $0 < m \leq 1 \leq M$. If $a \in \mathcal{R}_1$ (resp. $\mathcal{R}_2$, resp. $\mathcal{R}_3$) defined in (5.9), then by definition (5.9), $(p-1)(1+M) \leq D$ (resp. $(p-1)(1+m) \leq D \leq (p-1)(1+M)$, resp. $D \leq (p-1)(1+m)$). Since $0 \leq w_0(y, s_0) \leq \kappa + Ce^{-\frac{s_0}{3}}$ by Proposition 4.6, combining this with (7.21) concludes the proof of Lemma 5.5. ■

7.8. Details for the control of $w_a(y, s)$ for a in region $\mathcal{R}_3$

In this subsection, we prove Lemmas 5.10, 5.11 and 5.13.

Proof of Lemma 5.10. Consider $A \geq 1$ and $s_0 \geq s_{13}(A)$, together with the parameter $d = (d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}(A, s_0)$, where s_{13} and $\mathcal{D}$ are defined in Proposition 4.6. Consider also $w_0(y, s_0)$ defined in (4.3). Recalling that $\mathcal{D} \subset [-2, 2]^5$, we may write the following Taylor expansion:

$$\left|w_0(y,s) - \kappa\left[1 - \frac{X}{p-1} - \frac{e^{-s_0}P}{p-1}\right]\right| \leq C\{I + J^2 + \mathbb{1}_{\{1<p<\frac{3}{2}\}}J^{\frac{1}{p-1}}\}, \tag{7.22}$$

where

$$X(y_1, y_2, s_0) = \frac{p-1}{\kappa}[e^{-s_0}y_1^2y_2^2 + \delta e^{-2s_0}(y_1^6 + y_2^6)],$$

and

$$\begin{aligned} I &= Xe^{-s_0}|P| + e^{-2s_0}|Q|(1+X) + X^2(1 + e^{-s_0}|P| + e^{-2s_0}|Q|)\\ &\quad + Ae^{-2s_0}(1 + |y|^4) + As_0^2e^{-3s_0}(1 + |y|^6),\\ J &= X + e^{-s_0}|P| + I, \end{aligned}$$

the polynomials P and Q (of degrees 2 and 4, respectively) are given in (2.7) and (2.8) (see (2.9) and (2.10)), and the constant $\delta \ge 1$ was already fixed large enough at the beginning of Section 3.

Consider now some $m \in (0,1)$ and $a \in \mathcal{R}_3$ defined in (5.9), with a decomposed as in (5.14), for some $L \ge K \ge 0$, with $L + K \ge A$. Given some $r \ge 2$, we may use relation (5.15) together with (7.22) to derive an expansion for $w_a(y, s_0)$, showing error terms bounded by small terms in scales of $\frac{1}{L+K}$ and e^{-s_0}. In particular, the following expansions are useful:

$$\begin{aligned} P(y_1+K, y_2+L) &= \frac{2(p-1)}{\kappa}\big[K^2+L^2+2+2Kh_1(y_1)+2Lh_1(y_2) \\ &\qquad + h_2(y_1)+h_2(y_2)\big], \\ X(y_1+K, y_2+L, s_0) &= e^{-s_0}\big[y_1^2y_2^2+2Ly_1^2y_2+2Ky_1y_2^2+L^2y_1^2+4KLy_1y_2 \\ &\qquad + K^2y_2^2+2KL^2y_1+2K^2Ly_2+K^2L^2\big] \\ &\quad + \delta e^{-2s_0}(K^6+L^6)+O((K^5+L^5)e^{-2s_0}) \end{aligned}$$

in $L^r_\rho(\mathbb{R}^2)$. This latter estimate can be easily written in the Hermite polynomials basis (1.10). Since by definition (5.16) of ι and (5.17), it follows that

$$\iota \le \frac{m\kappa}{p-1} \le \frac{\kappa}{p-1}, \quad K+L \le 2\iota^{\frac{1}{6}}e^{\frac{s_0}{3}} \quad \text{and} \quad KL \le \sqrt{\iota}e^{\frac{s_0}{2}},$$

one can easily bound all the error terms by $O(\frac{\iota}{A})$ and $O(\iota^2)$, as required by the statement of the lemma. This concludes the proof of Lemma 5.10. ∎

Proof of Lemma 5.11. Take $A \ge 1$ and $s_0 \ge \max[s_{13}(A), s_{18}(A)]$, where s_{13} and s_{18} are defined in Proposition 4.6 and Lemma 5.10. Consider $(d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}$ defined in Proposition 4.6 and initial data $w(y, s_0)$ defined in (4.3). Consider also some $\eta^* \le \frac{\kappa}{p-1}$, $0 < m < \frac{\eta^*(p-1)}{\kappa} \le 1$ and $a \in \mathcal{R}_3$ from (5.9) given by (5.14) for some $L \ge K \ge 0$ such that $L + K \ge A$. In this case, Proposition 4.6 applies, and so does Lemma 5.10. In particular, the expansion given there holds for $w_a(y, s_0)$.

(i) Note that condition (5.19) holds from the choice of η^* and m. Therefore, using (5.17), we see that $\iota \le \frac{m\kappa}{p-1} \le \eta^*$. By definition (5.20) of s^*, it follows that $s^* = s_0 + \log\frac{\eta^*}{\iota} \ge s_0$.

(ii) Assume now that

$$\forall s \in [s_0, s_1], \quad \|w_a(s)\|_{L^\infty} \le 2\kappa, \tag{7.23}$$

for some $s_1 \ge s_0$. Introducing

$$v_a = w_a - \kappa \quad \text{and} \quad \bar{s} = \min(s^*, s_1), \tag{7.24}$$

we work in the following in the interval $[s_0, \bar{s}]$, and proceed in three steps in order to give the proof:

- In Step 1, we write an equation satisfied by v_a and project it on the various components $v_{a,i,j}$ defined in (3.17).
- In Step 2, we integrate those equations.
- In Step 3, we collect the previous information to conclude the proof.

Step 1: *Dynamics for v_a.* Since w_a satisfies equation (1.12), by definition (7.24), it follows that v_a satisfies the following equation:

$$\forall s \in [s_0, \bar{s}], \quad \partial_s v_a = \mathcal{L} v_a + \bar{B}(v_a), \tag{7.25}$$

where the linear operator $\mathcal{L}$ is introduced in (3.3) and

$$\bar{B}(v_a) = |\kappa + v_a|^{p-1}(\kappa + v_a) - \kappa^p - p\kappa^{p-1} v_a. \tag{7.26}$$

In this step, we project equation (7.25) in order to write differential inequalities satisfied by the various components $v_{a,i,j}$ defined in (3.17) as well as $\bar{P}(v_a)$, where $\bar{P}$ is the L^2_ρ orthogonal projector on

$$\bar{E} = \operatorname{span}\{h_i h_j \mid (i,j) \notin \{(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\\ (1,2),(2,1),(2,2)\}\}, \tag{7.27}$$

the orthogonal supplement of the directions appearing in the expansion of Lemma 5.10. This is our statement.

Lemma 7.3 (Projections of equation (7.25)). *Under the hypotheses of Lemma 5.11, for all $s \in [s_0, \bar{s}]$, for all $i \in \mathbb{N}$ and $j = 0, \dots, i$, it holds that*

$$\left| v'_{a,i,j}(s) - \left(1 - \frac{i}{2}\right) v_{a,i,j}(s) \right| \le C(i) \|v_a(s)\|^2_{L^2_\rho}. \tag{7.28}$$

In addition,

$$\frac{d}{ds}\|\bar{P}(v_a(s))\|_{L^2_\rho} \le -\frac{1}{2}\|\bar{P}(v_a(s))\|_{L^2_\rho} + \mathbb{1}_{\{s_0 \le s \le s_0+2\}} C_{28} \|v_a(s_0)\|^2_{L^4_\rho}\\ + \mathbb{1}_{\{s_0+2 \le s \le \bar{s}\}} C_{28} \|v_a(s-2)\|^2_{L^2_\rho} \tag{7.29}$$

for some universal constant C_{28}, where $\bar{s}$ is defined in (7.24) and the projector $\bar{P}$ is defined right before (7.27).

Proof. Note first from (7.23), (7.24) and (7.26) that

$$\forall s \in [s_0, \bar{s}], \quad |\bar{B}(v_a)| \le C|v_a|^2. \tag{7.30}$$

This way, the proof follows as for Proposition 4.2 proved in Section 7.2. More precisely, identity (7.28) follows from equation (7.25) and the quadratic estimate (7.30) exactly as for item (i) of that proposition. As for (7.29), arguing as for item (ii) of the same proposition, we write

$$\frac{d}{ds}\|\bar{P}(v_a(s))\|_{L^2_\rho} \le -\frac{1}{2}\|\bar{P}(v_a(s))\|_{L^2_\rho} + C\|v_a(s)^2\|_{L^2_\rho},$$

since $\lambda = -\frac{1}{2}$ is the largest eigenvalue of $\mathcal{L}$ corresponding to the components spanning $\bar{E}$ (see (7.27)), the image of the projector $\bar{P}$ (see (3.6)). Note that $\lambda = -\frac{1}{2}$ corresponds to the eigenfunctions $h_3 h_0$ and $h_0 h_3$.

The question then reduces to the control of $\|v_a(s)^2\|_{L^2_\rho} = \|v_a(s)\|^2_{L^4_\rho}$. Arguing as in Section 7.3 for the proof of Proposition 4.3, we can apply here items (i) and (ii) of Lemma 7.1 (with a delay time equal to 2). Indeed, $v_a(s)$, $v_a(s_0)$ and $\nabla v_a(s_0)$ are in $L^\infty(\mathbb{R}^2)$, thanks to (7.23) and Proposition 4.6, through the transformations (7.24) and (5.15). This concludes the proof of Lemma 7.3. ■

Step 2: *Integration of the equations of Lemma* 7.3. We claim the following.

Lemma 7.4 (Integration of equation (7.25)). *There exist M_{29}, $A_{29} \ge 1$ and $\eta_{29} > 0$ such that under the hypotheses of Lemma 5.11, if $A \ge A_{29}$ and $\eta^* \le \eta_{29}$, then for all $s \in [s_0, \bar{s}]$, it holds that for all $i = 0, \dots, 4$ and $j = 0, \dots, i$,*

$$|v_{a,i,j}(s) - e^{(1-\frac{i}{2})\tau} v_{a,i,j}(s_0)| \le M_{29}(\eta^* + A^{-2})e^\tau \iota,$$
$$\|\bar{P}(v_a(s))\|_{L^2_\rho} \le e^{-\frac{\tau}{2}} \|\bar{P}(v_a(s_0))\|_{L^2_\rho} + M_{29}(\eta^* + A^{-2})e^\tau \iota$$

provided that A and s_0 are large enough and η^* is small enough, where $\bar{s}$, ι and $\bar{P}$ are defined in (7.24), (5.16) and right before (7.27), with $\tau = s - s_0$.

Proof. This integration is at the heart of our argument, as it was already the case in our previous work [23] dedicated to the classification of the possible behaviors near non-isolated blow-up points for equation (1.1) (see in particular Section 4.4 in that paper). Looking in that paper is certainly convincing for the expert reader. To be nice to all readers, we summarize the integration argument below.

Consider $M_{29} > 0$ to be fixed later. For s_0 large enough, we note that both identities in Lemma 7.4 are true at $s = s_0$. Therefore, we may proceed by contradiction and assume that one of these identities does not hold at some time in the interval $[s_0, \bar{s}]$, where $\bar{s}$ is defined in (7.24). If s_{29} is the infimum of such times, then we see from continuity that both identities hold on the interval $[s_0, s_{29}]$ and that one of them has an equality case at $s = s_{29}$. In the following, we will prove that no equality case occurs, yielding a contradiction.

Since both identities hold for all $s \in [s_0, s_{29}]$, using the estimates at initial time $s = s_0$ given in Lemma 5.10 and using definition (7.24) of v_a, we derive the following bounds for s_0 large enough and for all $s \in [s_0, s_{29}]$ and $r \in \{2, 4\}$:

$$\|v_a(s_0)\|_{L^r_\rho} \le M'_{29}(\iota + J),$$
$$\|v_a(s)\|_{L^2_\rho} \le M'_{29}(e^\tau \iota + J) + M'_{29} M_{29}(\eta^* + A^{-2})e^\tau \iota,$$

where

$$J = e^{-s_0}(K^2 + L^2) \tag{7.31}$$

for some universal constant $M'_{29} > 0$, where $\tau = s - s_0$ hereafter. Taking $\eta^* > 0$ small enough and A large enough, we write for all $s \in [s_0, s_{29}]$,

$$\|v_a(s)\|_{L^2_\rho} \le 2M'_{29}(e^\tau \iota + J). \tag{7.32}$$

Using Lemma 7.3 together with (7.32), we write for s_0 large enough, for all $i = 0, \dots, 4$, $j = 0, \dots, i$ and $s \in [s_0, s_{29}]$,

$$\left|v'_{a,i,j}(s) - \Big(1 - \frac{i}{2}\Big) v_{a,i,j}(s)\right| \le C_{29} {M'_{29}}^2 (e^{2\tau}\iota^2 + J^2),$$
$$\frac{d}{ds}\|\bar{P}(v_a(s))\|_{L^2_\rho} \le -\frac{1}{2}\|\bar{P}(v_a(s))\|_{L^2_\rho} + C_{29} {M'_{29}}^2 (e^{2\tau}\iota^2 + J^2),$$

for some universal constant $C_{29} > 0$.

Integrating the first equation, we see that for all $s \in [s_0, s_{29}]$,

$$|v_{a,i,j}(s) - e^{(1-\frac{i}{2})\tau} v_{a,i,j}(s_0)| \le C_{29} {M'_{29}}^2 (e^{2\tau}\iota^2 + 2e^{\tau} J^2). \tag{7.33}$$

Integrating the second inequality, we see that for all $s \in [s_0, s_{29}]$,

$$\|\bar{P}(v_a(s))\|_{L^2_\rho} \le e^{-\frac{\tau}{2}}\|\bar{P}(v_a(s_0))\|_{L^2_\rho} + C_{29} {M'_{29}}^2 \Big(\frac{2}{5} e^{2\tau}\iota^2 + 2J^2\Big). \tag{7.34}$$

Recalling that $\tau = s - s_0 \le s_{29} - s_0 \le \bar{s} - s_0 \le s^* - s_0$, we see by definition (5.20) that

$$e^{\tau}\iota \le \eta^*. \tag{7.35}$$

Moreover, recalling that $K + L \ge A$ and $\delta \ge 1$ (see the beginning of Section 3), we write by definitions (7.31) and (5.16) of J and ι,

$$J^2 \le e^{-2s_0}\frac{(K+L)^2}{A^2}(K^2 + L^2)^2 \le e^{-2s_0}\frac{16}{A^2}(K^6 + L^6) \le \frac{16\iota}{\delta A^2} \le \frac{16\iota}{A^2}.$$

Using this together with (7.35), (7.33) and (7.34), we see that for s_0 large enough, we have for all $s \in [s_0, s_{29}]$, for all $i = 0, \dots, 4$ and $j = 0, \dots, i$,

$$|v_{a,i,j}(s) - e^{(1-\frac{i}{2})\tau} v_{a,i,j}(s_0)| \le 32 C_{29} {M'_{29}}^2 (\eta^* + A^{-2}) e^{\tau}\iota,$$
$$\|\bar{P}(v_a(s))\|_{L^2_\rho} \le e^{-\frac{\tau}{2}}\|\bar{P}(v_a(s_0))\|_{L^2_\rho} + 32 C_{29} {M'_{29}}^2 (\eta^* + A^{-2}) e^{\tau}\iota.$$

Fixing

$$M_{29} = 33 C_{29} {M'_{29}}^2,$$

we see that no equality case occurs in both identities shown in Lemma 7.4. A contradiction follows from the beginning of the proof. This concludes the proof of Lemma 7.4. ■

Step 3: *Conclusion of the proof of Lemma* 5.11. Recalling transformation (7.24), then using Lemma 7.4 together with Lemma 5.10 and the Cauchy–Schwarz inequality, we see that for all $s \in [s_0, \bar{s}]$,

$$\|w_a(\cdot, s) - (\kappa - e^{\tau}\iota)\|_{L^2_\rho} \le M''_{29}\Big\{M_{29}(\eta^* + A^{-2})e^{\tau}\iota + e^{\tau}\Big(\frac{\iota}{A} + \iota^2\Big) + (L^2 + K^2)e^{-s_0}\Big\},$$

for some universal constant $M''_{29} > 0$. Using bounds (5.18) and (5.17), then recalling that $\delta \ge 1$ from the beginning of Section 3, we see that

$$e^{-s_0}(L^2 + K^2) \le 2e^{-\frac{s_0}{3}}\Big(\frac{\kappa}{p-1}\Big)^{\frac{1}{3}}.$$

This concludes the proof of Lemma 5.11. ■

Proof of Lemma 5.13. Since $w_0(y,\sigma)=\varphi(y,\sigma)+q(y,\sigma)$ by relation (3.1), proceeding as in (5.15), we may introduce

$$\begin{aligned}\varphi_a(y,\sigma)&=\varphi(y+ae^{\frac{\sigma}{2}},\sigma)=\varphi(y_1+K',y_2+L',\sigma),\\ q_a(y,\sigma)&=q(y+ae^{\frac{\sigma}{2}},\sigma)=q(y_1+K',y_2+L',\sigma)\end{aligned}\tag{7.36}$$

(use (5.23)). This way, we write $w_a(y,\sigma)=\varphi_a(y,\sigma)+q_a(y,\sigma)$, and the proof of (5.25) follows by adding the expansions of $\varphi_a(y,\sigma)$ and $q_a(y,\sigma)$, performed in two steps.

Step 1: *The expansion of* $\varphi_a(y,\sigma)$. We claim that the expansion of $\varphi_a(y,\sigma)$ follows from Lemma 5.10. Indeed, the input in that lemma is initial data $w_0(y,s_0)$ in (4.3), and if one takes the parameters $(d_{0,0},d_{2,0},d_{4,0},d_{4,2},d_{6,0})=(0,0,0,0,0)$ and formally replaces s_0 by σ in definition (4.3) of initial data $w_0(y,s_0)$, then we recover $\varphi(y,\sigma)$ defined in (2.2). In addition, the point a we consider is given by (5.23) with $K'+L'=A$, which falls in the framework considered in Section 5.5.1. Therefore, Lemma 5.10 applies and we see that

$$\begin{aligned}\varphi_a(y,\sigma)=\kappa-\iota'-e^{-\sigma}\{&2K'L'^2h_1h_0+2K'^2L'h_1h_2+L'^2h_2h_0+4K'L'h_1h_1\\&+K'^2h_0h_2+2L'h_2h_1+2K'h_1h_2+h_2h_2\}\\&+O\Big(\frac{\iota'}{A}\Big)+O(\iota'^2)\end{aligned}\tag{7.37}$$

in L^r_ρ for any $r\ge 2$, where ι' is given in (5.26).

Step 2: *The expansion of* $q_a(y,\sigma)$. Take $r\ge 2$. Using decomposition (3.16) for $q(y,\sigma)$, we obtain from (7.36) that

$$q_a(y,\sigma)=\bar q_a(y,\sigma)+\underline{q}_a(y,\sigma),\tag{7.38}$$

where

$$\begin{aligned}\bar q_a(y,\sigma)&=\sum_{i=0}^{7}\sum_{j=0}^{i}q_{i,j}(\sigma)h_{i-j}(y_1+K')h_j(y_2+L'),\\ \underline{q}_a(y,\sigma)&=q_-(y_1+K',y_2+L',\sigma).\end{aligned}\tag{7.39}$$

Concerning $\bar q_a(y,\sigma)$, recalling that $q(\sigma)\in V_A(\sigma)$ defined in Definition 3.1, then proceeding as for the proof of Lemma 5.10 given at the beginning of this subsection, we derive that

$$\begin{aligned}\bar q_a(y,\sigma)=q_{6,2}(\sigma)\{&L'^4h_2h_0+K'^4h_0h_2+4L'^3h_2h_1+4K'^3h_1h_2+6(K'^2+L'^2)h_2h_2\\&+4K'h_3h_2+4L'h_2h_3+h_4h_2+h_2h_4\}+\Big(\frac{\iota'}{A}\Big)\end{aligned}\tag{7.40}$$

in L^r_ρ. As for $\underline{q}_a(y,\sigma)$, we can bound it thanks to the following parabolic regularity estimate on $q_-(\sigma)$.

Lemma 7.5 (Parabolic regularity for $q_-(\sigma)$). *Under the hypotheses of Lemma* 5.13*, it holds that for all* $r'\ge 2$*,*

$$\forall s\in[s_0,\sigma],\quad \|q_-(s)\|_{L^{r'}_\rho}\le C(r')A^2s^2e^{-3s}.$$

Indeed, using (7.39), we write

$$\int |q_{\underline{a}}(y,\sigma)|^r \rho(y)dy = \int |q_-(z,\sigma)|^r \rho(z_1 - K', z_2 - L')dz. \tag{7.41}$$

Then, by definition (1.18), we write

$$\rho(z_1 - K', z_2 - L') = \rho(z) e^{\frac{K' z_1 + L' z_2}{2}} e^{\frac{K'^2 + L'^2}{4}}.$$

Using the Cauchy–Schwarz inequality together with Lemma 7.5, we write

$$\begin{aligned}
&\int |q_-(z,\sigma)|^r \rho(z_1 - K', z_2 - L')dz \\
&\le e^{\frac{K'^2+L'^2}{4}} \Big(\int |q_-(z,\sigma)|^{2r} \rho(z)dz \Big)^{\frac12} \Big(\int e^{K' z_1 + L' z_2} \rho(z)dz \Big)^{\frac12} \\
&\le e^{\frac34 (K'^2+L'^2)} \Big(\int |q_-(z,\sigma)|^{2r} \rho(z)dz \Big)^{\frac12} \\
&\le C e^{\frac34 (K'^2+L'^2)} (A^2 \sigma^2 e^{-3\sigma})^r.
\end{aligned} \tag{7.42}$$

Using (5.24) and recalling definition (5.26) of ι', then taking s_0 large enough (remember that $\sigma \ge s_0$), we see that

$$\begin{aligned}
e^{\frac{3}{4r}(K'^2+L'^2)} A^2 \sigma^2 e^{-3\sigma} &\le e^{-2\sigma} \frac{A^6}{A} = e^{-2\sigma} \frac{(K' + L')^6}{A} \\
&\le \frac{2^6}{A} e^{-2\sigma} (K'^6 + L'^6) \le \frac{2^6 \iota'}{A\delta}.
\end{aligned} \tag{7.43}$$

Using (7.40), (7.41), (7.42) and (7.43), we obtain an expansion for $q_a(z,\sigma)$, by (7.38). Adding expansion (7.37), we obtain the desired expansion (5.25) in Lemma 5.13. It remains to justify Lemma 7.5.

Proof of Lemma 7.5. Assume here that $s_0 \ge s_{11}(A)$ defined in Proposition 4.2, so that Proposition 4.3 applies. Consider then some $r' \ge 2$. Proceeding as for item (ii) of Proposition 4.2, we project equation (3.2) for all $s \in [s_0, \sigma]$ as follows:

$$\partial_s q_- = \mathcal{L} q_- + G, \quad \text{where } G = P_-(Vq + B + R)$$

and P_- is the L^2_ρ projector on the subspace E_- from (7.1). Given some $\sigma_0 \in [s_0, \sigma]$, we may write a Duhamel formulation based on the kernel given in (7.2),

$$q_-(\sigma) = e^{(\sigma-\sigma_0)\mathcal{L}} q(\sigma_0) + \int_{\sigma_0}^{\sigma} e^{(\sigma-\sigma')\mathcal{L}} G(\sigma')d\sigma'.$$

Taking the $L^{r'}_\rho$ norm, we write

$$\|q_-(\sigma)\|_{L^{r'}_\rho} \le \mathrm{I} + \mathrm{II} \equiv \|e^{(\sigma-\sigma_0)\mathcal{L}} q(\sigma_0)\|_{L^{r'}_\rho} + \int_{\sigma_0}^{\sigma} \|e^{(\sigma-\sigma')\mathcal{L}} G(\sigma')\|_{L^{r'}_\rho} d\sigma'. \tag{7.44}$$

We start by bounding II. We claim that for any $\sigma' \in [s_0, \sigma]$, $G(\sigma')$ and its gradient have polynomial growth in y, allowing the application of item (ii) of Lemma 7.1. Indeed, by definition (3.3), together with the L^∞ bound on $q(s)$ from Definition 3.1 and the gradient estimate of Proposition 5.2, we see that $Vq + B(q) + R$ and its gradient have polynomial growth in y. By definition of the P_- operator (see (7.1) and (3.16)), so does G. Applying item (ii) of Lemma 7.1, we see that

$$|\mathrm{II}| \le \int_{\sigma_0}^{\sigma} \|G(\sigma')\|_{L^{r'}_\rho} d\sigma'. \tag{7.45}$$

Since

$$\|P_-(g)\|_{L^{r'}_\rho} \le C(r')\|g\|_{L^{r'}_\rho} \quad \text{for any } g \in L^{r'}_\rho$$

(see again (7.1) and (3.16)), we write

$$\|G(\sigma')\|_{L^{r'}_\rho} \le \|Vq(\sigma')\|_{L^{r'}_\rho} + \|B(q(\sigma'))\|_{L^{r'}_\rho} + \|R_-(\sigma')\|_{L^{r'}_\rho}.$$

Using Proposition 4.3 and proceeding as for item (ii) of Proposition 4.2, we see that

$$\|G(\sigma')\|_{L^{r'}_\rho} \le C(r')A\sigma' e^{-3\sigma'}. \tag{7.46}$$

As for the term I, introducing $\sigma^*(r') > 0$ and $C^*(r') > 0$ such that the following delay regularizing effect holds for any $v \in L^2_\rho$ (see item (i) of Lemma 7.1):

$$\|e^{\sigma^*(r')\mathcal{L}}(v)\|_{L^{r'}_\rho} \le C^*(r')\|v\|_{L^2_\rho}, \tag{7.47}$$

we distinguish two cases.

Case 1: $\sigma \ge s_0 + \sigma^*$. Fixing $\sigma_0 = \sigma - \sigma^*$, we see that $\sigma_0 \ge s_0$. Using (7.47), we see by definition (7.44) of I and Definition 3.1 of $V_A(s)$ that

$$|\mathrm{I}| \le C^*(r')\|q_-(\sigma - \sigma^*)\|_{L^2_\rho} \le C^*(r')A^2(\sigma - \sigma^*)^2 e^{-3(\sigma-\sigma^*)}.$$

Using (7.44), (7.45) and (7.46), we see that

$$\begin{aligned}\|q_-(\sigma)\|_{L^{r'}_\rho} &\le C^*(r')A^2\sigma^2 e^{-3(\sigma-\sigma^*)} + \sigma^* C(r')A(\sigma - \sigma^*)e^{-3(\sigma-\sigma^*)}\\ &\le \bar{C}(r')A^2\sigma^2 e^{-3\sigma},\end{aligned}$$

and the conclusion of Lemma 7.5 follows.

Case 2: $s_0 \le \sigma \le s_0 + \sigma^*$. Fixing $\sigma_0 = s_0$, and noting that $q(s_0)$ and $\nabla q(s_0)$ are bounded (see the hypotheses of Lemma 5.13 and Definition 3.1 of $V_A(s_0)$), we can apply item (ii) of Lemma 7.1 and write by definition (7.44) of I and Definition 3.1:

$$|\mathrm{I}| \le Ce^{\sigma-s_0}\|q_-(s_0)\|_{L^2_\rho} \le Ce^{\sigma-s_0}A^2 s_0^2 e^{-3s_0} \le Ce^{\sigma^*}A^2\sigma^2 e^{-3(\sigma-\sigma^*)},$$

and the conclusion follows as in Case 1. This concludes the proof of Lemma 7.5. ■

This concludes the proof of Lemma 5.13 too. ■

Appendix A. A classical parabolic regularity estimate for equation (1.12)

We prove Lemma 7.2 here. Since the argument was extensively used in our earlier papers, we would not give details.

Proof of Lemma 7.2. We proceed in two steps: we first justify the estimate locally in space, then we extend it to the whole space.

Step 1: *Proof of a local version, where* $\mathbb{R}^2$ *is replaced by* $B(0,1)$*, the unit ball of* $\mathbb{R}^2$. When restricting to $B(0,1)$, one can use the similarity variables' transformation to translate the problem into a regularity question for equation (1.1). Using the technique of [21, p. 1060, Step 2], which relies on [8, p. 406, Theorem 3], we get the result.

Step 2: *Extension to the whole space* $\mathbb{R}^2$. Introducing for any $a \in \mathbb{R}^2$,

$$w_{a,n}(y,s) = w_n(y + ae^{\frac{s}{2}}, s), \tag{A.1}$$

we see from the similarity variables transformation (1.4) that $w_{a,n}$ is also a solution of (1.12) defined for all $s \in [s_{0,n}, s_{2,n}]$ and satisfying the uniform bound (7.17). Applying the same local regularity technique on $w_{a,n}$ as in Step 1, we show that (7.19) holds also for $w_{a,n}$ with $\mathbb{R}^2$ replaced by $B(0,1)$, uniformly in $a \in \mathbb{R}^2$. Using (A.1) and varying a in the whole space $\mathbb{R}^2$, we recover the full estimate (7.19) (on $\mathbb{R}^2$) for w_n. This concludes the proof of Lemma 7.2. ∎

Appendix B. Stability results for equation 1.12

This appendix is devoted to the proof of Propositions 5.6 and 5.8.

Proof of Proposition 5.6. Consider a solution w of equation (1.12) defined for all $(y,s) \in \mathbb{R}^2 \times [0, \sigma_1]$ for some $\sigma_1 \geq 0$, with

$$|w(y,s)| \leq 2\kappa \quad \text{and} \quad \nabla w(0)(1+|y|)^{-k} \in L^\infty \tag{B.1}$$

for some $k \in \mathbb{N}$. We aim to prove that

$$\forall s \in [0, \sigma_1], \quad \|w(s)\|_{L^2_\rho} \leq M_0 \|w(0)\|_{L^2_\rho} e^{-\frac{s}{p-1}}, \tag{B.2}$$

provided that

$$\|w(0)\|_{L^2_\rho} \leq \varepsilon_0, \tag{B.3}$$

for some large $M_0 \geq 1$ and small $\varepsilon_0 > 0$.

We will assume that

$$\|w(0)\|_{L^2_\rho} > 0, \tag{B.4}$$

otherwise $w \equiv 0$ and (B.2) is trivial.

Since $\Delta w - \frac{1}{2}y \cdot \nabla w = \frac{1}{\rho}\nabla \cdot (\rho \nabla w)$ by definition (1.18) of ρ, multiplying equation (1.12) by $w\rho$, integrating in space, then using an integration by parts, we write for all $s \in [0, \sigma_1]$,

$$\frac{1}{2}\frac{d}{ds}\int w(y,s)^2\rho(y)dy \leq -\frac{1}{p-1}\int w(y,s)^2\rho(y)dy + \int |w(y,s)|^{p+1}\rho(y)dy. \quad \text{(B.5)}$$

Note that the fact that $w(0) \in L^\infty$ and $\nabla w(0)(1+|y|)^{-k} \in L^\infty$ for some $k \in \mathbb{N}$ (see formula (B.1)) is important to justify this integration by parts, as it is the case in item (ii) of Lemma 7.1. The conclusion will follow from two arguments: a rough estimate for general data, then a delicate estimate for small data. In the final step, we combine both arguments to conclude.

Step 1: *A rough estimate for general data.* Using (B.5) together with (B.1) and definition (1.5) of κ, we write for all $s \in [0, \sigma_1]$,

$$\frac{1}{2}\frac{d}{ds}\int w(y,s)^2\rho(y)dy \leq \frac{2^{p-1}-1}{p-1}\int w(y,s)^2\rho(y)dy,$$

hence,

$$\|w(s)\|_{L^2_\rho} \leq e^{\frac{(2^{p-1}-1)}{p-1}s}\|w(0)\|_{L^2_\rho}. \quad \text{(B.6)}$$

Step 2: *A delicate estimate for small data and large* σ_1. Using again equation (1.12), together with (B.1) and definitions (1.5) and (3.3) of κ and $\mathscr{L}$, we write for almost every $(y, s) \in \mathbb{R}^2 \times [0, \sigma_1]$,

$$\partial_s|w| \leq \Big(\mathscr{L} - 1 + \frac{(2^{p-1}-1)}{p-1}\Big)|w|.$$

Using the regularizing effect of Lemma 7.1, we derive the existence of $s_* > 0$ and $C^* > 0$ such that if $\sigma_1 \geq s_*$, then for all $s \in [s_*, \sigma_1]$, we have

$$\|w(s)\|_{L^{p+1}_\rho} \leq C^*\|w(s-s_*)\|_{L^2_\rho}. \quad \text{(B.7)}$$

Now, if $s \in [0, \min(s_*, \sigma_1)]$, we use the L^∞ bound (B.1) and definition (1.5) of κ to derive the following rough control of the nonlinear term:

$$\|w(s)\|^{p+1}_{L^{p+1}_\rho} \leq (2\kappa)^{p-1}\|w(s)\|^2_{L^2_\rho} = \frac{2^{p-1}}{p-1}\|w(s)\|^2_{L^2_\rho}.$$

Using (B.5) with these two controls of the nonlinear term, we write for all $s \in [0, \sigma_1]$, the following delay differential inequality:

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|w(s)\|^2_{L^2_\rho} \leq &-\frac{1}{p-1}\|w(s)\|^2_{L^2_\rho} + \mathbb{1}_{\{0\leq s\leq s_*\}}\frac{2^{p-1}}{p-1}\|w(s)\|^2_{L^2_\rho}\\ &+ \mathbb{1}_{\{s_*\leq s\leq \sigma_1\}}(C^*)^{p+1}\|w(s-s_*)\|^{p+1}_{L^2_\rho}. \quad \text{(B.8)}\end{aligned}$$

Step 3: *Conclusion of the proof.* Fixing $M_0 > 0$ and ε_0 such that

$$M_0 = 2\max(1, e^{\frac{2^{p-1}}{p-1}s_*}) \quad \text{and} \quad 4(C^*M_0)^{p+1}\varepsilon_0^{p-1}e^{\frac{2s_*}{p-1}} = M_0^2, \tag{B.9}$$

we are ready to finish the proof of (B.2) if (B.3) and (B.4) hold. By continuity of the L^2_ρ norm[1] of $w(s)$ and noting that $M_0 > 1$, if we proceed by contradiction and assume that identity (B.2) fails, then we may introduce $\bar{s} \in (0, \sigma_1]$ such that

$$\forall s \in [0, \bar{s}], \quad \|w(s)\|_{L^2_\rho} \le M_0\|w(0)\|_{L^2_\rho} e^{-\frac{s}{p-1}}, \tag{B.10}$$

$$\|w(\bar{s})\|_{L^2_\rho} = M_0\|w(0)\|_{L^2_\rho} e^{-\frac{\bar{s}}{p-1}}. \tag{B.11}$$

We will reach a contradiction in each of the two cases we consider in the following.

Case 1: $\bar{s} \le s_*$. In this case, using (B.6), assumption (B.4) and the choice of M_0 in (B.9), we write

$$\begin{aligned}\|w(\bar{s})\|_{L^2_\rho} &\le e^{\frac{(2^{p-1}-1)}{p-1}\bar{s}}\|w(0)\|_{L^2_\rho} \le e^{\frac{(2^{p-1}-1)}{p-1}s_*}\|w(0)\|_{L^2_\rho}\\ &\le \frac{M_0}{2}\|w(0)\|_{L^2_\rho} e^{-\frac{s_*}{p-1}} < M_0\|w(0)\|_{L^2_\rho} e^{-\frac{\bar{s}}{p-1}},\end{aligned}$$

and a contradiction follows by (B.11).

Case 2: $s_* \le \bar{s} \le \sigma_1$. In this case, identity (B.8) holds for any $s \in [0, \bar{s}]$. Using (B.6) when $0 \le s \le s_*$ and (B.10) when $s_* \le s \le \bar{s}$, we write for all $s \in [0, \bar{s}]$:

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\|w(s)\|^2_{L^2_\rho} \le &-\frac{1}{p-1}\|w(s)\|^2_{L^2_\rho} + \mathbb{1}_{\{0\le s\le s_*\}}\frac{2^{p-1}}{p-1}e^{\frac{2(2^{p-1}-1)}{p-1}s}\|w(0)\|^2_{L^2_\rho}\\ &+ \mathbb{1}_{\{s_*\le s\le \bar{s}\}}(C^*M_0\|w(0)\|_{L^2_\rho})^{p+1}e^{-\frac{(p+1)(s-s_*)}{p-1}}.\end{aligned}$$

Integrating this equation, we see that

$$\begin{aligned}\|w(\bar{s})\|^2_{L^2_\rho} &\le e^{-\frac{2\bar{s}}{p-1}}\{\|w(0)\|^2_{L^2_\rho} + e^{\frac{2\cdot 2^{p-1}s_*}{p-1}}\|w(0)\|^2_{L^2_\rho} + 2(C^*M_0\|w(0)\|_{L^2_\rho})^{p+1}e^{\frac{2s_*}{p-1}}\}\\ &\le e^{-\frac{2\bar{s}}{p-1}}\left\{\left(\frac{M_0}{2}\right)^2\|w(0)\|^2_{L^2_\rho} + \left(\frac{M_0}{2}\right)^2\|w(0)\|^2_{L^2_\rho} + \frac{M_0^2}{2}\|w(0)\|^2_{L^2_\rho}\right\}\\ &< \frac{3}{4}M_0^2 e^{-\frac{2\bar{s}}{p-1}}\|w(0)\|^2_{L^2_\rho}\end{aligned}$$

thanks to definition (B.9) of M_0 and ε_0, together with (B.4). Thus, a contradiction follows from (B.11). This concludes the proof of Proposition 5.6. ∎

Now, we give the proof of Proposition 5.8.

[1] This is a consequence of the continuity in L^∞ for equation (1.1), through transformation (1.4)

Proof of Proposition 5.8. Consider a solution w of equation (1.12) defined for all $(y,s) \in \mathbb{R}^2 \times [0,\sigma_1]$ for some $\sigma_1 \geq 0$, with

$$\nabla w(0)(1+|y|)^{-k} \in L^\infty \quad \text{and} \quad |w(y,s)| \leq 2\kappa, \tag{B.12}$$

for some $k \in \mathbb{N}$. We will prove that for some universal constant $M_1 \geq 1$, if

$$\|w(0) - \psi(\sigma^*)\|_{L^2_\rho} \equiv \varepsilon_1 \leq \frac{|\psi'(\sigma^*)|}{M_1} \tag{B.13}$$

for some $\sigma^* \in \mathbb{R}$, where ψ is defined in (5.8), then it holds that for all $s \in [0,\sigma_1]$,

$$\|w(s) - \psi(s+\sigma^*)\|_{L^2_\rho} \leq M_1 \|w(0) - \psi(\sigma^*)\|_{L_\rho^{\bar{p}+1}} \frac{|\psi'(s+\sigma^*)|}{|\psi'(\sigma^*)|}. \tag{B.14}$$

We may assume that

$$\varepsilon_1 > 0, \tag{B.15}$$

otherwise $w(y,s) = \psi(s+\sigma^*)$ for any $s \geq 0$, from the uniqueness of solutions to equation (1.12), and (B.14) is trivial.

Since w and ψ are both solutions of (1.12), introducing

$$\bar{\psi}(s) = \psi(s+\sigma^*) \quad \text{and} \quad v(y,s) = w(y,s) - \bar{\psi}(s), \tag{B.16}$$

we write the following PDE satisfied by v, for all $(y,s) \in \mathbb{R}^2 \times [0,\sigma_1]$:

$$\partial_s v = \Delta v - \frac{1}{2} y \cdot \nabla v - \frac{v}{p-1} + p|\tilde{w}|^{p-1} v, \tag{B.17}$$

where

$$\tilde{w}(y,s) \in [w(y,s), \bar{\psi}(s)]. \tag{B.18}$$

Arguing as for (B.5), we derive the following identity from (B.17), for all $s \in [0,\sigma_1]$:

$$\frac{1}{2} z'(s) \leq -\frac{z(s)}{p-1} + p \int |\tilde{w}(y,s)|^{p-1} v(y,s)^2 \rho(y) dy, \tag{B.19}$$

where

$$z(s) = \int v(y,s)^2 \rho(y) dy.$$

The fact that $\nabla w(0)(1+|y|)^{-k} \in L^\infty$ is useful to justify the integration by parts in (B.19) and elsewhere. We proceed in two steps, first deriving a differential inequality for $z(s)$, then using a Gronwall argument to conclude.

Step 1: *A differential inequality on* $z(s)$. Since $\tilde{w}$ and $\bar{\psi}$ are bounded by (5.8), (B.18) and (B.12), using definitions (B.18) and (B.16) of $\tilde{w}$ and v, we write by continuity

$$||\tilde{w}(y,s)|^{p-1} - \bar{\psi}^{p-1}| \leq C_0 |\tilde{w}(y,s) - \bar{\psi}(s)|^{\bar{p}-1} \leq C_0 |v(y,s)|^{\bar{p}-1}$$

for some $C_0 > 0$, where $\bar{p} = \min(p, 2)$. Plugging this in (B.19), we write

$$\frac{1}{2}z'(s) \le \Big[-\frac{1}{p-1} + p\bar{\psi}(s)^{p-1}\Big]z(s) + C_0\int |v(y,s)|^{\bar{p}+1}\rho(y)dy. \tag{B.20}$$

Let us now bound $\|v(s)\|_{L^{\bar{p}+1}_\rho}$. Using equation (B.17), bound (B.12), definitions (B.18) and (5.8) of $\tilde{w}$ and ψ, together with definitions (1.5) and (3.3) of κ and $\mathcal{L}$, we write for almost every $(y, s) \in \mathbb{R}^2 \times [0, \sigma_1]$,

$$\partial_s|v| \le \Big(\mathcal{L} - 1 + \frac{(2^{p-1}p-1)}{p-1}\Big)|v|. \tag{B.21}$$

Arguing as for (B.7), we see that if $\sigma_1 \ge s_*$, then we have for all $s \in [s_*, \sigma_1]$,

$$\|v(s)\|_{L^{\bar{p}+1}_\rho} \le \bar{C}\|v(s-s_*)\|_{L^2_\rho} = \bar{C}z(s-s_*)^{\frac{1}{2}}, \tag{B.22}$$

for some possibly different $s_*(p) > 0$ and $\bar{C} > 0$. Now, if $s \in [0, \min(s_*, \sigma_1)]$, using (B.12) and definition (5.8) of ψ, we see by definition (B.16) of v that $|v| \le 3\kappa$ and $\nabla v(0)(1+|y|)^{-k} \in L^\infty$. Therefore,

$$\int |v(y,s)|^{\bar{p}+1}\rho(y)dy \le (3\kappa)^{\bar{p}-1}\int v(y,s)^2\rho(y)dy. \tag{B.23}$$

In addition, using (B.21), we see that we can apply item (ii) of Lemma 7.1 and get from (B.13)

$$\forall s \in [0, \min(s_*, \sigma_1)], \quad \|v(s)\|_{L^2_\rho} \le C^*\|v(0)\|_{L^2_\rho} = C^*\varepsilon_1.$$

Using this together with (B.23), (B.20) and (B.22), we see that for all $s \in [0, \sigma_1]$,

$$\begin{aligned}\frac{1}{2}z'(s) \le \Big[-\frac{1}{p-1} + p\bar{\psi}(s)^{p-1}\Big]z(s) &+ C_1\mathbb{1}_{\{0\le s\le s_*\}}\varepsilon_1^2\\ &+ C_1\mathbb{1}_{\{s_*\le s\le \sigma_1\}}z(s-s_*)^{\frac{\bar{p}+1}{2}}\end{aligned} \tag{B.24}$$

for some $C_1 > 0$.

Step 2: *A Gronwall estimate.* Let us define

$$\bar{z}_p(s) = z_p(s+\sigma^*), \quad \text{where } z_p(s) = \frac{e^{-\frac{2s}{p-1}}}{(1+e^{-s})^{\frac{2p}{p-1}}} = \Big[\frac{p-1}{\kappa}\psi'(s)\Big]^2, \tag{B.25}$$

and ψ is defined in (5.8). Since $\psi(s)$ satisfies equation (1.12), it follows that $\bar{z}_p(s)$ is a solution of the linear part of (B.24), namely

$$\bar{z}_p'(s) = 2\Big(-\frac{1}{p-1} + p\bar{\psi}(s)^{p-1}\Big)\bar{z}_p(s). \tag{B.26}$$

Then, we introduce the following barrier:

$$\bar{z}(s) = \frac{M_1'\varepsilon_1^2}{z_p(\sigma^*)}\bar{z}_p(s), \tag{B.27}$$

where $M_1' > 1$ will be fixed large enough later. With this definition and recalling definition (B.19) of $z(s)$, we suggest to prove that

$$\forall s \in [0, \sigma_1], \quad z(s) \le \overline{z}(s), \tag{B.28}$$

if ε_1 defined in (B.13) is small enough, which clearly implies (B.14), by definition (B.25) of z_p. We proceed by contradiction and assume that identity (B.28) fails. Since

$$0 < z(0) = \varepsilon_1^2 < M_1'\varepsilon_1^2 = \overline{z}(0), \tag{B.29}$$

by (B.19), (B.16), (B.13), (B.15), (B.27) and (B.25), using the continuity in time of the L^2_ρ norm of $v(s)$ solution of equation (B.17) (which is a consequence of the continuity in L^∞ for equation (1.1), through transformations (1.4) and (B.16)), we see that (B.28) holds at least on a small interval to the right of 0. Hence, we may introduce $\overline{s} \in (0, \sigma_1]$ such that

$$\forall s \in [0, \overline{s}], \quad z(s) \le \overline{z}(s), \tag{B.30}$$

$$z(\overline{s}) = \overline{z}(\overline{s}). \tag{B.31}$$

Using the differential inequality (B.24) together with the auxiliary function $\overline{z}_p$ which satisfies equation (B.26), (B.29) and (B.30), we write the following Gronwall estimate:

$$z(\overline{s}) \le \overline{z}_p(\overline{s})\Big\{\frac{\varepsilon_1^2}{\overline{z}_p(0)} + C_1\varepsilon_1^2 J_1 + C_1\Big(\frac{M_1'\varepsilon_1^2}{z_p(\sigma^*)}\Big)^{\frac{\overline{p}+1}{2}} J_2\Big\},$$

where

$$J_1 = \int_0^{s_*} \frac{d\sigma}{\overline{z}_p(\sigma)} \quad \text{and} \quad J_2 = \int_{s_*}^{\overline{s}} \frac{\overline{z}_p(\sigma - s_*)^{\frac{\overline{p}+1}{2}}}{\overline{z}_p(\sigma)} d\sigma.$$

By definition (B.25), we see that for all $s' \in [0, s_*]$, we have

$$z_p(\sigma^* + s') \ge z_p(\sigma^*) e^{-2\frac{(p+1)}{p-1}s_*},$$

hence

$$J_1 = \int_{\sigma^*}^{\sigma^*+s_*} \frac{d\sigma'}{z_p(\sigma')} \le \frac{s_* e^{2\frac{(p+1)}{p-1}s_*}}{z_p(\sigma^*)}.$$

We also have

$$J_2 = \int_{\sigma^*+s_*}^{\sigma^*+\overline{s}} \frac{z_p(\sigma' - s_*)^{\frac{\overline{p}+1}{2}}}{z_p(\sigma')} d\sigma' \le \int_{-\infty}^{\infty} \frac{z_p(\sigma' - s_*)^{\frac{\overline{p}+1}{2}}}{z_p(\sigma')} d\sigma' \equiv C_2(s^*).$$

Imposing that

$$\varepsilon_1^2 \le \frac{z_p(\sigma^*)}{M_1'(C_1 C_2(s_*))^{\frac{2}{\overline{p}-1}}}, \tag{B.32}$$

we see that

$$z(\overline{s}) \le \frac{\varepsilon_1^2}{z_p(\sigma^*)} \overline{z}_p(\overline{s})[1 + C_1 s_* e^{2\frac{(p+1)}{p-1}s_*} + 1].$$

Fixing

$$M_1' = 3 + C_1 s_* e^{2\frac{(p+1)}{p-1}s_*},$$

we see that a contradiction follows from (B.31), (B.30) and (B.27) (note that $\bar{z}(\bar{s}) > 0$ by definition (B.27), together with (B.25) and (B.15)). Thus, (B.28) holds. Since $z_p(s) = C|\psi'(s)|^2$ from (B.25), using definitions (B.19) and (B.27) of $z(s)$ and $\bar{z}(s)$, together with condition (B.32), we conclude the proof of Proposition 5.8. ∎

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