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Martin Balko · David Chodounský · Natasha Dobrinen · Jan Hubička ·
Matěj Konečný · Lluís Vena · Andy Zucker

Exact big Ramsey degrees for finitely constrained binary free amalgamation classes

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Abstract. We characterize the big Ramsey degrees of free amalgamation classes in finite binary languages defined by finitely many forbidden irreducible substructures, thus refining the recent upper bounds given by Zucker. Using this characterization, we show that the Fraïssé limit of each such class admits a big Ramsey structure satisfying the infinite Ramsey theorem, implying that the automorphism group of the Fraïssé limit has a metrizable universal completion flow.

Keywords: big Ramsey degrees, Fraïssé structures, binary free amalgamation classes.

1. Introduction

Set-theoretic and model-theoretic notation throughout the paper is mostly standard. We write $\omega = \{0, 1, 2, \dots\}$ for the least infinite ordinal, and we identify $m < \omega$ with the set $\{0, \dots, m-1\}$, though sometimes we will write the latter for emphasis. If $m < n < \omega$, we

Martin Balko: Department of Applied Mathematics (KAM), Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic; balko@kam.mff.cuni.cz

David Chodounský: Department of Applied Mathematics (KAM), Charles University, Malostranské nám. 25, 118 00 Praha 1; Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic; chodounsky@math.cas.cz

Natasha Dobrinen: Department of Mathematics, University of Notre Dame, 255 Hurley Bldg, Notre Dame, IN 46556, USA; ndobrine@nd.edu

Jan Hubička: Department of Applied Mathematics (KAM), Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic; hubicka@kam.mff.cuni.cz

Matěj Konečný: Institute of Algebra, TU Dresden, Helmholtzstraße 10, 01069 Dresden, Germany; Department of Applied Mathematics (KAM), Charles University, Malostranské nám. 25, 118 00 Praha 1, Czech Republic; matej.konecny@tu-dresden.de

Lluís Vena: Departamento de Matemáticas, Universitat Politècnica de Catalunya, Jordi Girona 31, 08034 Barcelona, Spain; lluis.vena@upc.edu, lluis.vena@gmail.com

Andy Zucker: Department of Pure Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, ON N2L 3G1, Canada; a3zucker@uwaterloo.ca

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write $[m,n] = \{r < \omega : m \leq r \leq n\}$, and similarly for (m,n), $(m,n]$, and $[m,n)$. Given a first-order language $\mathcal{L}$, we typically denote $\mathcal{L}$-structures in bold letters and use the unbolded letter to denote the underlying set, i.e., A, B, C are the underlying sets of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, etc. If $\mathbf{A}$ is an $\mathcal{L}$-structure and $B \subseteq A$, then $\mathbf{A}|_B$ denotes the $\mathcal{L}$-structure on underlying set B induced from $\mathbf{A}$. Given $\mathcal{L}$-structures $\mathbf{A}$ and $\mathbf{B}$, an *embedding* of $\mathbf{A}$ into $\mathbf{B}$ is an injection from A to B which preserves relations, non-relations, functions, and constants. We write $\mathrm{Emb}(\mathbf{A},\mathbf{B})$ for the set of embeddings of $\mathbf{A}$ into $\mathbf{B}$, and we write $\mathbf{A} \leq \mathbf{B}$ when $\mathrm{Emb}(\mathbf{A},\mathbf{B}) \neq \emptyset$.

Definition 1. Given structures $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ with $\mathbf{A} \leq \mathbf{B} \leq \mathbf{C}$ and integers $r > \ell \geq 1$, we write

$$\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_{r,\ell}$$

if for every $\gamma\colon \mathrm{Emb}(\mathbf{A},\mathbf{C}) \to r$, there is some $g \in \mathrm{Emb}(\mathbf{B},\mathbf{C})$ with $|\gamma[g \circ \mathrm{Emb}(\mathbf{A},\mathbf{B})]| \leq \ell$. When $\ell = 1$, we omit it from the notation.

Given a class $\mathcal{K}$ of finite $\mathcal{L}$-structures and $\mathbf{A} \in \mathcal{K}$, the *Ramsey degree* of $\mathbf{A}$ in $\mathcal{K}$, denoted by $\mathrm{RD}(\mathbf{A}, \mathcal{K})$, is the least $\ell < \omega$ (or ∞ if no such ℓ exists) such that for any integer $r > \ell$ (equivalently, $r = \ell + 1$) and any $\mathbf{B} \in \mathcal{K}$ with $\mathbf{A} \leq \mathbf{B}$, there is $\mathbf{C} \in \mathcal{K}$ with $\mathbf{B} \leq \mathbf{C}$ and $\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_{r,\ell}$. The class $\mathcal{K}$ has *finite Ramsey degrees* if $\mathrm{RD}(\mathbf{A}, \mathcal{K}) < \infty$ for every $\mathbf{A} \in \mathcal{K}$, and $\mathcal{K}$ has the *Ramsey property* if $\mathrm{RD}(\mathbf{A}, \mathcal{K}) = 1$ for every $\mathbf{A} \in \mathcal{K}$.

Given an infinite $\mathcal{L}$-structure $\mathbf{K}$ and a finite structure $\mathbf{A} \leq \mathbf{K}$, the *big Ramsey degree of* $\mathbf{A}$ *in* $\mathbf{K}$, denoted by $\mathrm{BRD}(\mathbf{A},\mathbf{K})$, is the least positive integer ℓ (or ∞ if no such ℓ exists) such that for every integer $r > \ell$ (equivalently, $r = \ell + 1$), we have $\mathbf{K} \to (\mathbf{K})^{\mathbf{A}}_{r,\ell}$. We say that $\mathbf{K}$ *satisfies the infinite Ramsey theorem* (IRT) if every finite $\mathbf{A} \leq \mathbf{K}$ satisfies $\mathrm{BRD}(\mathbf{A},\mathbf{K}) = 1$.

For instance, the classical finite Ramsey theorem [43] says exactly that the class of finite linear orders has the Ramsey property, and the classical infinite Ramsey theorem says exactly that the structure $\langle \omega, < \rangle$ satisfies IRT. Some authors call Ramsey degrees by the name *small* Ramsey degrees to distinguish them from big Ramsey degrees.

We briefly mention that many authors formulate Definition 1 with respect to *copies* rather than embeddings, where a *copy* is simply the image of an embedding. Comparing (big) Ramsey degrees when using copies versus embeddings is straightforward. Letting $\mathrm{RD}^{\mathrm{copy}}(\mathbf{A}, \mathcal{K})$ and $\mathrm{BRD}^{\mathrm{copy}}(\mathbf{A}, \mathbf{K})$ be defined in the obvious way, one can show that $\mathrm{RD}(\mathbf{A}, \mathcal{K}) = \mathrm{RD}^{\mathrm{copy}}(\mathbf{A}, \mathcal{K}) \cdot |\mathrm{Aut}(\mathbf{A})|$ and $\mathrm{BRD}(\mathbf{A},\mathbf{K}) = \mathrm{BRD}^{\mathrm{copy}}(\mathbf{A},\mathbf{K}) \cdot |\mathrm{Aut}(\mathbf{A})|$; see, for instance, [49, Proposition 4.4]. In this paper, we always refer to the embedding version unless explicitly mentioned otherwise.

Structural Ramsey theory has a long and rich history. It was initiated in the 1970's when Nešetřil and Rödl, and independently Abramson and Harrington, proved that the class of finite ordered graphs has the Ramsey property [1, 37]. In fact, the Nešetřil–Rödl theorem is much stronger, implying the Ramsey property for several classes of finite ordered relational structures. More recently, the Kechris–Pestov–Todorčević correspondence connecting structural Ramsey theory and topological dynamics as well as Nešetřil's classification program of Ramsey classes [35] boosted the development of the area which led to discoveries of many more classes with the Ramsey property. The most general result on classes with the Ramsey property is due to Hubička and Nešetřil [28], who extended the earlier proof techniques (Nešetřil and Rödl's partite construction [36,38,39]) to structures in languages containing both relations and functions and gave a sufficient structural condition which can be used to show that a given class is Ramsey, effectively providing a "black box" for the combinatorial arguments involved.

While most classes of finite unordered structures do not have the Ramsey property, often enriching a given class by adding a linear order or finitely many new relations produces an expansion class with the Ramsey property. When this is possible, the original class will have finite small Ramsey degrees. For instance, the class of finite ordered graphs has the Ramsey property, while the class of finite graphs has finite Ramsey degrees. One way to show that a class $\mathcal{K}$ has finite Ramsey degrees, developed by Kechris–Pestov–Todorčević [29] and Nguyen Van Thé [40], is to show that $\mathcal{K}$ admits a *precompact expansion class* $\mathcal{K}^*$ which has the Ramsey property. For example, the class of finite sets has the class of finite linear orders as a precompact expansion, and the class of finite graphs has the class of finite ordered graphs as a precompact expansion. Zucker showed in [49] that, in fact, every *Fraïssé class* $\mathcal{K}$ with finite Ramsey degrees admits a precompact expansion class with the Ramsey property.

Recall that a class $\mathcal{K}$ of finite structures is *Fraïssé* if it is closed under isomorphism, contains countably many structures up to isomorphism, contains structures of arbitrarily large finite size, and satisfies the following three properties:

- Hereditary property (HP): If $\mathbf{A} \leq \mathbf{B}$ and $\mathbf{B} \in \mathcal{K}$, then $\mathbf{A} \in \mathcal{K}$.
- Joint embedding property (JEP): If $\mathbf{A}, \mathbf{B} \in \mathcal{K}$, there is $\mathbf{C} \in \mathcal{K}$ with both $\mathbf{A} \leq \mathbf{C}$ and $\mathbf{B} \leq \mathbf{C}$.
- Amalgamation property (AP): For any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{K}$ and embeddings $f\colon \mathbf{A} \to \mathbf{B}$ and $g\colon \mathbf{A} \to \mathbf{C}$, there are $\mathbf{D} \in \mathcal{K}$ and embeddings $r\colon \mathbf{B} \to \mathbf{D}$ and $s\colon \mathbf{C} \to \mathbf{D}$ with $r \circ f = s \circ g$.

Given a Fraïssé class $\mathcal{K}$, we can use the members of $\mathcal{K}$ to build a generic countably infinite structure, the *Fraïssé limit* of $\mathcal{K}$, denoted by $\mathrm{Flim}(\mathcal{K})$. Writing $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$, then $\mathbf{K}$ is characterized up to isomorphism by the following two properties:

- $\mathcal{K} = \mathrm{Age}(\mathbf{K}) := \{\mathbf{A} \text{ finite} : \mathbf{A} \leq \mathbf{K}\}$.
- If $\mathbf{A} \subseteq \mathbf{K}$ is finite and $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{K})$, there is $g \in \mathrm{Aut}(\mathbf{K})$ with $g|_{\mathbf{A}} = f$.

Until Section 6, $\mathcal{K}$ denotes a Fraïssé class of $\mathcal{L}$-structures and $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$. We will add assumptions to $\mathcal{K}$ and $\mathbf{K}$ later, but at a minimum this will always hold.

A routine compactness argument yields that if $\mathbf{A} \in \mathcal{K}$ and $\ell < \omega$, then $\mathrm{RD}(\mathbf{A}, \mathcal{K}) = \ell$ if and only if for any $\mathbf{A} \leq \mathbf{B} \in \mathcal{K}$ and any positive integer r, we have $\mathbf{K} \to (\mathbf{B})^{\mathbf{A}}_{r,\ell}$. In particular, $\mathrm{RD}(\mathbf{A}, \mathcal{K}) \leq \mathrm{BRD}(\mathbf{A}, \mathbf{K})$. As is commonly done, we refer to the big Ramsey degree of $\mathbf{A}$ *in* $\mathcal{K}$ and write $\mathrm{BRD}(\mathbf{A}, \mathcal{K})$ for $\mathrm{BRD}(\mathbf{A}, \mathbf{K})$. We say that $\mathcal{K}$ has *finite big Ramsey degrees* if $\mathrm{BRD}(\mathbf{A}, \mathcal{K}) < \infty$ for every $\mathbf{A} \in \mathcal{K}$. If $\mathcal{K}$ has finite big Ramsey degrees, then $\mathcal{K}$ has finite small Ramsey degrees. We refer to Dobrinen's ICM paper [20] for background on historical and recent results on big Ramsey degrees.

It is natural to ask, in analogy to how every Fraïssé class with finite Ramsey degrees admits a precompact expansion with the Ramsey property, if something similar can be done for classes with finite big Ramsey degrees. In this spirit, Zucker in [50] defines a *big Ramsey structure*, whose definition we recall (and add to) in Definition 3.

Definition 2. Given first-order languages $\mathcal{L}^* \supseteq \mathcal{L}$ and an $\mathcal{L}^*$-structure $\mathbf{M}^*$, the *$\mathcal{L}$-reduct* $\mathbf{M}|_{\mathcal{L}}$ is the $\mathcal{L}$-structure on the same underlying set as $\mathbf{M}^*$ and with the same interpretations of symbols from $\mathcal{L}$ as in $\mathbf{M}$. Conversely, if $\mathbf{M}$ is an $\mathcal{L}$-structure, an *$\mathcal{L}^*$-expansion* of $\mathbf{M}$ is some $\mathcal{L}^*$-structure $\mathbf{M}^*$ on underlying set M with $\mathbf{M}^*|_{\mathcal{L}} = \mathbf{M}$. Given an $\mathcal{L}$-structure $\mathbf{M}$, $\mathbf{B} \leq \mathbf{M}$, and an $\mathcal{L}^*$-expansion $\mathbf{M}^*$ of $\mathbf{M}$, we set

$$\mathbf{M}^*(\mathbf{B}) := \{\mathbf{B}^* : \mathbf{B}^* \text{ is an } \mathcal{L}^*\text{-expansion of } \mathbf{B} \text{ with } \mathbf{B}^* \leq \mathbf{M}^*\}.$$

If $f \in \mathrm{Emb}(\mathbf{B}, \mathbf{M})$, we write $\mathbf{M}^* \cdot f$ for the unique $\mathbf{B}^* \in \mathbf{M}^*(\mathbf{B})$ with $f \in \mathrm{Emb}(\mathbf{B}^*, \mathbf{M}^*)$.

Definition 3. If $\mathcal{K}$ has finite big Ramsey degrees, a *big Ramsey structure* for $\mathcal{K}$ is an $\mathcal{L}^*$-expansion $\mathbf{K}^*$ of $\mathbf{K}$ for some first-order language $\mathcal{L}^* \supseteq \mathcal{L}$ satisfying the following:

- For every $\mathbf{A} \in \mathrm{Age}(\mathbf{K})$, we have $|\mathbf{K}^*(\mathbf{A})| = \mathrm{BRD}(\mathbf{A}, \mathcal{K})$.
- On $\mathrm{Emb}(\mathbf{A}, \mathbf{K})$, the coloring $f \to \mathbf{K}^* \cdot f$ witnesses that $\mathrm{BRD}(\mathbf{A}, \mathcal{K}) \geq |\mathbf{K}^*(\mathbf{A})|$.

For example, $\langle \omega, < \rangle$ is a big Ramsey structure for the class of finite sets. In fact, all known examples of Fraïssé classes with finite big Ramsey degrees admit big Ramsey structures satisfying IRT.

While big Ramsey degrees (and even in a sense, the big Ramsey structures) of the Rado graph and, more generally, unrestricted structures with finitely many binary relations were fully understood by 2006 via the work of Sauer [47] and its sequel [33] by Laflamme, Sauer, and Vuksanovic, the big Ramsey degrees for the k-clique-free analogs of the Rado graph, called Henson graphs, were not fully characterized until the present paper. Vertices were shown to have big Ramsey degree one for the triangle-free [30] and all k-clique-free [22] Henson graphs. Moreover, Sauer had shown by 1998 that the big Ramsey degree for edges is two in triangle-free Henson graph [44]. However, a general theorem proving finite big Ramsey degrees in the Henson graphs remained elusive, due to the fact that the standard techniques using Milliken's theorem were known not to work in this setting. Answering questions of Sauer, Dobrinen [15, 19] developed the new techniques of coding trees and set-theoretic forcing on these trees to show (in ZFC) that the classes of finite triangle-free graphs and more generally, finite k-clique-free graphs for each $k \geq 3$, have finite big Ramsey degrees. Generalizing these techniques to obtain a simpler proof of finite big Ramsey degrees, Zucker [51] considered Fraïssé classes of the following form. Let $\mathcal{L}$ be a relational language. An $\mathcal{L}$-structure $\mathbf{F}$ is *irreducible* if every $a \neq b \in F$ is contained in some relation, meaning there are $R \in \mathcal{L}$ and $\vec{c}$ from F with $R^{\mathbf{F}}(\vec{c})$ and $a, b \in \vec{c}$. Given a set $\mathcal{F}$ of finite irreducible $\mathcal{L}$-structures such that each $\mathbf{F} \in \mathcal{F}$ has size at least two, let

$$\mathrm{Forb}(\mathcal{F}) = \{\mathbf{A} : \mathbf{A} \text{ is a finite } \mathcal{L}\text{-structure and } \forall\, \mathbf{F} \in \mathcal{F}\ (\mathbf{F} \not\leq \mathbf{A})\}.$$

The class $\mathrm{Forb}(\mathcal{F})$ is always a Fraïssé class which additionally satisfies a strengthening of the amalgamation property called *free amalgamation*. This means that given an amalgamation problem $f\colon \mathbf{A} \to \mathbf{B}$ and $g\colon \mathbf{A} \to \mathbf{C}$, we can find a solution $r\colon \mathbf{B} \to \mathbf{D}$ and $s\colon \mathbf{C} \to \mathbf{D}$ satisfying the following:

- $D = \mathrm{Im}(r) \cup \mathrm{Im}(s)$.
- $\mathrm{Im}(r) \cap \mathrm{Im}(s) = \mathrm{Im}(r \circ f) = \mathrm{Im}(s \circ g)$.
- If $a \neq b \in D$ are contained in a relation, then $\{a, b\} \subseteq \mathrm{Im}(r)$ or $\{a, b\} \subseteq \mathrm{Im}(s)$.

Conversely, if $\mathcal{K}$ is a Fraïssé free amalgamation class, then $\mathcal{K} = \mathrm{Forb}(\mathcal{F})$ for some set $\mathcal{F}$ of finite irreducible $\mathcal{L}$-structures. If $\mathcal{L}$ is a finite relational language with symbols of arity at most two and $\mathcal{F}$ is a *finite* set of finite irreducible $\mathcal{L}$-structures, we call $\mathrm{Forb}(\mathcal{F})$ a *finitely-constrained binary free amalgamation class*. In [51], Zucker proved that every finitely-constrained binary free amalgamation class has finite big Ramsey degrees. We mention that by examples of Sauer [46], the condition that $\mathcal{F}$ is finite is necessary.

Dobrinen conjectured that the upper bounds obtained in [15, 19] for triangle-free and k-clique-free graphs were exact. This turns out not to be the case in general and some slight modifications were needed to obtain the exact values. While these were being prepared in [16], Balko, Chodounský, Hubička, Konečný, and Vena independently obtained exact values derived from the upper bounds obtained in [27]. Slightly before this, general upper bounds for binary finitely-constrained classes appeared in [51]. Hence the seven authors decided to come together and produce this work. Here, the structural properties responsible for exact big Ramsey degrees will be developed in the general framework due to Zucker in [51]. This paper characterizes exact big Ramsey degrees for finitely-constrained binary free amalgamation classes, culminating and concluding work in [15, 19, 22, 23, 25, 27, 30, 33, 42, 47, 51].

The following is our main theorem.

Theorem 4. *There is an algorithm which, when given a finitely-constrained binary free amalgamation class $\mathcal{K}$ and $\mathbf{A} \in \mathcal{K}$, outputs* $\mathrm{BRD}(\mathbf{A}, \mathcal{K})$.

A key step in our characterization is finding the lower bounds, i.e., producing *unavoidable* colorings of $\mathrm{Emb}(\mathbf{A}, \mathbf{K})$ for each $\mathbf{A} \in \mathrm{Age}(\mathbf{K})$. Our construction of these colorings yields the following.

Theorem 5. *Let $\mathcal{K}$ be a finitely-constrained binary free amalgamation class. Then $\mathcal{K}$ admits a big Ramsey structure in a finite relational language which satisfies IRT. In particular,* $\mathrm{Aut}(\mathbf{K})$ *has a metrizable universal completion flow.*

Universal completion flows were introduced by Zucker in [50]. While it is unknown whether every topological group admits a universal completion flow, it is proven in [50] that if $\mathcal{K}$ admits a big Ramsey structure, then $\mathrm{Aut}(\mathbf{K})$ has a metrizable universal completion flow. Indeed, in a suitable space of expansions of $\mathbf{K}$, the universal completion flow is formed by simply taking the orbit closure of any big Ramsey structure.

1.1. Big Ramsey degrees and trees

As we remarked above, $\langle \omega, < \rangle$ is a big Ramsey structure for the class of finite sets. One might hope that adding a well-order is enough to obtain a big Ramsey structure in general. However, this is not the case; as soon as the class has non-trivial binary relations, trees start appearing in the study of its big Ramsey behavior. The following paragraphs give some intuition about this, and we hope that they will give the reader a better "big picture" understanding of what is happening in the main part of this paper.

Let $\mathscr{G}$ denote the class of finite graphs, let $\mathbf{G} = \langle G, E^{\mathbf{G}} \rangle = \mathrm{Flim}(\mathscr{G})$ be the *random graph* (or *Rado graph*), and suppose that we want to construct a "bad" coloring of $\mathrm{Emb}(\mathbf{A}, \mathbf{G})$ for some $\mathbf{A} \in \mathscr{G}$ (in the sense that every copy of $\mathbf{G}$ in $\mathbf{G}$ attains many colors). We can assume that $G = \omega$. Considering vertex 0, the remaining vertices of $\mathbf{G}$ split into two *types* – those which are connected to 0 and those which are not. In general, for each set of vertices $\{0, \dots, n-1\} \subseteq G$, there are 2^n many types: Each function $f : n \to 2$ corresponds to the collection of all vertices $k \geq n$ in G such that for each $i < n$, k has an edge with i if and only if $f(i) = 1$. The set of all types over initial segments of G can thus be identified with the set $T = 2^{<\omega}$ of all finite binary strings. There are three natural partial orders on T: the lexicographic order $\preceq$ (which is a dense linear order), the order $\sqsubseteq$ defined by $s \sqsubseteq t$ if and only if s is an initial segment of t, which defines a tree ordering on T, and the partial order $\leq_\ell$ of *relative levels* given by $s \leq_\ell t$ if and only if $\mathrm{dom}(s) \leq \mathrm{dom}(t)$. Given $s, t \in T$, we define $s \wedge t$ to be the longest common prefix of s and t and call it their *meet*. We also form a function $c : \omega \to T$ where given $v \in \boldsymbol{\omega}$, we have $\mathrm{dom}(c(v)) = v$ and $c(v)(i) = 1$ if and only if $E^{\mathbf{G}}(v, i)$ holds.

Notice that just from fixing an enumeration, we obtained a tree with levels and a meet operation as well as a map c which maps vertices of $\mathbf{G}$ into this tree; this is the *coding tree* of $\mathbf{G}$. In particular, we can use this tree to color $\mathrm{Emb}(\mathbf{A}, \mathbf{G})$, where given $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{G})$, we record certain information about the function $c \circ f : \mathbf{A} \to T$. Writing $S = \mathrm{Im}(f) \subseteq \omega$ and $\mathrm{Crit}(S) := S \cup \{\mathrm{dom}(c(i) \wedge c(j)) : i, j \in S\} := \{i_0 < \cdots < i_{n-1}\}$, we obtain for each $a \in A$ a function $\pi_f(a) \in 2^{<n}$ with $\mathrm{dom}(\pi_f(a)) = m$ if and only if $f(a) = i_m$ and with $\pi_f(a)(j) = (c \circ f(a))(i_j)$, for each $j < m$. We color f based on the map $a \to \pi_f(a)$.

It turns out that this coloring is not *unavoidable*. In addition to proving that the random graph has finite big Ramsey degrees, Sauer [47] considered the above coloring and constructed an $\eta \in \mathrm{Emb}(\mathbf{G}, \mathbf{G})$ which reduced the number of colors, and soon after, Laflamme, Sauer and Vuksanovic [33] proved that Sauer's η eliminates as many colors as possible. The remaining colors are such that, with notation as above, the $\sqsubseteq$-downwards closure of $\mathrm{Im}(\pi_f)$ has exactly one "interesting event" per level, i.e., exactly one node v with either $\pi_f(a) = v$ for some $a \in A$ (a "coding event") or $\pi_f(a) \wedge \pi_f(b)$ for some $a, b \in A$ (a "splitting event"). In particular, the coding nodes form an antichain in the tree.

Hence by enumerating the random graph, we create two kinds of "interesting events" which we can use to construct unavoidable colorings, namely splitting and coding. For the random graph, these are the only interesting events. However, consider the class $\mathscr{G}_3$ of finite triangle-free graphs, and set $\mathbf{G}_3 = \mathrm{Flim}(\mathscr{G}_3)$. We once again fix a well-order of vertices of $\mathbf{G}_3$, identify its vertex set with ω. Recall that a type is a set of all vertices of $\mathbf{G}_3$

connected in the same way to a given initial segment of vertices of $\mathbf{G}_3$, and so we can ask which finite graphs embed into the substructure of $\mathbf{G}_3$ induced on this set (called the *age* of the type). For $\mathbf{G}$ the answer is always $\mathcal{G}$. In $\mathbf{G}_3$, however, the situation is more complex. If the type has no edges to the corresponding initial segment, then its age is $\mathcal{G}_3$, but once the type attains a neighbor, its age shrinks to the class of all finite graphs with no edges; since all vertices of the type have a common neighbor, no two of them can be connected by an edge. Thus the appearance of the first neighbor of a vertex is an interesting event, and the definition of $\mathrm{Crit}(S)$ in this setting must account for these. In general, one has to consider ages of not only single types, but of several types combined. For example, given two different types in $\mathbf{G}_3$ over the same initial segment, an interesting event we must track is when these two types have an edge to a *common* vertex of the initial segment. This corresponds to the notion of *parallel ones* from [15, 44]. Here, we will broadly call events like this *age changes*, and they form a crucial part of our characterization of exact big Ramsey degrees.

These ideas can be traced back to the work of Sauer and his co-authors who, in a series of papers [12, 22, 23, 31, 41, 46, 48], studied these questions in the special case of vertex colorings. He introduced the poset of ages and used it to characterize the big Ramsey degrees of vertices of many homogeneous structures [46], including the examples which will be discussed in Section 1.2. Characterizing big Ramsey degrees for larger finite substructures requires a generalization of this partial order which we develop in Section 2.

1.2. Organization of the paper

Section 2 introduces the key concept of gluings, which we use to define the higher-dimensional posets of ages. Section 3 defines *diaries*, the tree-like objects which are induced by the key "unavoidable" aspects of the coding trees discussed in Section 1.1 responsible for the big Ramsey degrees. In particular, every diary codes an $\mathcal{L}$-structure embeddable into $\mathbf{K}$, and any diary which codes an $\mathcal{L}$-structure isomorphic to $\mathbf{K}$ can be encoded as an $\mathcal{L}^*$-expansion of $\mathbf{K}$ for some suitable $\mathcal{L}^*$. Using ideas from a recent preprint of Dobrinen and Zucker [21], Proposition 3.4.11 constructs a diary coding $\mathbf{K}$.[1]

The main goal of the rest of the paper is to show that diaries which encode $\mathbf{K}$ are big Ramsey structures for $\mathcal{K}$ which satisfy IRT. The argument has two main components, namely lower and upper bounds for big Ramsey degrees. Section 4 proves the lower bounds by proving the stronger Theorem 3.4.12, which says that any two diaries coding $\mathbf{K}$ are bi-embeddable. Section 5 proves the upper bounds for big Ramsey degrees using the general upper bound theorem from [51]. This along with the bi-embeddability from Theorem 3.4.12 suffice to show that any diary coding $\mathbf{K}$ is a big Ramsey structure for $\mathcal{K}$ satisfying IRT. Lastly, Section 6 collects some open questions and previews some related ongoing work.

[1] This is a key difference between this version of the paper and the first version which appeared on arXiv. It allows us to eliminate the concept of a "diagonal substructure", which dramatically simplifies the upper bound proof, now the content of Section 5.

Throughout, we also develop a few key examples to help illustrate the definitions and the results. These examples are the following:

- Given $\ell \geq 3$, $\mathcal{G}_\ell$ denotes the class of finite K_ℓ-free graphs. We denote its Fraïssé limit by $\mathbf{G}_\ell$. We will especially focus on the case $\ell = 3$.
- $\mathcal{G}_\mathbf{T}$ denotes the class of finite oriented graphs (i.e., no loops and no 2-cycles) which do not embed the oriented 3-cycle, which we denote by $\mathbf{T}$. Write $\mathbf{G}_\mathbf{T}$ for the Fraïssé limit.
  This example differs from $\mathcal{G}_\ell$ since the single unary is *free* (Definition 3.3.2). Roughly speaking, this says that there is a non-trivial way of extending members of the class by one new point, with all old points having the same *non-trivial* relation to the new point, and such that the result stays in the class. Another example of a class with a single unary which is free is $\mathcal{G}$ (Section 1.1), but it is interesting to develop an example with both a free relation and a non-trivial forbidden substructure.
- $\mathcal{H}$ denotes the class of finite graphs with two types of edges, which we call red or blue (i.e., a given pair of vertices can be unrelated, red-related, or blue-related), and which forbid monochromatic triangles. We let $\mathbf{H}$ denote the Fraïssé limit.
  This class differs from both of the above classes in that the (one-dimensional) poset of ages has more than one *maximal path* (Definition 2.3.1). As mentioned earlier in the introduction, Sauer in [46] used the poset of ages, and in particular maximal paths through it, to compute the big Ramsey degrees of vertices. In particular, vertices in $\mathcal{H}$ will have big Ramsey degree 2.

We mention that all of these examples only have a single unary. While we do give the proof in full generality for any number of unary predicates, we opt to keep the examples relatively simple while illustrating key differences between big Ramsey and small Ramsey degrees.

2. $\mathcal{L}$-structures, gluings, and age classes

As indicated in the introduction, understanding the poset of ages is one of the key ingredients in our characterization of big Ramsey degrees. Since members of this poset can be described by which fragments of forbidden structures one can glue to various structures without creating a forbidden structure, we spend this section developing the necessary formalism for being able to do this abstractly.

Until Section 6, $\mathcal{L}$ is a finite relational language with symbols of arity at most two, $\mathcal{F}$ is a finite set of finite irreducible $\mathcal{L}$-structures such that each $\mathbf{F} \in \mathcal{F}$ has size at least 2, $\mathcal{K} = \mathrm{Forb}(\mathcal{F})$, and $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$. Let $\|\mathcal{F}\| = \max\{|F| : \mathbf{F} \in \mathcal{F}\}$. We can arrange, by changing $\mathcal{L}$ and $\mathcal{F}$ as necessary, that all of the following hold for any $\mathbf{A} \in \mathcal{K}$:

- For any $a \in A$, there is exactly one unary predicate $U \in \mathcal{L}$ such that $U^\mathbf{A}(a)$ holds.
- For any $a \in A$ and any binary $R \in \mathcal{L}$, we have $\neg R^\mathbf{A}(a, a)$.
- For any $a \neq b \in A$, there is at most one $R \in \mathcal{L}$ with $R^\mathbf{A}(a, b)$.

- There is a map Flip: $\mathcal{L} \to \mathcal{L}$ such that for $a \neq b \in A$, $R^{\mathbf{A}}(a, b)$ if and only if $\mathrm{Flip}(R)^{\mathbf{A}}(b, a)$.

Writing $\mathcal{L}^{\mathsf{u}} = \{U_i : i < \mathsf{U}\}$ and $\mathcal{L}^{\mathsf{b}} = \{R_i : i < k\}$ for the unary and binary predicates, respectively, we treat $U^{\mathbf{A}}\colon A \to \mathsf{U}$ and $R^{\mathbf{A}}\colon A^2\backslash\{(a, a) : a \in A\} \to k$ as functions, i.e., $U^{\mathbf{A}}(a) = i$ if and only if $U_i^{\mathbf{A}}(a)$ holds, etc.

The relational symbol R_0 will play the role of "no relation", and this is the sense in which we interpret notions such as "irreducible", "free amalgam", etc.

We let $\{V_n : n < \omega\}$ be new unary symbols not in $\mathcal{L}^{\mathsf{u}}$, and we set $\mathcal{L}_d := \mathcal{L}^{\mathsf{b}} \cup \{V_i : i < d\}$. We always assume that the following items hold for any $\mathcal{L}_d$-structure $\mathbf{B}$ that we discuss:

- $B \cap \omega = \emptyset$.
- For any $a \in B$, there is exactly one $i < d$ such that $V_i^{\mathbf{B}}(a)$ holds.
- Conventions for the binary symbols are exactly the same as those for $\mathcal{L}$-structures.

We similarly treat $V^{\mathbf{B}}\colon B \to d$ as a function. Write $\mathrm{Fin}(\mathcal{L}_d)$ for the class of finite $\mathcal{L}_d$-structures.

2.1. Gluings

Our reason for the convention that $B \cap \omega = \emptyset$ whenever $\mathbf{B}$ is an $\mathcal{L}_d$-structure is that we will attach members of $\mathrm{Fin}(\mathcal{L}_d)$ to $\mathcal{L}$-structures to form new $\mathcal{L}$-structures, and we will often consider $\mathcal{L}$-structures whose underlying set is a subset of ω.

Definition 2.1.1. A *gluing* is a triple $\gamma := (\mathbf{X}, \rho, \eta)$, where

- $\mathbf{X}$ is a finite $\mathcal{L}$-structure with $X \subseteq \omega$. We call X the *underlying set* of γ and $\mathbf{X}$ the *structure* of γ.
- There is $d < \omega$ such that $\rho\colon d \to \mathsf{U}$ is a function. We call d the *rank* of γ and ρ the *sort* of γ. More generally, we can call any function from a natural number to U a *sort*.
- $\eta\colon d \times X \to k$ is a function called the *attachment map* of γ.

Given $\rho\colon d \to \mathsf{U}$, we let $\mathrm{Glue}(\rho)$ be the set of gluings with sort ρ, and we let $\widetilde{\rho} \in \mathrm{Glue}(\rho)$ be the gluing $(\emptyset, \rho, \emptyset)$.

Given $\mathbf{B}$ an $\mathcal{L}_d$-structure and $\gamma = (\mathbf{X}, \rho, \eta)$ a rank d gluing, the $\mathcal{L}$-structure $\gamma(\mathbf{B})$ is formed on underlying set $X \cup B$ such that

- $\gamma(\mathbf{B})|_X = \mathbf{X}$.
- The $\mathcal{L}^{\mathsf{b}}$-part of $\gamma(\mathbf{B})$ on B is induced from $\mathbf{B}$.
- If $b \in B$, we have $U^{\gamma(\mathbf{B})}(b) = \rho \circ V^{\mathbf{B}}(b)$.
- If $b \in B$ and $x \in X$, we have $R^{\gamma(\mathbf{B})}(b, x) = \eta(V^{\mathbf{B}}(b), x)$.

Remark. $\mathcal{L}_d$-structures and gluings enable a precise description of the different ways a given finite $\mathcal{L}$-structure can be extended to another $\mathcal{L}$-structure. The $\mathcal{L}_d$-structures are structures $\mathbf{B}$ with binary relations in $\mathcal{L}^{\mathsf{b}}$ with an accompanying partition of B into d-many (possibly empty) labeled pieces $\{B_i : i < d\}$. Given a finite $\mathcal{L}$-structure $\mathbf{X}$, a gluing

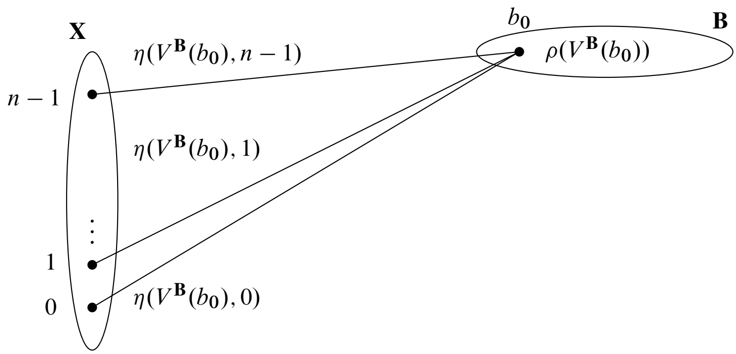

Fig. 1. A gluing $(\mathbf{X}, \rho, \eta)$ with $X = \{0, \ldots, n-1\}$.

$\gamma = (\mathbf{X}, \rho, \eta)$ of rank d is a set of instructions for gluing any $\mathcal{L}_d$-structure $\mathbf{B}$ to $\mathbf{X}$ to obtain an $\mathcal{L}$-structure extending $\mathbf{X}$. Specifically, for each $i < d$, γ assigns each element of B_i the same unary relation $U_{\rho(i)}$ in $\mathcal{L}^{\mathsf{u}}$, and for each x in X, γ assigns the same $\mathcal{L}^{\mathsf{b}}$-relation between x and each element of B_i (see Figure 1).

As the (abuse of) notation suggests, the gluing γ gives rise to a map from the class of $\mathcal{L}_d$-structures to the class of $\mathcal{L}$-structures. This suggests the following convention.

Convention. Given a class $\mathcal{C}$ of $\mathcal{L}'$-structures ($\mathcal{L}' \in \{\mathcal{L}, \mathcal{L}_d\}$), we identify $\mathcal{C}$ with its characteristic function, i.e., $\mathcal{C}(\mathbf{A}) = 1$ if and only if $\mathbf{A} \in \mathcal{C}$.

Definition 2.1.2. Given a gluing γ of rank d, we define the class of finite $\mathcal{L}_d$-structures

$$\mathcal{K}\cdot\gamma := \{\mathbf{B} \in \mathrm{Fin}(\mathcal{L}_d) : \gamma(\mathbf{B}) \in \mathcal{K}\}.$$

Note that by treating $\mathcal{K}$ and $\mathcal{K}\cdot\gamma$ as characteristic functions, the notation becomes quite suggestive, i.e., $(\mathcal{K}\cdot\gamma)(\mathbf{B}) = \mathcal{K}(\gamma(\mathbf{B}))$.

A rank d gluing γ *admits all unaries* if $\mathcal{K}\cdot\gamma$ contains every singleton $\mathcal{L}_d$-structure. In other words, writing $\gamma = (\mathbf{X}, \rho, \eta)$, then $\mathbf{X} \in \mathcal{K}$ and for each $i < d$, we can extend $\mathbf{X}$ to a new member of $\mathcal{K}$ by adding one new vertex with unary $\rho(i)$ and connecting this new vertex to $\mathbf{X}$ with binary relations determined by $\eta|_{\{i\}\times X}$.

Given a sort $\rho\colon d \to \mathsf{U}$, we set

$$P(\rho) := \{\mathcal{K}\cdot\gamma : \gamma \in \mathrm{Glue}(\rho) \text{ admits all unaries}\}.$$

We treat $P(\rho)$ as a partial order under inclusion. In the case $\mathsf{U} = 1$, there is a unique sort with domain d, so we just write $P(d)$.

Remark. When $\mathsf{U} = 1$ and we discuss $P(1)$, it is helpful to keep in mind that even though members of $P(1)$ are classes of $\mathcal{L}_1$-structures, each such class is bi-interpretable with a subclass of $\mathcal{K}$, where the interpretation interchanges the unary U_0 with the unary V_0. In particular, the maximal member of $P(1)$ is always bi-interpretable with $\mathcal{K}$. Going forward, we will perform these bi-interpretations without explicit mention, and simply regard members of $P(1)$ as subclasses of $\mathcal{K}$.

Example 2.1.3. Let $\mathcal{K} = \mathcal{G}_{\mathbf{T}}$ be the class of *oriented* graphs which do not embed the oriented 3-cycle $\mathbf{T}$ as in Section 1.2. Here $\mathsf{U} = 1$ and $k = 3$, with $\mathrm{Flip}(1) = 2$ and $\mathrm{Flip}(2) = 1$. The poset $P(1)$ contains only $\mathcal{G}_{\mathbf{T}}$ (up to bi-interpretability), while the poset $P(2)$ contains 4 members – the class of finite $\mathbf{T}$-free oriented graphs $\mathbf{B}$ with $V_0^{\mathbf{B}}$, $V_1^{\mathbf{B}}$ arbitrary; the class of finite $\mathbf{T}$-free oriented graphs $\mathbf{B}$ with no edges from $V_0^{\mathbf{B}}$ to $V_1^{\mathbf{B}}$; the class of finite $\mathbf{T}$-free oriented graphs $\mathbf{B}$ with no edges from $V_1^{\mathbf{B}}$ to $V_0^{\mathbf{B}}$, and the class of finite $\mathbf{T}$-free oriented graphs $\mathbf{B}$ with no edges between $V_0^{\mathbf{B}}$ and $V_1^{\mathbf{B}}$. We can encode these 4 classes via the 4 *directed* graphs (i.e., no loops, but 2-cycles are ok) on vertex set 2. Because $\mathbf{T}$ has 3 vertices, we have that for $d \geq 3$, $P(d)$ is determined by what is allowed on any pair $V_i^{\mathbf{B}}$, $V_j^{\mathbf{B}}$. Hence the members of $P(d)$ are in one-to-one correspondence with *directed* graphs on vertex set d.

Proposition 2.1.4. *For any sort $\rho\colon d \to \mathsf{U}$, $P(\rho)$ is finite.*

Proof. Fix $\gamma = (\mathbf{X}, \rho, \eta) \in \mathrm{Glue}(\rho)$. Observe that since $\mathcal{K}$ is a free amalgamation class, so is $\mathcal{K}\cdot\gamma$. Thus $\mathcal{K}\cdot\gamma = \mathrm{Forb}(\mathcal{E})$ for some set $\mathcal{E}$ of finite irreducible $\mathcal{L}_d$-structures. We can assume that distinct $\mathbf{A}, \mathbf{B} \in \mathcal{E}$ do not embed into one another. So fix $\mathbf{B} \in \mathcal{E}$. As $\gamma(\mathbf{B}) \notin \mathcal{K}$, find $\mathbf{F} \subseteq \gamma(\mathbf{B})$ isomorphic to a member of $\mathcal{F}$. Towards showing that $B \subseteq F$, fix $b \in B$, and consider $\mathbf{C} \subsetneq \mathbf{B}$ induced on $B\setminus\{b\}$. By our assumption on $\mathcal{E}$, we have $\gamma(\mathbf{C}) \in \mathcal{K}$. In particular, $F \nsubseteq C \cup X$. As $F \subseteq B \cup X$, it follows that we must have $b \in F$, so $B \subseteq F$ as desired. Hence up to isomorphism, there are only finitely many possibilities for $\mathbf{B}$, so also for $\mathcal{E}$. ■

Since $\mathcal{K}$ has free amalgamation, the following construction on gluings is quite natural.

Definition 2.1.5. We call $\gamma_0 = (\mathbf{X}_0, \rho, \eta_0)$ and $\gamma_1 = (\mathbf{X}_1, \rho, \eta_1) \in \mathrm{Glue}(\rho)$ *disjoint* if $X_0 \cap X_1 = \emptyset$. Given disjoint $\gamma_0, \gamma_1 \in \mathrm{Glue}(\rho)$ as above, the *union gluing* $\gamma_0 \cup \gamma_1$ is the gluing $(\mathbf{X}_0 \sqcup \mathbf{X}_1, \rho, \eta_0 \cup \eta_1) \in \mathrm{Glue}(\rho)$, where $\mathbf{X}_0 \sqcup \mathbf{X}_1$ is the $\mathcal{L}$-structure on $X_0 \cup X_1$ with $\mathbf{X}_0$ and $\mathbf{X}_1$ as induced substructures and with $R^{\mathbf{X}_0 \sqcup \mathbf{X}_1}(x_0, x_1) = 0$ for every $x_0 \in X_0$ and $x_1 \in X_1$.

Lemma 2.1.6. *For any sort ρ and disjoint gluings $\gamma_0, \gamma_1 \in \mathrm{Glue}(\rho)$, then writing $\gamma = \gamma_0 \cup \gamma_1$, we have $\mathcal{K}\cdot\gamma = \mathcal{K}\cdot\gamma_0 \cap \mathcal{K}\cdot\gamma_1$.*

Proof. This is an immediate consequence of free amalgamation in $\mathcal{K}$. ■

Proposition 2.1.7. *For any sort $\rho\colon d \to \mathsf{U}$, $P(\rho)$ is closed under intersections.*

Proof. As $P(\rho)$ is finite, it is enough to consider $\mathcal{A}, \mathcal{B} \in P(\rho)$ and show that $\mathcal{A} \cap \mathcal{B} \in P(\rho)$. Let $\gamma_{\mathcal{A}} = (\mathbf{X}_{\mathcal{A}}, \rho, \eta_{\mathcal{A}})$ and $\gamma_{\mathcal{B}} = (\mathbf{X}_{\mathcal{B}}, \rho, \eta_{\mathcal{A}})$ be disjoint gluings with $\mathcal{A} = \mathcal{K}\cdot\gamma_{\mathcal{A}}$, and $\mathcal{B} = \mathcal{K}\cdot\gamma_{\mathcal{B}}$. Form $\gamma_{\mathcal{A}} \cup \gamma_{\mathcal{B}}$ and apply Lemma 2.1.6. ■

Example 2.1.8. Fix $\ell \geq 3$ and consider the class $\mathcal{G}_\ell$ of K_ℓ-free graphs. So $\mathsf{U} = 1$ and $k = 2$. Given $d < \omega$, let $\binom{d}{<\ell}$ denote the collection of subsets of $\{0, \ldots, d-1\}$ of size at most $\ell - 1$. Each member of $P(d)$ is described by a function $z\colon \binom{d}{<\ell} \to \{1, 2, \ldots, \ell - 1\}$ satisfying the following properties:

(1) z is monotone: if $S, T \in \binom{d}{<\ell}$ and $S \subseteq T$, then $z(S) \leq z(T)$,

(2) z is consistent: for each $T \in \binom{d}{<\ell}$, there is a function $\sigma\colon T \to \omega$ with $\sum_{i \in T} \sigma(i) = z(T)$ and such that for any $S \subseteq T$, we have $\sum_{i \in S} \sigma(i) \leq z(S)$.

We call a function z satisfying the above two items a (d, ℓ)*-graph-age function*, and an ℓ*-graph-age function* is just a (d, ℓ)-graph-age function for some $d < \omega$. Given a (d, ℓ)-graph-age function z, we define the class $\mathcal{G}_z$ via

$$\mathcal{G}_z := \left\{\mathbf{B} \in \mathrm{Fin}(\mathcal{L}_d) : \forall T \in \binom{d}{<\ell}\Big[\bigcup_{i \in T} V_i^{\mathbf{B}} \text{ contains no } (z(T)+1)\text{-clique}\Big]\right\}.$$

So the number $z(T)$ describes the largest clique which is allowed to appear in $\bigcup_{i \in T} V_i^{\mathbf{B}}$.

We sketch the argument that $P(d)$ is indeed characterized by classes of this form. First, fix a (d, ℓ)-graph-age function z. To see that $\mathcal{G}_z \in P(d)$, build a gluing $\gamma = (\mathbf{X}, \rho, \eta)$ as follows. For each $T \in \binom{d}{<\ell}$, add an $(\ell - 1 - z(T))$-clique $\mathbf{X}_T$ to $\mathbf{X}$, so that

$$X = \bigsqcup_{T \in \binom{d}{<\ell}} X_T.$$

Then define $\eta\colon d \times X \to 2$ by setting $\eta(i, x) = 1$ if and only if for some $T \in \binom{d}{<\ell}$ we have $i \in T$ and $x \in X_T$. It is straightforward to check that $\mathcal{G}_\ell \cdot \gamma = \mathcal{G}_z$. Conversely, if $\mathcal{A} \in P(d)$, then letting $z_{\mathcal{A}}\colon \binom{d}{<\ell} \to \{1, \ldots, \ell - 1\}$ be such that $z_{\mathcal{A}}(T)$ describes the largest possible clique in $\mathcal{A}$ with unaries from T, then one can check that $\mathcal{A} = \mathcal{G}_{z_{\mathcal{A}}}$.

Unfortunately, checking whether a given function z satisfies item (2) of the definition of (d, ℓ)-graph-age function seems to be a challenging combinatorial problem. When $\ell = 3$, however, item (1) implies item (2). We briefly mention that since the single unary in each $\mathcal{G}_\ell$ is not *free* (Definition 3.3.2), the subset of $P(d)$ corresponding to those (d, ℓ)-graph-age functions with $z(\{i\}) \leq \ell - 2$ for each $i < d$ will be of special importance. When $\ell = 3$, we can think of this subset of (d, ℓ)-graph-age functions as graphs; each singleton must get value 1, and for pairs, we can think of value 2 as an edge and value 1 as a non-edge. Conversely, any graph on vertex set d can be encoded by a $(d, 3)$-graph-age function in this way.

2.2. *Manipulating classes of structures*

When we introduce aged coding trees in Section 3, each level set of size d from such a tree will be endowed with a class of $\mathcal{L}_d$-structures. Passing to a subset of this level set will then correspond to a restriction operation on the class. Using a general form of this restriction operation, we discuss how to shrink a given class of $\mathcal{L}_d$-structures along some $S \subseteq d$ to obtain a new, smaller class of $\mathcal{L}_d$-structures. We then analyze what it means for such an operation to result in a *consecutive* pair of classes.

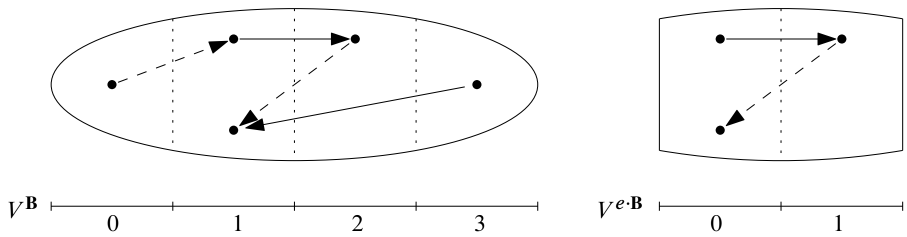

Fig. 2. An example of forming $e{\cdot}\mathbf{B}$ from e and $\mathbf{B}$. Left: $\mathcal{L}_4$-structure $\mathbf{B}$, right: $\mathcal{L}_2$-structure $e \cdot \mathbf{B}$, where $e\colon 4 \rightharpoonup 2$ with $\mathrm{dom}(e) = \{1, 2\}$, $e(1) = 0$, $e(2) = 1$.

Convention. When discussing partial functions $e\colon d_0 \rightharpoonup d$ with $d_0, d < \omega$, we think of e as equipped with the knowledge of what d_0 is, even if $\mathrm{dom}(e) \subsetneq d_0$, and of what d is, even if e is not surjective.

Definition 2.2.1. Let $d_0, d < \omega$ and let $e\colon d_0 \rightharpoonup d$ be a partial function. If $\mathbf{B}$ is an $\mathcal{L}_{d_0}$-structure, then $e{\cdot}\mathbf{B}$ is the $\mathcal{L}_d$-structure on the underlying set $\{x \in \mathbf{B} : V^{\mathbf{B}}(x) \in \mathrm{dom}(e)\}$, with $\mathcal{L}^{\mathrm{b}}$-part induced from $\mathbf{B}$, and with $V^{e\cdot\mathbf{B}}(x) = e(V^{\mathbf{B}}(x))$. In other words, we take $\mathbf{B}$, throw away the points whose unary is not in $\mathrm{dom}(e)$, then relabel the unaries according to the function e (see Figure 2). If $\mathcal{A}$ is a class of finite $\mathcal{L}_d$-structures, we set

$$\mathcal{A}{\cdot}e := \{\mathbf{B} \in \mathrm{Fin}(\mathcal{L}_{d_0}) : e{\cdot}\mathbf{B} \in \mathcal{A}\}.$$

Treating $\mathcal{A}$ and $\mathcal{A}{\cdot}e$ as characteristic functions, we have $(\mathcal{A}{\cdot}e)(\mathbf{B}) = \mathcal{A}(e{\cdot}\mathbf{B})$.

Notation 2.2.2. Given an injection $e\colon d_0 \to d$, we treat $e^{-1}\colon d \rightharpoonup d_0$ as a partial function with domain $\mathrm{Im}(e)$. By abusing notation, we can identify a finite set $S \subseteq \omega$ with its increasing enumeration. When doing this, we interpret S as a total function with domain $|S|$ and with codomain given from context, i.e., writing $S \subseteq d$ indicates that the codomain is d. If $S \subseteq d$, $\mathcal{A}$ is a class of finite $\mathcal{L}_d$-structures, and keeping in mind the convention before Definition 2.2.1, then we can write $\mathcal{A}{\cdot}S$, $\rho \circ S$, etc. We write $\mathcal{A}^i$ for $\mathcal{A}{\cdot}\{i\}$. We write $\mathrm{id}_{S,d}\colon d \rightharpoonup d$ for the partial function with domain S which is the identity on S.

Proposition 2.2.3. *Let $d < \omega$, and fix a sort $\rho\colon d \to \mathsf{U}$ and $\mathcal{A} \in P(\rho)$.*

(1) *If $d_0 < \omega$ and $e\colon d_0 \to d$ is a total function, then $\mathcal{A}{\cdot}e \in P(\rho \circ e)$.*

(2) *If $f\colon d \rightharpoonup d_0$ is a partial function, $\theta\colon d_0 \to \mathsf{U}$ is a sort satisfying $\theta \circ f = \rho|_{\mathrm{dom}(f)}$, and $\mathcal{B} \in P(\theta)$, then $\mathcal{A} \cap \mathcal{B}{\cdot}f \in P(\rho)$.*

Proof. Let $\mathcal{A} = \mathcal{K} \cdot \gamma_{\mathcal{A}}$ and $\mathcal{B} = \mathcal{K} \cdot \gamma_{\mathcal{B}}$, where $\gamma_{\mathcal{A}} = (\mathbf{X}_{\mathcal{A}}, \rho, \eta_{\mathcal{A}})$ and $\gamma_{\mathcal{B}} = (\mathbf{X}_{\mathcal{B}}, \theta, \eta_{\mathcal{B}})$ both admit all unaries. We can assume that $X_{\mathcal{A}} \cap X_{\mathcal{B}} = \emptyset$.

To see that $\mathcal{A}{\cdot}e \in P(\rho \circ e)$, we can define the gluing $\gamma_{\mathcal{A}}{\cdot}e = (\mathbf{X}_{\mathcal{A}}, \rho \circ e, \eta_{\mathcal{A}}{\cdot}e)$, where given $c < d_0$ and $x \in X_{\mathcal{A}}$, we have $(\eta_{\mathcal{A}}{\cdot}e)(c, x) = \eta_{\mathcal{A}}(e(c), x)$. It is routine to check that $\gamma_{\mathcal{A}}{\cdot}e$ is a gluing which admits all unaries and with $\mathcal{K}{\cdot}(\gamma_{\mathcal{A}}{\cdot}e) = (\mathcal{K}{\cdot}\gamma_{\mathcal{A}}){\cdot}e = \mathcal{A}{\cdot}e$.

To see that $\mathcal{A} \cap \mathcal{B}\cdot f \in P(\rho)$, first form the gluing $\gamma_{\mathcal{B}}\cdot f = (\mathbf{X}_{\mathcal{B}}, \rho, \eta_{\mathcal{B}}\cdot f)$, where given $c < d$ and $x \in X_{\mathcal{B}}$, we have

$$(\eta_{\mathcal{B}}\cdot f)(c,x) = \begin{cases} 0 & \text{if } c \notin \mathrm{dom}(f), \\ \eta_{\mathcal{B}}(f(c),x) & \text{if } c \in \mathrm{dom}(f). \end{cases}$$

Letting $\gamma = \gamma_{\mathcal{A}} \cup \gamma_{\mathcal{B}}\cdot f$ denote the union gluing, it is routine to check that γ admits all unaries and $\mathcal{K}\cdot\gamma = \mathcal{A} \cap \mathcal{B}\cdot f$. ∎

Definition 2.2.4. Fix a sort $\rho: d \to \mathsf{U}$. We say that $\mathcal{A} \supseteq \mathcal{B} \in P(\rho)$ are *consecutive* if there is no $\mathcal{C} \in P(\rho)$ with $\mathcal{A} \supsetneq \mathcal{C} \supsetneq \mathcal{B}$. Write $\mathrm{Con}(\rho) = \{(\mathcal{A},\mathcal{B}) \in P(\rho)^2 : \mathcal{A} \supseteq \mathcal{B}$ are consecutive$\}$. By Proposition 2.2.3, if $(\mathcal{A},\mathcal{B}) \in \mathrm{Con}(\rho)$ and $S \subseteq d$, then either $(\mathcal{A}\cdot S, \mathcal{B}\cdot S) \in \mathrm{Con}(\rho)$ or $\mathcal{A}\cdot S = \mathcal{B}\cdot S$.

We say that $(\mathcal{A},\mathcal{B}) \in \mathrm{Con}(\rho)$ is *essential* on some $S \subseteq d$ if there is a minimal under inclusion $\mathbf{C} \in \mathcal{A}\backslash\mathcal{B}$ such that $S = \{V^{\mathbf{C}}(x) : x \in C\}$. We say that $(\mathcal{A},\mathcal{B}) \in \mathrm{Con}(\rho)$ is *essential* if it is essential on d. Write $\mathrm{ECon}(\rho) = \{(\mathcal{A},\mathcal{B}) \in \mathrm{Con}(\rho) : (\mathcal{A},\mathcal{B})$ is essential$\}$.

Fact 2.2.5. By arguments very similar to those of Proposition 2.1.4, such a $\mathbf{C}$ must satisfy $|\mathbf{C}| < \|\mathcal{F}\|$, so in particular, $|S| < \|\mathcal{F}\|$.

Proposition 2.2.6. *With notation as in Definition 2.2.4, if $(\mathcal{A},\mathcal{B}) \in \mathrm{Con}(\rho)$, then there is a unique $S \subseteq d$ on which $(\mathcal{A},\mathcal{B})$ is essential. Fixing this S, if $S' \subseteq d$ and $S \subseteq S'$, then $(\mathcal{A}\cdot S', \mathcal{B}\cdot S') \in \mathrm{Con}(\rho \circ S')$, and if $S \not\subseteq S'$, we have $\mathcal{A}\cdot S' = \mathcal{B}\cdot S'$.*

Proof. Towards a contradiction, suppose $(\mathcal{A},\mathcal{B})$ is essential on both $S_0 \neq S_1 \subseteq d$. Then for each $i < 2$, we have $\mathcal{A} \supsetneq \mathcal{A} \cap (\mathcal{B}\cdot \mathrm{id}_{S_i,d}) \supseteq \mathcal{B}$, so as $(\mathcal{A},\mathcal{B}) \in \mathrm{Con}(\rho)$, we have $\mathcal{B} = \mathcal{A} \cap (\mathcal{B}\cdot \mathrm{id}_{S_i,d})$. In particular, if $\mathcal{A} = \mathcal{K}\cdot\gamma$ for some gluing γ, then as in the proof of Proposition 2.2.3, we can for each $i < 2$ write $\mathcal{B} = \mathcal{K}\cdot(\gamma \cup \delta_i)$, where $\delta_i = (\mathbf{X}_i, \rho, \eta_i)$ is a gluing with the property that $\eta_i(c,x) = 0$ whenever $c \notin S_i$.

Now without loss of generality, assume that $S_0\backslash S_1 \neq \emptyset$, and let $\mathbf{C}_0 \in \mathcal{A}\backslash\mathcal{B}$ witness that $(\mathcal{A},\mathcal{B})$ is essential on S_0. However, considering the gluing $\gamma \cup \delta_1$, and since $\mathcal{K}$ has free amalgamation, we see that if $\mathbf{C}_0 \notin \mathcal{B} = \mathcal{K}(\gamma \cup \delta_1)$, then a proper substructure of $\mathbf{C}_0$ (namely, the substructure induced on points with unary in S_1) must also fail to be in $\mathcal{B}$, that is a contradiction.

The remaining claims about $S' \subseteq d$ are straightforward. ∎

2.3. Paths and path sorts

Given $i < \omega$, let $\iota_i : 1 \to \omega$ be the function with $\iota_i(0) = i$. Sometimes the intended codomain of ι_i is some $d < \omega$ (for the purposes of forming the partial function ι_i^{-1}), and this will be clear from the context. If $i < \mathsf{U}$, then ι_i is also a sort, and we write P_i for $P(\iota_i)$.

Definition 2.3.1. A *path* through P_i is any subset $\mathfrak{p} \subseteq P_i$ linearly ordered by inclusion. Write Path_i for the set of paths through P_i and $\mathsf{Path} := \bigsqcup_{i<\mathsf{U}} \mathsf{Path}_i$. A *maximal* path is maximal under inclusion; write MP_i for the set of maximal paths through P_i and $\mathsf{MP} :=$

$\bigsqcup_{i<\mathsf{U}} \mathsf{MP}_i$. We equip MP with an arbitrary linear order $\leq_{\mathsf{MP}}$. We write $\mathsf{u}\colon \mathsf{MP} \to \mathsf{U}$ for the map sending $\mathfrak{p} \in \mathsf{MP}$ to the $i < \mathsf{U}$ with $\mathfrak{p} \in \mathsf{MP}_i$.

A *full* path is an initial segment of a maximal path; write Full_i for the set of full paths through P_i and $\mathsf{Full} := \bigsqcup_{i<\mathsf{U}} \mathsf{Full}_i$. Given $\mathfrak{p} \in \mathsf{Full}_i$, $\max(\mathfrak{p})$ and $\min(\mathfrak{p})$ denote the maximal and minimal members of $\mathfrak{p}$ under inclusion. We note that $\min(\mathfrak{p}) = \min(P_i) = \bigcap_{\mathcal{A}\in P_i} \mathcal{A}$ (Proposition 2.1.7). When $\mathfrak{p} \in \mathsf{MP}_i$, then $\max(\mathfrak{p}) = \max(P_i) = \mathcal{K}\cdot\tilde{\iota}_i$, where we recall that $\tilde{\iota}_i$ is the gluing $(\emptyset, \iota_i, \emptyset)$. Up to bi-interpretation, $\mathcal{K}\cdot\tilde{\iota}_i$ is the class of all structures in $\mathcal{K}$ with all vertices having unary relation i. Given $\mathfrak{p} \in \mathsf{Full}_i$, we write $\mathfrak{p}' \in \mathsf{Full}_i$ for the full path $\mathfrak{p}\setminus\{\max(\mathfrak{p})\}$.

Example 2.3.2. Let $\mathcal{H}$ be as defined in Section 1.2. So $\mathsf{U} = 1$ and $k = 3$, where 1 and 2 represent blue and red edges, respectively. Then $P(1) = P_0$ has four elements, which up to bi-interpretability are the following:

- $\mathcal{H}$,
- $\mathcal{H}_r$, the class of finite, triangle-free graphs with all red edges,
- $\mathcal{H}_b$, the class of finite, triangle-free graphs with all blue edges,
- $\mathcal{H}_0$, the class of graphs with no edges.

We have $\mathsf{MP} = \mathsf{MP}_0 = \{\mathfrak{p}_r, \mathfrak{p}_b\}$, where $\mathfrak{p}_r := \{\mathcal{H}_0, \mathcal{H}_r, \mathcal{H}\}$ and $\mathfrak{p}_b := \{\mathcal{H}_0, \mathcal{H}_b, \mathcal{H}\}$. We declare that $\mathfrak{p}_r \leq_{\mathsf{MP}} \mathfrak{p}_b$.

For $d > 1$, a typical element of $P(d)$ is described by a pair of $(d, 3)$-graph-age functions as in Example 2.1.8, one for red cliques and one for blue cliques. While the single unary in the class $\mathcal{H}$ also is not free (Definition 3.3.2), we postpone the description of which subset of $P(d)$ will be of interest until Example 3.4.6.

It will be useful to work with a more general notion of sort.

Definition 2.3.3. A *path sort* is a function $\rho\colon d \to \mathsf{MP}$ for some $d < \omega$. We set

$$\widetilde{\rho} := (\emptyset, \mathsf{u}\circ\rho, \emptyset) \in \mathrm{Glue}(\mathsf{u}\circ\rho),$$
$$P(\rho) := \{\mathcal{A} \in P(\mathsf{u}\circ\rho) : \forall\, i < d\ (\mathcal{A}^i \in \rho(i))\}.$$

Note that if two elements of $P(\rho)$ are consecutive in $P(\rho)$, then they are also consecutive in $P(\mathsf{u}\circ\rho)$. Thus we can unambiguously define $\mathrm{Con}(\rho) = \mathrm{Con}(\mathsf{u}\circ\rho) \cap P(\rho)^2$ and $\mathrm{ECon}(\rho) = \mathrm{ECon}(\mathsf{u}\circ\rho) \cap P(\rho)^2$.

3. Lower bounds: Diaries

Given $\mathbf{A} \in \mathcal{K}$, recall that a finite coloring γ of $\mathrm{Emb}(\mathbf{A}, \mathbf{K})$ is *unavoidable* if for every $\eta \in \mathrm{Emb}(\mathbf{K}, \mathbf{K})$, we have $\mathrm{Im}(\gamma\circ\eta) = \mathrm{Im}(\gamma)$. In particular, $\mathrm{BRD}(\mathbf{A}, \mathcal{K}) \geq \ell$ if and only if there is an unavoidable coloring $\gamma\colon \mathrm{Emb}(\mathbf{A}, \mathbf{K}) \to \ell$. This section will produce a finite relational language $\mathcal{L}^* \supseteq \mathcal{L}$ and an $\mathcal{L}^*$-expansion $\mathbf{K}^*$ of $\mathbf{K}$ such that for every $\mathbf{A} \in \mathcal{K}$, the map $\chi_{\mathbf{A}}\colon \mathrm{Emb}(\mathbf{A}, \mathbf{K}) \to \mathbf{K}^*(\mathbf{A})$ given by $\chi_{\mathbf{A}}(f) = \mathbf{K}^*\cdot f$ is unavoidable (Definition 2), showing that $\mathrm{BRD}(\mathbf{A}, \mathcal{K}) \geq |\mathbf{K}^*(\mathbf{A})|$. To do this, we introduce a method of coding structures embeddable into $\mathbf{K}$ using a tree-like object called a *diary*.

3.1. Conventions about trees

Let $(L, \leq_L)$ be either $(\mathsf{U}, \leq)$ or $(\mathsf{MP}, \leq_{\mathsf{MP}})$. Given $t \in L \times k^{<\omega}$, we write $t = (t^{\mathsf{p}}, t^{\mathsf{seq}})$ with $t^{\mathsf{p}} \in L$ and $t^{\mathsf{seq}} \in k^{<\omega}$. If $L = \mathsf{MP}$, write $t^{\mathsf{u}} := \mathsf{u}(t^{\mathsf{p}})$, and if $L = \mathsf{U}$, set $t^{\mathsf{u}} = t^{\mathsf{p}}$ (here p refers to "path" and u refers to "unary"). Write $\ell(t) = \mathrm{dom}(t^{\mathsf{seq}})$, which we call the *level* of t, and given $m < \ell(t)$, we often abuse notation and write $t(m)$ for $t^{\mathsf{seq}}(m)$. Similarly abusing notation, if $q < k$, we write $t^\frown q$ for $(t^{\mathsf{p}}, (t^{\mathsf{seq}})^\frown q)$.

Definition 3.1.1. We define the partial orders $\leq_\ell$, $\sqsubseteq$ and $\preceq_{\mathrm{lex}}$, and the partial binary operation $\wedge$ on $L \times k^{<\omega}$; let $s, t \in L \times k^{<\omega}$.

(1) We write $s \leq_\ell t$ if and only if $\ell(s) \leq \ell(t)$.

(2) If $m < \ell(s)$, we write $s|_m$ or $\pi_m(s)$ for the initial segment of s at level m. We write $s \sqsubseteq t$ if $s^{\mathsf{p}} = t^{\mathsf{p}}$ and $s^{\mathsf{seq}} = t^{\mathsf{seq}}|_{\ell(s)}$.

(3) We set $s \preceq_{\mathrm{lex}} t$ if and only if $s^{\mathsf{p}} <_L t^{\mathsf{p}}$ or $s^{\mathsf{p}} = t^{\mathsf{p}}$ and $s^{\mathsf{seq}} \preceq_{\mathrm{lex}} t^{\mathsf{seq}}$ (where the latter $\preceq_{\mathrm{lex}}$ is the usual lexicographic order on $k^{<\omega}$).

(4) If $s^{\mathsf{p}} = t^{\mathsf{p}}$, we write $s \wedge t$ for the largest common initial segment of s and t.

A *level subset* of $L \times k^{<\omega}$ is simply a subset of $L \times k^m$ for some $m < \omega$. Given a level subset $S = \{s_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} s_{d-1}\} \subseteq L \times k^m$, write $\ell(S) = m$ and $\mathrm{Sort}(S)\colon d \to L$ for the sort or path sort (depending on L) given by $\mathrm{Sort}(S)(i) = s_i^{\mathsf{p}}$.

If $X \subseteq L \times k^{<\omega}$, then the $\sqsubseteq$-downwards closure of X is denoted by $X{\downarrow}$.

A *subtree* $\Delta \subseteq L \times k^{<\omega}$ is any $\sqsubseteq$-downwards-closed subset; thus necessarily subtrees are $\wedge$-closed as well. The *height* of Δ is $\mathrm{ht}(\Delta) := \{\ell(t) : t \in \Delta\} \leq \omega$. Given $m < \mathrm{ht}(\Delta)$, we put $\Delta(m) = \{t \in \Delta : \ell(t) = m\}$. Given $x \in L$, we write $\Delta_x = \{t \in \Delta : t^{\mathsf{p}} = x\}$ and $\Delta_x(m) = \Delta_x \cap \Delta(m)$, and given $i < \mathsf{U}$, write $\Delta_i = \{t \in \Delta : t^{\mathsf{u}} = i\}$ and $\Delta_i(m) = \Delta_i \cap \Delta(m)$. Given $m < n < \mathrm{ht}(\Delta)$ and $S \subseteq \Delta(m)$, we put $\mathrm{Succ}_\Delta(S, n) = \{t \in \Delta(n) : t \sqsupset s \text{ for some } s \in S\}$; if $n = m + 1$, we write $\mathrm{IS}_\Delta(S)$ for $\mathrm{Succ}_\Delta(S, m + 1)$; these are the *immediate successors* of S in Δ. When $S = \{s\}$, we simply write $\mathrm{Succ}(s, n)$ or $\mathrm{IS}_\Delta(s)$. A *splitting node* of Δ is any $t \in \Delta$ with $|\mathrm{IS}_\Delta(t)| > 1$; write $\mathrm{SpNd}(\Delta) = \{t \in \Delta : t \text{ a splitting node of } \Delta\}$. Given $s \in \Delta$ and $n > \ell(s)$, we put $\mathrm{Left}_\Delta(s, n)$ to be the $\preceq_{\mathrm{lex}}$-least $t \in \Delta(n)$ with $t \sqsupset s$. Similarly, if $S \subseteq \Delta$ and $n > \ell(s)$ for each $s \in S$, we can write $\mathrm{Left}_\Delta(S, n) = \{\mathrm{Left}_\Delta(s, n) : s \in S\}$. If the Δ subscript is omitted in various notations, we intend $\Delta = L \times k^{<\omega}$.

Definition 3.1.2. An *aged coding tree* is a subtree $\Delta \subseteq L \times k^{<\omega}$, where additionally

- We designate a subset $\mathrm{CdNd}(\Delta) \subseteq \Delta$ of *coding nodes* such that for every $m < \mathrm{ht}(\Delta)$, we have $|\mathrm{CdNd}(\Delta) \cap \Delta(m)| \leq 1$. We write $c^\Delta\colon |\mathrm{CdNd}(\Delta)| \to \mathrm{CdNd}(\Delta)$ for the function which enumerates the coding nodes in increasing height and $\ell^\Delta(n)$ for $\ell(c^\Delta(n))$.
- We assign to each $m < \mathrm{ht}(\Delta)$ a class of $\mathcal{L}_{|\Delta(m)|}$-structures in $P(\mathrm{Sort}(\Delta(m)))$ which we denote by $\mathrm{Age}_\Delta(m)$.

If $S \subseteq \Delta(m) = \{s_0 \preceq_{\text{lex}} \cdots \preceq_{\text{lex}} s_{d-1}\}$ and $I = \{i < d : s_i \in S\}$, we can write $\text{Age}_\Delta(S)$ for $\text{Age}_\Delta(m)\cdot I$, and if $d_0 < \omega$ and $e\colon d_0 \to \Delta(m)$ is any function, we define $\text{Age}_\Delta(e) = \text{Age}_\Delta(m)\cdot\overline{e}$, where $\overline{e}$ is defined to satisfy $e(i) = s_{\overline{e}(i)}$. If $s \in \Delta$, we write $\text{Age}_\Delta(s)$ for $\text{Age}_\Delta(\{s\})$. We remark that $\text{Age}_\Delta(s) \in s^{\mathsf{p}}$. We let $\text{Path}_\Delta(s) = \{\text{Age}_\Delta(s|_m) : m \leq \ell(s)\}$; for the aged coding trees we consider in this paper, we will always have $\text{Path}_\Delta(s) \in \mathsf{Path}_i$.

Definition 3.1.3. Given an aged coding tree Δ and $X \subseteq \Delta$, we define the set $\text{Crit}^\Delta(X)$ of *critical levels* of X in Δ by declaring that $m \in \text{Crit}^\Delta(X)$ if and only if any of the following happens:

- There is $x \in X$ with $m = \ell(x)$.
- There are $x, y \in X$ with $m = \ell(x \wedge y)$.
- $\text{Age}_\Delta(\pi_m[X]) \neq \text{Age}_\Delta(\pi_{m+1}[X])$.

Note that the above events need not be mutually exclusive.

Definition 3.1.4. Given aged coding trees $\Theta, \Delta \subseteq L \times k^{<\omega}$ (the same L for both Θ and Δ), an *aged embedding* $\varphi\colon \Theta \to \Delta$ is an injection satisfying all of the following:

(1) φ preserves $\sqsubseteq$, $\wedge$, $\preceq_{\text{lex}}$, and $\leq_\ell$ (in both the positive and negative sense). Write $\widetilde{\varphi}\colon \text{ht}(\Theta) \to \text{ht}(\Delta)$ for the function satisfying $\varphi[\Theta(m)] \subseteq \Delta(\widetilde{\varphi}(m))$.

(2) $\varphi[\text{CdNd}(\Theta)] \subseteq \text{CdNd}(\Delta)$.

(3) For each $x \in L$, $\varphi[\Theta_x] \subseteq \Delta_x$.

(4) For each $t \in \Theta$ and $m < \ell(t)$, we have $t(m) = \varphi(t)(\widetilde{\varphi}(m))$.

(5) For each $m < \text{ht}(\Theta)$, $\text{Age}_\Theta(m) = \text{Age}_\Delta(\varphi[\Theta(m)])$.

Write $\text{AEmb}(\Theta, \Delta)$ for the set of aged embeddings from Θ to Δ.

In this paper, we work with two types of aged coding trees. First, we recall that an *enumerated structure* is simply a structure $\mathbf{A}$ with $A = |A|$. The following is a mild modification of [51, Definition 2.1].

Definition 3.1.5. Given an enumerated structure $\mathbf{A} \leq \mathbf{K}$, we define $c^{\mathbf{A}}\colon A \to \mathsf{U} \times k^{<\omega}$ via $c^{\mathbf{A}}(n) = (U^{\mathbf{A}}(n), \langle R^{\mathbf{A}}(n,0), \ldots, R^{\mathbf{A}}(n, n-1)\rangle)$. We define the *coding tree of* $\mathbf{A}$ to be $\text{CT}^{\mathbf{A}} := \text{Im}(c^{\mathbf{A}})\!\downarrow$, and we set $\text{CdNd}(\text{CT}^{\mathbf{A}}) = \text{Im}(c^{\mathbf{A}})$. Hence $c^{\mathbf{A}} = c^{\text{CT}^{\mathbf{A}}}$, though we always write the former. To assign each level an age, consider some $n < A$, and write $\text{CT}^{\mathbf{A}}(n) = \{s_0 \preceq_{\text{lex}} \cdots \preceq_{\text{lex}} s_{d-1}\}$. Consider the rank d gluing $\gamma = (\mathbf{A}|_n, \rho, \eta)$, where given $i < d$, we have $\rho(i) = s_i^{\mathsf{p}}$, and given $i < d$ and $m < n$, we have $\eta(i, m) = s_i(m)$. Note that γ admits all unaries since $s_i \in \text{Im}(c^{\mathbf{A}})\!\downarrow$ for each $i < d$. We set $\text{Age}_{\text{CT}^{\mathbf{A}}}(n) := \text{Age}_{\mathbf{A}}(n) = \mathcal{K}\cdot\gamma \in P(\text{Sort}(\text{CT}^{\mathbf{A}}(m)))$. More generally, we write $\text{Succ}_{\mathbf{A}}(S, n)$, $\text{IS}_{\mathbf{A}}(S)$, etc., though we note that always $\text{Left}_{\mathbf{A}}(S, n) = \text{Left}(S, n)$ for $n < A$. In addition to the notational conventions for general aged coding trees, in the case of $\text{CT}^{\mathbf{A}}$, we define $\text{Age}_{\mathbf{A}}(e)$ for any $e\colon d_0 \to \text{CT}^{\mathbf{A}}$ (even when the image of e is not a level set) by choosing some $n \geq \max(\{\ell(e(i)) : i < d_0\})$ and setting $\text{Age}_{\mathbf{A}}(e) = \text{Age}_{\mathbf{A}}(\text{Left}(e(-), n))$.

We differ slightly from the definition of $\text{CT}^{\mathbf{A}}$ given in [51] in two ways. First, we now define $\text{CT}^{\mathbf{A}}$ to be an aged coding tree rather than just the coding function. Second, instead of recording the unary predicates just at the coding nodes, we now separate out this information at the very start by working with $\mathsf{U} \times k^{<\omega}$ instead of just $k^{<\omega}$.

The second type of aged coding tree we will consider is a *diary*, whose definition we develop in this section. As trees, diaries will be subtrees of $\mathsf{MP} \times k^{<\omega}$. In contrast to $\mathrm{CT}^{\mathbf{A}}$, where the age of each level was completely determined by the placement of the coding nodes, we have some freedom with the age that we assign to each level of a diary. However, since we want these ages to say something meaningful about the structure we are coding (see Proposition 3.4.3), we will need the assignment of ages to follow some rules which more or less assert that our assignment is compatible with the coding and splitting of the diary and with the class $\mathcal{K}$. These rules also ensure that diaries are the correct objects for encoding exact big Ramsey degrees.

3.2. *Controlled coding*

Definition 3.2.1. Given $j < d$, a function $\varphi\colon (d-1) \to k$, and an $\mathcal{L}_{d-1}$-structure $\mathbf{B}$, we define $\mathrm{Add}_{j,\varphi}(\mathbf{B})$ an $\mathcal{L}_d$-structure with underlying set $B \cup \{x\}$, where $x \notin B$ is some new point. On B, $\mathrm{Add}_{j,\varphi}(\mathbf{B})$ induces the structure $(d\setminus\{j\})\cdot\mathbf{B}$. As for the new point x, we set $V^{\mathrm{Add}_{j,\varphi}(\mathbf{B})}(x) = j$, and given $b \in B$, we set $R^{\mathrm{Add}_{j,\varphi}(\mathbf{B})}(b, x) = \varphi(V^{\mathbf{B}}(b))$. In words, $\mathrm{Add}_{j,\varphi}(\mathbf{B})$ is the $\mathcal{L}_d$-structure obtained from $\mathbf{B}$ by shifting the V-values from $d-1$ to $d\setminus\{j\}$ via increasing bijection, preserving the binary relations on pairs of vertices in B, giving one new vertex x the V-value j, and letting φ determine the binary relations between x and the vertices in B. Thus $\mathrm{Add}_{j,\varphi}(\mathbf{B})$ contains exactly one point with unary j, namely x.

Given a class $\mathcal{A}$ of $\mathcal{L}_d$-structures, we define

$$\mathcal{A}\cdot\mathrm{Add}_{j,\varphi} = \{\mathbf{B} \in \mathrm{Fin}(\mathcal{L}_{d-1}) : \mathrm{Add}_{j,\varphi}(\mathbf{B}) \in \mathcal{A}\}.$$

With our convention identifying classes of structures with their characteristic function, we have $(\mathcal{A}\cdot\mathrm{Add}_{j,\varphi})(\mathbf{B}) = \mathcal{A}(\mathrm{Add}_{j,\varphi}(\mathbf{B}))$.

We note that if $\mathbf{A} \leq \mathbf{K}$ is an enumerated structure, $S = \{s_0, \dots, s_{d-1}\} \subseteq \mathrm{CT}^{\mathbf{A}}(m)$ satisfies $\mathrm{Age}_{\mathbf{A}}(S) = \mathcal{A}$, $s_j = c^{\mathbf{A}}(m)$, and $T = \{t_0, \dots, t_{d-1}\} \subseteq \mathrm{CT}^{\mathbf{A}}(m+1)$ is such that $t_i = (s_{(d\setminus\{j\})(i)})^\frown\varphi(i)$ for each $i < d-1$, then $\mathrm{Age}_{\mathbf{A}}(T) = \mathcal{A}\cdot\mathrm{Add}_{j,\varphi}$.

More generally, if $\mathcal{A} = \mathcal{K}\cdot\gamma$ for some gluing γ, then there is also a gluing γ' such that $\mathcal{A}\cdot\mathrm{Add}_{j,\varphi} = \mathcal{K}\cdot\gamma'$.

Definition 3.2.2. Given a path sort $\rho\colon d \to \mathsf{MP}$, $\mathcal{A} \in P(\rho)$, $j < d$, and $\varphi\colon (d-1) \to k$, we call $(\mathcal{A}, j, \varphi)$ a *controlled coding triple* if the following all hold:

(1) $\mathcal{A}\cdot\mathrm{Add}_{j,\varphi} = \mathcal{A}\cdot(d\setminus\{j\})$.

(2) Writing $i = \mathsf{u} \circ \rho(j) < \mathsf{U}$, we have $\mathcal{A}^j = \min(P_i)$.

(3) If $\mathcal{B} \in P(\rho)$ satisfies $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{B}\cdot(d\setminus\{j\}) = \mathcal{B}\cdot\mathrm{Add}_{j,\varphi} = \mathcal{A}\cdot(d\setminus\{j\})$, then $\mathcal{B} = \mathcal{A}$.

Call $(\mathcal{A}, j)$ a *controlled coding pair* if there is some function $\varphi\colon (d-1) \to k$ making $(\mathcal{A}, j, \varphi)$ a controlled coding triple.

Example 3.2.3. Consider the class $\mathcal{G}_\ell$ of K_ℓ-free finite graphs. Fix $d < \omega$, a (d, ℓ)-graph-age function z (see Example 2.1.8), $j < d$, and $\varphi\colon (d-1) \to 2$. In order to understand when $(\mathcal{G}_z, j, \varphi)$ is a controlled coding triple, we first need to understand when $\mathcal{G}_z{\cdot}\mathrm{Add}_{j,\varphi} = (\mathcal{G}_z){\cdot}(d\backslash\{j\})$. Writing $X = (d\backslash\{j\})[\varphi^{-1}(\{1\})]$, one can show that this happens exactly when both of the following occur:

(1) For every $T \in \binom{X}{<\ell}$, we have $z(T) < \ell - 1$.

(2) For every $T \in \binom{X}{<(\ell-1)}$, we have $z(T \cup \{j\}) = z(T) + 1$.

Now suppose $(\mathcal{G}_z, j, \varphi)$ satisfies the above two items. When is it a controlled coding triple? First, we need $z(\{j\}) = 1$ so that item (1) of Definition 3.2.2 holds. Aside from this, we need to enforce item (2) by ensuring that z is "as small as possible." Given $S \in \binom{d}{<\ell}$, call S *determined* if either $S \subseteq d\backslash\{j\}$ or $S \subseteq X \cup \{j\}$; we demand for every $T \in \binom{d}{<\ell}$ that $z(T) = \max\{z(S) : S \subseteq T \text{ is determined}\}$.

In summary, $(\mathcal{G}_z, j)$ is a controlled coding pair if and only if all of the following hold:

(1) $z(\{j\}) = 1$.

(2) Writing $X = \{\alpha \in d\backslash\{j\} : z(\{\alpha, j\}) = z(\{\alpha\}) + 1\}$, then for every $T \in \binom{X}{<\ell}$, we have $z(T) < \ell - 1$, and for every $T \in \binom{X}{<(\ell-1)}$, we have $z(T \cup \{j\}) = z(T) + 1$.

(3) With the same notion of "determined" as above, we have for every $T \in \binom{d}{<\ell}$ that $z(T) = \max\{z(S) : S \subseteq T \text{ is determined}\}$.

It turns out that whenever $(\mathcal{G}_z, j)$ is a controlled coding pair, then there is a unique φ making $(\mathcal{G}_z, j, \varphi)$ a controlled coding triple; φ is defined for $\alpha < (d-1)$ by setting $\varphi(\alpha) = 1$ if and only if $(d\backslash\{j\})(\alpha) \in X$.

We will discuss the motivation for controlled coding triples after Theorem 3.4.12. We first show that they exist in abundance.

Lemma 3.2.4. *Given a path sort $\rho\colon d \to \mathsf{MP}$, $\mathcal{A} \in P(\rho)$, $j < d$, and $\varphi\colon (d-1) \to k$ such that $\mathcal{A}{\cdot}\mathrm{Add}_{j,\varphi}$ contains every singleton $\mathcal{L}_{d-1}$-structure, then there is a unique $\mathcal{B} \in P(\rho)$ with $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{B}{\cdot}\mathrm{Add}_{j,\varphi} = \mathcal{A}{\cdot}\mathrm{Add}_{j,\varphi}$, and with $(\mathcal{B}, j, \varphi)$ a controlled coding triple.*

We denote this unique $\mathcal{B}$ by $\langle \mathcal{A}, j, \varphi\rangle$.

Proof. Write $\mathcal{C} = \mathcal{A}{\cdot}\mathrm{Add}_{j,\varphi}$. We let

$$\mathcal{B} = \bigcap\{\mathcal{D} \in P(\rho) : \mathcal{D}{\cdot}\mathrm{Add}_{j,\varphi} = \mathcal{C}\}.$$

We have $\mathcal{B} \in P(\rho)$ by Proposition 2.1.4, and we also have $\mathcal{B}{\cdot}\mathrm{Add}_{j,\varphi} = \mathcal{C}$. We also see that for any $\mathcal{B}' \in P(\rho)$ with $\mathcal{B}' \subsetneq \mathcal{B}$, we must have $\mathcal{B}'{\cdot}\mathrm{Add}_{j,\varphi} \subsetneq \mathcal{C}$, showing that item (3) of Definition 3.2.2 holds. To see that items (1) and (2) hold, write $i = \mathsf{u} \circ \rho(j)$, and consider the classes $\mathcal{A} \cap \min(P_i){\cdot}\iota_j^{-1}$ and $\mathcal{A} \cap \mathcal{C}{\cdot}(d\backslash\{j\})^{-1}$. Letting $\mathcal{D}$ denote either of these classes, we have that $\mathcal{D} \in P(\rho)$ by Proposition 2.2.3 (in the second case, this uses that $\mathcal{C}$ contains every singleton $\mathcal{L}_{d-1}$-structure as well as the observation immediately before Definition 3.2.2), and it is straightforward to show that $\mathcal{D}{\cdot}\mathrm{Add}_{j,\varphi} = \mathcal{C}$. This implies that $\mathcal{B}^j = \min(P_i)$ and $\mathcal{B}{\cdot}(d\backslash\{j\}) = \mathcal{C}$, showing that items (1) and (2) hold. ∎

Next, we show that restricting a controlled coding triple to a subset of coordinates gives rise to another controlled coding triple.

Lemma 3.2.5. *Let $\rho\colon d \to \mathsf{MP}$ be a path sort, let $\mathcal{A} \in P(\rho)$, $j < d$, and let $\varphi\colon (d-1) \to k$ be such that $(\mathcal{A}, j, \varphi)$ is a controlled coding triple. Let $e\colon d_0 \to d$ be an injection and $j_0 < d_0$ satisfy $e(j_0) = j$. Write $\sigma\colon (d_0 - 1) \to (d-1)$ for the injection satisfying $e \circ (d_0 \backslash \{j_0\}) = (d \backslash \{j\}) \circ \sigma$. Then $(\mathcal{A}{\cdot}e, j_0, \varphi \circ \sigma)$ is a controlled coding triple.*

Proof. We verify the three items of Definition 3.2.2.

For item (1), we have

$$(\mathcal{A}{\cdot}e){\cdot}(d_0\backslash\{j_0\}) = (\mathcal{A}{\cdot}(d\backslash\{j\})){\cdot}\sigma = (\mathcal{A}{\cdot}\mathrm{Add}_{j,\varphi}){\cdot}\sigma = (\mathcal{A}{\cdot}e)(j_0, \varphi \circ \sigma).$$

For item (2), writing $i = \mathsf{u} \circ \rho(j)$, we have $\mathcal{A}^j = \min(P_i)$, so $(\mathcal{A}{\cdot}e)^{j_0} = \min(P_i)$ as well.

For item (3), suppose $\mathcal{B} \subseteq \mathcal{A}{\cdot}e$ satisfies

$$\mathcal{B}{\cdot}(d_0\backslash\{j_0\}) = \mathcal{B}(j_0, \varphi \circ \sigma) = (\mathcal{A}{\cdot}e)(j_0, \varphi \circ \sigma).$$

We need to show that $\mathcal{B} = \mathcal{A}{\cdot}e$. To that end, form the class $\mathcal{C} := \mathcal{A} \cap \mathcal{B}{\cdot}e^{-1}$. It is routine to check that $\mathcal{C}{\cdot}(d\backslash\{j\}) = \mathcal{A}{\cdot}(d\backslash\{j\})$. To check that $\mathcal{C}{\cdot}\mathrm{Add}_{j,\varphi} = \mathcal{A}{\cdot}\mathrm{Add}_{j,\varphi}$, the left-to-right inclusion is clear. For the right-to-left inclusion, fix $\mathbf{D} \in \mathcal{A}{\cdot}\mathrm{Add}_{j,\varphi}$. This means that $\mathrm{Add}_{j,\varphi}(\mathbf{D}) \in \mathcal{A}$, and we want to show that it is also in $\mathcal{C}$. To that end, we note that $e^{-1}{\cdot}(\mathrm{Add}_{j,\varphi}(\mathbf{D})) = (j_0, \varphi \circ \sigma)(\sigma^{-1}{\cdot}\mathbf{D})$. Then $\sigma^{-1}{\cdot}\mathbf{D} \in (\mathcal{A}{\cdot}e)(j_0, \varphi \circ \sigma) = \mathcal{B}(j_0, \varphi \circ \sigma)$. So $e^{-1}{\cdot}(\mathrm{Add}_{j,\varphi}(\mathbf{D})) = (j_0, \varphi \circ \sigma)(\sigma^{-1}{\cdot}(\mathbf{D})) \in \mathcal{B}$, which means that $\mathrm{Add}_{j,\varphi}(\mathbf{D}) \in \mathcal{C}$. Because $(\mathcal{A}, j, \varphi)$ is a controlled coding triple, we must have $\mathcal{A} = \mathcal{C}$, implying that $\mathcal{A}{\cdot}e = \mathcal{B}$. ∎

3.3. Abstract splitting events

Before proceeding further, it will help to assume that every unary predicate in $\mathcal{L}$ interacts with suitably interesting members of $\mathcal{K}$.

Definition 3.3.1. Fix $i < \mathsf{U}$. We call i *non-degenerate for* $\mathcal{K}$ if there is some $\mathbf{A} \in \mathcal{K}$ with $A = \{a, b\}$, $U^{\mathbf{A}}(a) = i$, and $R^{\mathbf{A}}(b, a) \neq 0$. Otherwise, we call i *degenerate*.

A good example to keep in mind is the class $\mathcal{K}$ of finite sets, where the single unary is degenerate. Degenerate unaries will behave rather differently than non-degenerate ones, but will be easy to deal with at the very end. So *until Section 4.4, we will assume that every $i < \mathsf{U}$ is non-degenerate for $\mathcal{K}$.*

Given a path sort $\rho\colon d \to \mathsf{MP}$, a class $\mathcal{A} \in P(\rho)$, and some $i < d$, we discuss an abstract notion of what it should mean to "split" the class $\mathcal{A}$ at position i. This is the operation that will occur at "splitting levels" of diaries. First, let $\mathrm{Sp}(d, i)\colon (d+1) \to d$ be the unique non-decreasing surjection with $\mathrm{Sp}(d, i)^{-1}(\{i\}) = \{i, i+1\}$. If σ is any function with domain d, we can write $\mathrm{Sp}(\sigma, i) := \sigma \circ \mathrm{Sp}(d, i)$. In particular, we obtain a new sort $\mathrm{Sp}(\rho, i)$.

Our goal is to take $\mathcal{A} \in P(\rho)$ and form $\mathrm{Sp}(\mathcal{A}, i)$ (read "split $\mathcal{A}$ at i"). The definition of $\mathrm{Sp}(\mathcal{A}, i)$ will depend on whether or not $\mathsf{u} \circ \rho(i)$ is a *free* or *non-free* unary.

Definition 3.3.2. Fix $i < \mathsf{U}$. We say that i is *free* if there are $j < \mathsf{U}$ and $1 \leq q < k$ such that, letting $\gamma_{j,i,q} = (\mathbf{X}, \iota_i, \eta)$ denote the gluing with $X = \{0\}$, $U^{\mathbf{X}}(0) = j$, and $\eta(0,0) = q$, then $\mathcal{K}\cdot\gamma_{j,i,q} = \mathcal{K}\cdot\tilde{\iota}_i = \max(P_i)$. We call such (j, q) a *free pair* for i.

Otherwise, we call i a *non-free* unary. Let $\mathsf{U}_{\mathrm{fr}} \subseteq \mathsf{U}$ denote the set of free unaries and $\mathsf{U}_{\mathrm{non}} = \mathsf{U} \setminus \mathsf{U}_{\mathrm{fr}}$ the set of non-free unary predicates.

We call $\mathfrak{p} \in \mathsf{MP}$ a *free path* or a *non-free path* depending on whether $\mathsf{u}(\mathfrak{p})$ is free or non-free. Write $\mathsf{MP}_{\mathrm{fr}}$ for the free paths and $\mathsf{MP}_{\mathrm{non}}$ for the non-free paths.

Example 3.3.3. For $\ell \geq 3$, the single unary $i = 0$ for the class $\mathcal{G}_\ell$ of finite K_ℓ-free graphs (Example 2.1.8) is non-free. The only choice for (j, q) would be $(0, 1)$, but the class $\mathcal{K}\cdot\gamma_{0,0,1}$ is bi-interpretable with $\mathcal{G}_{\ell-1}$, the class of finite graphs which forbid $\mathcal{K}_{\ell-1}$.

The single unary $i = 0$ in the class $\mathcal{G}_{\mathbf{T}}$ of finite directed graphs forbidding cyclic triangles (Example 2.1.3) is free, as witnessed by $(j, q) = (0, 1)$ or $(0, 2)$.

Fact 3.3.4. For $i \in \mathsf{U}_{\mathrm{non}}$, while we cannot find $j < \mathsf{U}$ and $1 \leq q < k$ with $\mathcal{K}\cdot\gamma_{j,i,q} = \max(P_i)$, we do have the next best thing: If $\mathfrak{p} \in \mathsf{MP}_i$, then there are $j < \mathsf{U}$ and $1 \leq q < k$ so that $\mathcal{K}\cdot\gamma_{j,i,q} = \max(\mathfrak{p}')$.

Definition 3.3.5. With notation as above, $\mathrm{Sp}(\mathcal{A}, i) \in P(\mathrm{Sp}(\rho, i))$ is defined as follows:

(1) If either $\rho(i) \in \mathsf{MP}_{\mathrm{fr}}$ or $\mathcal{A}^i \neq \max(\mathfrak{p})$, we set $\mathrm{Sp}(\mathcal{A}, i) = \mathcal{A}\cdot\mathrm{Sp}(d, i)$.

(2) If $\rho(i) \in \mathsf{MP}_{\mathrm{non}}$ and $\mathcal{A}^i = \max(\mathfrak{p})$, we set $\mathrm{Sp}(\mathcal{A}, i) = \mathcal{A}\cdot\mathrm{Sp}(d, i) \cap \max(\mathfrak{p}')\cdot\iota_{i+1}^{-1}$.

In case (2), notice that $\mathrm{Sp}(\mathcal{A}, i)^{i+1} = \max(\mathfrak{p}')$. We remark that this is one of the main ways that the difference between free and non-free paths will arise.

The difference between these two cases is motivated by what happens in coding trees. If $\mathbf{A} \leq \mathbf{K}$ is an enumerated structure, $i \in \mathsf{U}_{\mathrm{non}}$, and $v = (i, 0^n) \in \mathrm{CT}^{\mathbf{A}}$, then $\mathrm{Age}_{\mathbf{A}}(v) = \mathcal{K}\cdot\tilde{\iota}_i$, but for any $0 \neq q < k$ with $v^\frown q \in \mathrm{CT}^{\mathbf{A}}$, we must have $\mathrm{Age}_{\mathbf{A}}(v^\frown q) \subsetneq \mathcal{K}\cdot\tilde{\iota}_i$. Hence starting from v, splitting and going right must result in an age change. As diaries are defined to mimic certain properties of coding trees, we must incorporate this feature into our definition of a diary.

Let us clarify one aspect of case (2), which we phrase a touch more generally.

Lemma 3.3.6. *Let $\rho\colon d \to \mathsf{MP}$ be a path sort, fix $i < d$, and suppose $\rho(i) = \mathfrak{p} \in \mathsf{MP}_{\mathrm{non}}$. Then if $\mathcal{B} \in P(\rho)$ is such that $\mathcal{B}^i = \max(\mathfrak{p})$, then*

$$(\mathcal{B}, \mathcal{B} \cap \max(\mathfrak{p}')\cdot\iota_i^{-1}) \in \mathrm{Con}(\rho).$$

Proof. Write $\mathcal{D} = \mathcal{B} \cap \max(\mathfrak{p}')\cdot\iota_i^{-1}$. Suppose $\mathcal{C} \in P(\rho)$ satisfied $\mathcal{D} \subseteq \mathcal{C} \subseteq \mathcal{B}$. We first observe that since $\mathcal{D}\cdot(d\setminus\{i\}) = \mathcal{B}\cdot(d\setminus\{i\})$, we have $\mathcal{C}\cdot(d\setminus\{i\}) = \mathcal{B}\cdot(d\setminus\{i\})$. Write $\mathcal{C} = \mathcal{K}\cdot\gamma$ for some gluing $\gamma = (\mathbf{X}, \rho, \eta)$. If $\eta(i, x) \neq 0$ for some $x \in X$, then as $\mathfrak{p} \in \mathsf{MP}_{\mathrm{non}}$, we would have $\mathcal{C}^i \subsetneq \max(\mathfrak{p})$, implying that $\mathcal{C}^i \subseteq \max(\mathfrak{p}')$, and hence $\mathcal{C} = \mathcal{D}$. However, if $\eta(i, x) = 0$ for every $x \in X$, then free amalgamation in $\mathcal{K}$ along with our first observation imply that $\mathcal{C} = \mathcal{B}$. ∎

We conclude the subsection by discussing what happens upon restricting our attention to a subset of $d+1$.

Proposition 3.3.7. *With notation as in Definition 3.3.5, fix $d_0 < \omega$ and an increasing injection $e\colon d_0 \to (d+1)$.*

(1) *If $\{i, i+1\} \subseteq \mathrm{Im}(e)$, then letting $i_0 < d_0$ be such that $e(i_0) = i$, and letting $e_0\colon (d_0-1) \to d$ be the map with $e_0 \circ \mathrm{Sp}(d_0-1, i_0) = \mathrm{Sp}(d, i) \circ e$, we have $\mathrm{Sp}(\mathcal{A}, i)\cdot e = \mathrm{Sp}(\mathcal{A}\cdot e_0, i_0)$.*

(2) *If $\{i, i+1\} \not\subseteq \mathrm{Im}(e)$, then if either $i+1 \notin \mathrm{Im}(e)$ or if $i+1 \in \mathrm{Im}(e)$ and we are in case (1) of Definition 3.3.5, then $\mathrm{Sp}(\mathcal{A}, i)\cdot e = \mathcal{A}\cdot(\mathrm{Sp}(i, d) \circ e)$.*

(3) *If $i \notin \mathrm{Im}(e)$, $i+1 \in \mathrm{Im}(e)$, and we are in case (2) of Definition 3.3.5, then if $i_0 < d_0$ satisfies $e(i_0) = i+1$, then $\mathrm{Sp}(\mathcal{A}, i)\cdot e = \mathcal{A}\cdot(\mathrm{Sp}(i, d) \circ e) \cap \max(\mathfrak{p}')\cdot\iota_{i_0}^{-1}$. In particular, by Lemma 3.3.6 we have $(\mathcal{A}\cdot(\mathrm{Sp}(d, i) \circ e), \mathrm{Sp}(\mathcal{A}, i)\cdot e) \in \mathrm{Con}(\mathrm{Sp}(\rho, i) \circ e)$.*

Proof. All three items are straightforward checking of the various cases involved. We omit the details. ■

3.4. Diaries and their embeddings

Definition 3.4.1. A *diary* is an aged coding tree $\Delta \subseteq \mathsf{MP} \times k^{<\omega}$ satisfying the following:

(1) If $m < \mathrm{ht}(\Delta)$, then $\Delta(m)$ contains at most one node t with $|\mathrm{IS}_\Delta(t)| \neq 1$.
   - If there is $t \in \Delta(m)$ with $\mathrm{IS}_\Delta(t) = \emptyset$, we call m a *coding level* of Δ.
   - If there is $t \in \Delta(m)$ with $|\mathrm{IS}_\Delta(t)| > 1$, then $|\mathrm{IS}_\Delta(t)| = 2$, and we call m a *splitting level* of Δ.
   - If every $t \in \Delta(m)$ satisfies $|\mathrm{IS}_\Delta(t)| = 1$, we call m an *age-change* level of Δ.

   We write $\mathrm{Sp}(\Delta)$, $\mathrm{Cd}(\Delta)$, and $\mathrm{AC}(\Delta)$ for the splitting, coding, and age-change levels of Δ, respectively. We let $\mathrm{CdNd}(\Delta) \subseteq \Delta$ be the set of terminal nodes of Δ.

(2) $\mathrm{CdNd}(\Delta)$ is $\sqsubseteq$-upwards cofinal in Δ.

(3) Writing $\rho = \mathrm{Sort}(\Delta(0))$, we have $\mathrm{Age}_\Delta(0) = \mathcal{K}\cdot\tilde{\rho} = \max(P(\rho))$.

(4) If $\Delta(m) = \{t_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} t_{d-1}\}$ and t_i is the splitting node, then we have $\Delta(m+1) = \{t_j{}^\frown 0 : j < d\} \cup \{t_i{}^\frown 1\}$ and $\mathrm{Age}_\Delta(m+1) = \mathrm{Sp}(\mathrm{Age}_\Delta(m), i)$ (Definition 3.3.5).

(5) If $\Delta(m) = \{t_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} t_{d-1}\}$ and t_j is a coding node, then writing $\Delta(m+1) = \{u_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} u_{d-2}\}$ and defining $\varphi\colon (d-1) \to k$ via $\varphi(i) = u_i(m)$, we have that $(\mathrm{Age}_\Delta(m), j, \varphi)$ is a controlled coding triple (Definition 3.2.2) and $\mathrm{Age}_\Delta(m+1) = \mathrm{Age}_\Delta(m)\cdot\mathrm{Add}_{j,\varphi} = \mathrm{Age}_\Delta(m)\cdot(d\backslash\{j\})$.

(6) If $m \in \mathrm{AC}(\Delta)$, then we have that $\Delta(m+1) = \{t^\frown 0 : t \in \Delta(m)\}$, and writing $\rho = \mathrm{Sort}(\Delta(m))$, we have $(\mathrm{Age}_\Delta(m), \mathrm{Age}_\Delta(m+1)) \in \mathrm{Con}(\rho)$ (Definition 2.2.4).

Fact 3.4.2. (1) If Δ is a diary and $\mathfrak{p} \in \mathsf{MP}_{\mathrm{non}}$, then for any $s \in \Delta_\mathfrak{p}$ with $s \neq (\mathfrak{p}, 0^{\ell(s)})$, $\mathrm{Age}_\Delta(s) \subsetneq \max(\mathfrak{p})$. This follows from case (2) of Definition 3.3.5 and part (1) of Definition 3.2.2.

(2) For any $t \in \mathrm{CdNd}(\Delta)$, we have $\mathrm{Path}_\Delta(t) = t^{\mathsf{p}}$. By part (2) of Definition 3.2.2, we have $\mathrm{Age}_\Delta(t) = \min(P_i)$. Item (6) of Definition 3.4.1 then implies that $\mathrm{Path}_\Delta(t) \in \mathsf{MP}$. But since $\mathrm{Age}_\Delta(t|_m) \in P(\mathrm{Sort}(t)) = t^{\mathsf{p}}$ for every $m \leq \ell(t)$, we must have $\mathrm{Path}_\Delta(t) = t^{\mathsf{p}}$.

Every level subset of a diary has a distinguished subset of *critical nodes*. If Δ is a diary and $m < \mathrm{ht}(\Delta)$, we define $\mathrm{CritNd}_\Delta(m) \subseteq \Delta(m)$ as follows. If $m \in \mathrm{Sp}(\Delta)$, we set $\mathrm{CritNd}_\Delta(m) = \mathrm{SpNd}(\Delta) \cap \Delta(m)$. If $m \in \mathrm{Cd}(\Delta)$, we set $\mathrm{CritNd}_\Delta(m) = \mathrm{CdNd}(\Delta) \cap \Delta(m)$. If $m \in \mathrm{AC}(\Delta)$ and $\Delta(m) = \{s_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} s_{d-1}\}$, then $(\mathrm{Age}_\Delta(m), \mathrm{Age}_\Delta(m+1))$ is essential on a unique $S \subseteq d$ (Proposition 2.2.6), and we set $\mathrm{CritNd}_\Delta(m) = \{s_i : i \in S\}$.

Given a diary Δ, the *structure coded by* Δ, denoted by $\mathbf{Str}(\Delta)$, is the $\mathcal{L}$-structure on underlying set $\mathrm{CdNd}(\Delta)$ such that given $s, t \in \mathrm{CdNd}(\Delta)$. We set $U^{\mathbf{Str}(\Delta)}(t) = t^{\mathsf{u}}$, and if $\ell(t) > \ell(s)$, we put $R^{\mathbf{Str}(\Delta)}(t, s) = t(\ell(s))$. We let $\mathbf{Str}^{\#}(\Delta)$ denote the enumerated structure isomorphic to $\mathbf{Str}(\Delta)$ via c^Δ. More generally, given $m < \mathrm{ht}(\Delta)$ and $S = \{s_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} s_{d-1}\} \subseteq \Delta(m)$, we define the $\mathcal{L}_d$-structure $\mathbf{Str}(\Delta)/S$ on underlying set $\{t \in \mathrm{CdNd}(\Delta) : t \sqsupseteq s \text{ for some } s \in S\}$ by setting $V^{\mathbf{Str}(\Delta)/S}(t) = j$ if and only if $t \sqsupseteq s_j$ and with the $\mathcal{L}^{\mathsf{b}}$-part induced from $\mathbf{Str}(\Delta)$. Much of the motivation behind Definition 3.4.1 is to guarantee a nice cohesion between the ages assigned to each level of Δ and $\mathbf{Str}(\Delta)$. We clarify this in Proposition 3.4.3.

Proposition 3.4.3. *Let Δ be a diary, $m < \mathrm{ht}(\Delta)$, and $S \subseteq \Delta(m)$.*

(1) $\mathrm{Age}(\mathbf{Str}(\Delta)/S) \subseteq \mathrm{Age}_\Delta(S)$. *In particular, we always have* $\mathrm{Age}(\mathbf{Str}(\Delta)) \subseteq \mathcal{K}$.

(2) *Let $n < \omega$ satisfy $\ell^\Delta(n-1) < m$. Define $\eta\colon |S| \times n \to k$ via $\eta(i, a) = s_i(\ell^\Delta(a))$. Then writing $\gamma = (\mathbf{Str}^{\#}(\Delta)|_n, \mathrm{Sort}(S), \eta)$, we have* $\mathrm{Age}_\Delta(S) \subseteq \mathcal{K}{\cdot}\gamma$.

Proof. (1) Given a finite $\mathbf{B} \subseteq \mathbf{Str}(\Delta)/S$, we prove $\mathbf{B} \in \mathrm{Age}_\Delta(S)$ by induction on $|B|$ for every level set $S \subseteq \Delta$ simultaneously. For $|B| = 1$, the result is clear. So suppose $|B| > 1$, and let $b \in B$ be $\leq_\ell$-least. Write $\pi_{\ell(b)}[B] = \{t_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} t_{\alpha-1}\} := T$, and let $j < \alpha$ satisfy $b = t_j$. For each $i \in \alpha \backslash \{j\}$, let $q_i < k$ be unique with $t_i^\frown q_i \in \Delta$, and let $\varphi\colon (\alpha - 1) \to k$ be the function with $\varphi(i) = q_{(\alpha\backslash\{j\})(i)}$. By induction, viewing $\mathbf{B}\backslash\{b\}$ as an $\mathcal{L}_\alpha$-structure, we have $\mathbf{B}\backslash\{b\} \in \mathrm{Age}_\Delta(T)$. Then, since $(\mathrm{Age}_\Delta(T), j, \varphi)$ is a controlled coding triple, we have, viewing $\mathbf{B}$ as an $\mathcal{L}_\alpha$-structure, that $\mathbf{B} \in \mathrm{Age}_\Delta(T)$. If $S = \{s_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} s_{d-1}\}$ and $\pi\colon \alpha \to d$ denotes the function with $t_i \sqsupseteq s_{\pi(i)}$ for each $i < \alpha$, then by the properties of splitting, age-change, and coding levels of diaries, we have $\mathrm{Age}_\Delta(T) \subseteq \mathrm{Age}_\Delta(S){\cdot}\pi$. This implies that $\mathbf{B}$, as an $\mathcal{L}_d$-structure, belongs to $\mathrm{Age}_\Delta(S)$.

(2) We induct on n for every level set $S \subseteq \Delta$ simultaneously. For $n = 0$, there is nothing to show. If $n > 0$, it suffices to consider the case where $S = \{s_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} s_{d-2}\} \subseteq \Delta(\ell^\Delta(n-1)+1)$. Write $\pi_{\ell^\Delta(n-1)}[S] \cup \{c^\Delta(n-1)\} := T := \{t_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} t_{d-1}\}$, and suppose $j < d$ satisfies $c^\Delta(n) = t_j$. Write $\varphi\colon (d-1) \to k$ for the function $\varphi(i) = s_i(\ell^\Delta(n-1))$. Since $(\mathrm{Age}_\Delta(T), j, \varphi)$ is a controlled coding triple by Lemma 3.2.5, we have $\mathrm{Age}_\Delta(S) = \mathrm{Age}_\Delta(T){\cdot}(d\backslash\{j\}) = \mathrm{Age}_\Delta(T){\cdot}\mathrm{Add}_{j,\varphi}$. Consider the gluing $\gamma' = (\mathbf{Str}^{\#}(\Delta)|_{n-1}, \mathrm{Sort}(T), \eta')$, where $\eta'\colon (d \times (n-1)) \to k$ is given by $\eta'(i, a) = t_i(\ell^\Delta(a))$. By induction, we have $\mathrm{Age}_\Delta(T) \subseteq \mathcal{K}{\cdot}\gamma'$. It is routine to check that $(\mathcal{K}{\cdot}\gamma'){\cdot}\mathrm{Add}_{j,\varphi} = \mathcal{K}{\cdot}\gamma$, where γ is as in the proposition statement. Hence $\mathrm{Age}_\Delta(S) \subseteq \mathcal{K}{\cdot}\gamma$ as desired. ■

We now discuss finite diaries for our three main examples; the reader who wishes to skip them can skip to Definition 3.4.7. One quirk of all three examples is that controlled coding pairs uniquely determine controlled coding triples. In practice, this means that if Δ is a diary, $m \in \mathrm{Cd}(\Delta)$, $t \in \Delta(m)$ is the coding node, and $v \in \Delta(>m)$, then $v(m)$ is completely determined by $\mathrm{Age}_\Delta(\{v|_m, t\})$. For classes where controlled coding pairs uniquely determine controlled coding triples, diaries are in one-to-one correspondence with certain sequences of structures (typically in another language). It is not always the case that controlled coding pairs uniquely determine controlled coding triples; an example where this is not the case is the class $\mathcal{G}$ of finite graphs. For these classes, at coding levels one must explicitly describe the "passing number" information.

Example 3.4.4. We describe all finite diaries for the class $\mathcal{G}_3$ of triangle-free graphs. By Fact 3.4.2 (1), if $\Delta(m) = \{s_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} s_{d-1}\}$ and $i < d$ is such that $\mathrm{Age}_\Delta(s_i) = \mathcal{G}_3{\cdot}\tilde{\iota}_0$, then $i = 0$, and furthermore, $\mathrm{Age}_\Delta(m)$ can be recovered from $\mathrm{Age}_\Delta(\{s_1 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} s_{d-1}\})$. Thus, given a diary Δ, we can recover Δ just from the nodes of Δ without maximum age. As in Example 2.1.8, if z is a $(d, 3)$-graph-age function with $z(\{i\}) = 1$ for every $i < d$, we can encode z using a graph on vertex set d. Using these ideas, we obtain the following general procedure to produce a diary coding a finite enumerated triangle-free graph:

(1) At stage 0, we let G_0 be the empty graph.

(2) Suppose at stage $m < \omega$, we have produced a finite linearly-ordered graph G_m. If $m > 0$ and G_m is the empty graph, we can choose to stop.

(3) If at stage m we choose to (or must) continue, we perform exactly one of the following to produce G_{m+1}:

  (a) Introduce a new least vertex which is adjacent to every vertex from G_m (the node 0^m has maximum age and splits).

  (b) If $G_m \neq \emptyset$, choose $v \in G_m$ and duplicate it; the two new vertices will be order-consecutive and non-adjacent in G_{m+1}, and will have the same neighbors in $G_m \setminus \{v\}$ as v did (a node of non-maximal age splits).

  (c) Delete an edge in G_m (an age change level).

  (d) Pick $v \in G_m$ whose neighbors form an anti-clique and delete v (a coding level).

Finite diaries for the class $\mathcal{G}_3$ are in one-to-one correspondence with finite runs of this procedure. The size of the enumerated triangle-free graph which is coded by this procedure is exactly the number of vertex deletion events, and the edges of this triangle-free graph are determined by the edges in G_m at the time of the vertex deletion events. For instance, if we wish to code an enumerated anti-clique, this is the same as demanding that we only delete isolated vertices from G_m. Thus the number of diaries coding the n-element anti-clique for $n = 1, 2, 3, 4$ are 1, 5, 161, 134397. It would be an interesting problem in enumerative combinatorics to obtain an exact formula, or to even understand how this function grows asymptotically (see Section 6.5).

Example 3.4.5. We describe all finite diaries for the class $\mathcal{G}_{\mathbf{T}}$ of finite *oriented* graphs which forbid the 3-cycle $\mathbf{T}$. From Example 2.1.3, members of $P(d)$ are in one-to-one correspondence with *directed* graphs on vertex set d. Thus we have the following general procedure to produce a diary coding a finite enumerated $\mathbf{T}$-free oriented graph; note the slight differences from Example 3.4.4 in the starting and ending conditions since here, the unary is free:

(1) At stage 0, we let G_0 be the singleton graph.

(2) Suppose that at stage $m < \omega$, we have produced a finite linearly-ordered *directed* graph G_m. If $G_m = \emptyset$, we must stop.

(3) If $G_m \neq \emptyset$, we perform exactly one of the following to produce G_{m+1}:

  (a) Choose $v \in G_m$ and duplicate it. The two new vertices will be order-consecutive and form a 2-cycle, and they will have the same relations with vertices in $G_m \backslash \{v\}$ as v did (a node splits).

  (b) Delete an edge in G_m (an age-change level).

  (c) Pick $v \in G_m$ which does not participate in any 2-cycles and such that there are no edges from the out-neighbors of v to the in-neighbors of v and delete it (a coding level).

Finite diaries for the class $\mathcal{G}_{\mathbf{T}}$ are in one-to-one correspondence with finite runs of this procedure. The size of the enumerated oriented $\mathbf{T}$-free graph coded by this procedure is exactly the number of vertex deletion events, and the oriented edges are determined by the edges in G_m at the time of vertex deletion events.

Example 3.4.6. The description of finite diaries for $\mathcal{H}$ is slightly more complicated since $|\mathsf{MP}| = 2$. As promised in Example 2.3.2, we describe the subset of $P(d)$ which is relevant to our description of diaries. We do so implicitly by describing the structures that appear in a sequence describing a typical diary. These will be linearly ordered graphs with red and blue edges where *loops are allowed* and pairs can be unrelated, red related, blue related, *or both*. In addition, there will be two disjoint unary predicates P_r and P_b which partition the vertex set, and P_r will form an initial segment of vertices in the linear order. Let us call structures of this form *RB-graphs*. With these structures, we can describe finite diaries for $\mathcal{H}$ via finite runs of the following procedure:

(1) At stage 0, G_0 is the empty RB-graph.

(2) Suppose at stage $m < \omega$, we have produced an RB-graph G_m. If $m > 0$ and $G_m = \emptyset$, we can choose to stop.

(3) If at stage m we choose to (or must) continue, we perform exactly one of the following to produce G_{m+1}:

  (a) Introduce a new least vertex among those with unary P_r. This vertex will be red adjacent to every member of P_r (including itself), and will be red-and-blue adjacent to every member of P_b (the node $(\mathfrak{p}_r, 0^m)$ has maximum age and splits).

  (b) Introduce a new least vertex among those with unary P_b. Same as above, switching red and blue (the node $(\mathfrak{p}_b, 0^m)$ has maximum age and splits).

(c) If $G_m \neq \emptyset$, choose $v \in G_m$ and duplicate it. The two new vertices will be order-consecutive, will have the same unary relation and loop edges as v did, and will have the same binary relations with vertices of $G_m \setminus \{v\}$ as v did. If $v \in P_r$, the two new vertices will be red adjacent, and if $v \in P_b$, they will be blue adjacent (a node of non-maximal age splits).

(d) Delete a loop edge (age change).

(e) If $u \neq v \in G_m$ are connected by a red edge and neither u nor v have a red loop, we may delete this edge. Likewise, for blue (age change).

(f) If $v \in G_m$ has no loops, is not red-and-blue related to any vertex, its red neighbors contain no red edges among them, and its blue neighbors contain no blue edges among them, we can delete v (coding level).

Definition 3.4.7. Let Θ and Δ be diaries. A *diary embedding* of Θ into Δ is any $\varphi \in \mathrm{AEmb}(\Theta, \Delta)$ which additionally satisfies the following property:

- If $m \in \mathrm{AC}(\Theta)$, then $\widetilde{\varphi}(m) \in \mathrm{AC}(\Delta)$. Defining $\varphi^+ \colon \Theta(m+1) \to \Delta(\widetilde{\varphi}(m)+1)$ via $\varphi^+(s^\frown 0) = \varphi(s)^\frown 0$, we have $\mathrm{Age}_\Theta(m+1) = \mathrm{Age}_\Delta(\varphi^+[\Theta(m+1)])$.

Write $\mathrm{DEmb}(\Theta, \Delta)$ for the set of diary embeddings of Θ into Δ.

Remark. Given $\varphi \in \mathrm{DEmb}(\Theta, \Delta)$, we can define φ^+ on every level. Given any $m < \mathrm{ht}(\Theta) - 1$, define $\varphi^+ \colon \Theta(m+1) \to \Delta(\widetilde{\varphi}(m)+1)$, where given $s \in \Theta(m)$ and $i < k$ with $s^\frown i \in \Theta(m+1)$, we have $\varphi^+(s^\frown i) = \varphi(s)^\frown i$; this is well defined since $\varphi \in \mathrm{AEmb}(\Theta, \Delta)$. When $m \notin \mathrm{AC}(\Delta)$, one can show that

$$\mathrm{Age}_\Theta(m+1) = \mathrm{Age}_\Delta(\varphi^+[\Theta(m+1)]),$$

and for $m \in \mathrm{AC}(\Delta)$, we explicitly demand this in the definition of diary embedding.

Note that if $\varphi \colon \Theta \to \Delta$ is a diary embedding, then $\varphi|_{\mathbf{Str}(\Theta)} \colon \mathbf{Str}(\Theta) \to \mathbf{Str}(\Delta)$ is an embedding of $\mathcal{L}$-structures, and given $S \subseteq \Theta(m)$, $\varphi|_{\mathbf{Str}(\Theta)/S} \colon \mathbf{Str}(\Theta)/S \to \mathbf{Str}(\Delta)/\varphi[S]$ is an embedding of $\mathcal{L}_d$-structures. Conversely, we have the following.

Proposition 3.4.8. *Given any diary Δ and $X \subseteq \mathrm{CdNd}(\Delta)$, there are a unique diary $\Delta\|_X$ and $\varphi_{\Delta,X} \in \mathrm{DEmb}(\Theta, \Delta)$ with $\varphi[\mathrm{CdNd}(\Delta\|_X)] = X$.*

Along with notation from the proposition statement, the map $\pi_{\Delta,X} \colon X \to \Delta\|_X$ defined in the proof (a partial inverse to $\varphi_{\Delta,X}$) will also be used later.

Proof. Write $|\mathrm{Crit}^\Delta(X)| = N \leq \omega$ and $\mathrm{Crit}^\Delta(X) = \{a_m : m < N\} \subseteq \omega$ with $a_{m-1} < a_m$ for each $1 \leq m < N$. We define a map $\pi_{\Delta,X} \colon X \to \mathsf{MP} \times k^{<\omega}$ where given $x \in X$, we set $\pi_{\Delta,X}(x)^{\mathsf{p}} = x^{\mathsf{p}}$, $\ell(\pi_{\Delta,X}(x)) = |\ell(x) \cap \mathrm{Crit}^\Delta(X)|$, and given $m < \ell(\pi_{\Delta,X}(x))$, we set $\pi_{\Delta,X}(x)(m) = x(a_m)$. One should think of $\pi_{\Delta,X}(x)$ as being formed from x by deleting all of the levels that are not in $\mathrm{Crit}^\Delta(X)$. Note that $\pi_{\Delta,X}$ preserves both $\leq_\ell$ and $\preceq_{\mathrm{lex}}$ and that $\mathrm{Im}(\pi_{\Delta,X})$ is a $\sqsubseteq$-antichain.

We set $\Delta\|_X = \mathrm{Im}(\pi_{\Delta,X}){\downarrow}$. Given $m < \mathrm{ht}(\Delta\|_X)$, we define $\mathrm{Age}_\Delta\|_X(m)$ in the following way. Pick $\{x_i : i < d\} \in X$ such that $\ell(\pi_{\Delta,X}(x_i)) \geq m$ for each $i < d$ and with $\Delta\|_X(m) = \{\pi_{\Delta,X}(x_0)|_m \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} \pi_{\Delta,X}(x_{d-1})|_m\}$. Then set $\mathrm{Age}_{\Delta\|_X}(m) = \mathrm{Age}_\Delta(\{x_i|_{a_m} : i < d\})$. That this turns $\Delta\|_X$ into a diary follows from Propositions 2.2.6 and 3.3.7, and Lemma 3.2.5.

We define $\varphi_{\Delta,X}\colon \Delta\|_X \to \Delta$ as follows. Consider $y \in \Delta\|_X(m)$, and suppose $x \in X$ satisfies $y \sqsubseteq \pi_{\Delta,X}(x)$. We set $\varphi_{\Delta,X}(y) = x|_{a_m}$. It is straightforward, if a bit tedious, to check that $\varphi_{\Delta,X}\colon \Delta\|_X \to \Delta$ is a diary embedding. ■

Our next goal, Proposition 3.4.11, is to construct a diary coding **K**.

Definition 3.4.9. Given a path sort $\rho\colon 2 \to \mathsf{MP}$, $\mathcal{A} \in P(\rho)$, $j < 2$, and $q < k$, we say that q is a *valid passing number* for $(\rho, \mathcal{A}, j)$ if $\mathcal{A}\cdot\mathrm{Add}_{j,\iota_q} \in \rho(1-j)$. In particular, 0 is always a valid passing number for $(\rho, \mathcal{A}, j)$.

If $\Delta \subseteq \mathsf{MP} \times k^{<\omega}$ is an aged coding tree, $m < \mathrm{ht}(\Delta)$, and $s \neq t \in \Delta(m)$, we say that $q < k$ is a *valid passing number* for (Δ, s, t) if, writing $\{s,t\} = \{s_0, s_1\}$ with $s_0 \preceq_{\mathrm{lex}} s_1$ and letting $j < 2$ satisfy $t = s_j$, we have that q is a valid passing number for $(\mathrm{Sort}(\{s,t\}), \mathrm{Age}_\Delta(\{s,t\}), j)$.

Lemma 3.4.10. *Suppose ρ, $\mathcal{A}$, and $j < 2$ are as in Definition 3.4.9 and that $q < k$ is a valid passing number for $(\rho, \mathcal{A}, j)$. Suppose $\rho(1-j) = \mathfrak{p} \in \mathsf{MP}_{\mathrm{non}}$ and $\mathcal{A}^{1-j} = \max(\mathfrak{p})$. Then q is a valid passing number for $(\rho, \mathcal{A} \cap \max(\mathfrak{p}')\cdot\iota_{1-j}^{-1}, j)$.*

Proof. Without loss of generality, suppose $j = 1$ and $q \neq 0$. Then $\mathcal{A}\cdot\mathrm{Add}_{1,\iota_q} \in \mathfrak{p}$, and in particular $\mathcal{A}\cdot\mathrm{Add}_{1,\iota_q} \subseteq \mathcal{A}^0 = \max(\mathfrak{p})$. Since $\mathcal{A} \in P(\rho)$ as witnessed by some gluing, we must have $\mathcal{A} \subseteq \mathcal{K}\cdot\widetilde{\rho}$. By the definition of $\mathcal{A}\cdot\mathrm{Add}_{1,\iota_q}$ and since $\mathfrak{p} \in \mathsf{MP}_{\mathrm{non}}$, we must have $\mathcal{A}\cdot\mathrm{Add}_{1,\iota_q} \subseteq \max(\mathfrak{p}')$. Hence we have $(\mathcal{A} \cap \max(\mathfrak{p}')\cdot\iota_0^{-1})\cdot\mathrm{Add}_{1,\iota_q} = \mathcal{A}\cdot\mathrm{Add}_{1,\iota_q}$. ■

Proposition 3.4.11. *There is a diary Δ with* $\mathbf{Str}(\Delta) \cong \mathbf{K}$.

Proof. We start by setting $\Delta(0) = \mathsf{MP} \times \{\emptyset\}$ and $\mathrm{Age}_\Delta(0) = \mathcal{K}\cdot\mathrm{Sort}(\Delta(0))$. Fix $n < \omega$, and assume inductively that we have build Δ up to level

$$m := \begin{cases} \ell^\Delta(n-1)+1 & \text{if } n \geq 1,\\ 0 & \text{if } n = 0.\end{cases}$$

Fix $t \in \Delta(m)$. This is the node we will extend to become $c^\Delta(n)$. The choice of t is arbitrary, so long as we eventually arrange that item (2) of Definition 3.4.1 holds. Our construction up to level $\ell^\Delta(n)+1$ now proceeds as follows:

- t is a splitting node. Write $u = t^\frown 1$.
- For every $s \in \Delta(m+1)\setminus\{u\}$, let I_s denote the set of valid passing numbers for (s, u). The next several levels are splitting levels satisfying item (4) of Definition 3.4.1. For each $s \in \Delta(m+1)\setminus\{u\}$, $|I_s| - 1$ of these splitting levels will have a splitting node extending s. Let $M > m$ denote the level we are at after these splitting levels, and write $\Delta(M) = \{s_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} s_{d-1}\}$. Note that for $s \in \Delta(m+1)\setminus\{u\}$, we have $|\mathrm{Succ}_\Delta(s, M)| = |I_s|$, and we have $\mathrm{Succ}_\Delta(u, M) := \{s_j\}$ a singleton.
  Write $\varphi\colon (d-1) \to k$ for the function such that for each $s \in \Delta(m+1)\setminus\{u\}$, writing $X_s = \{i < d-1 : s \sqsubseteq s_{(d\setminus\{j\})(i)}\}$, $\varphi|_{X_s}$ is the $\preceq_{\mathrm{lex}}$-increasing bijection onto I_s.

- By Lemma 3.4.10, $\mathrm{Age}_\Delta(M)\cdot\mathrm{Add}_{j,\varphi}$ contains every singleton $\mathcal{L}_{d-1}$-structure. Write $\mathrm{Age}_\Delta(M) = \mathcal{A}_0$, and find $\mathcal{A}_0 \supseteq \cdots \supseteq \mathcal{A}_r = \langle \mathcal{A}_0, j, \varphi\rangle$ with $(\mathcal{A}_i, \mathcal{A}_{i+1}) \in \mathrm{Con}(\rho)$. Then the levels in $[M, M + r)$ are age-change levels with $\mathrm{Age}_\Delta(M + i) = \mathcal{A}_i$ for $i < r$.
- Now for every $i < d$, we have $\mathrm{Succ}_\Delta(s_i, M + r) = \{\mathrm{Left}(s_i, M + r)\}$. We set $\ell^\Delta(n) = M + r$ and $c^\Delta(n) = \mathrm{Left}(s_j, \ell^\Delta(n))$. We set

$$\Delta(\ell^\Delta(n) + 1) = \{\mathrm{Left}(s_{(d\setminus\{j\})(i)}, \ell^\Delta(n))^\frown\varphi((d\setminus\{j\})(i)) : i < d - 1\}$$

  and $\mathrm{Age}_\Delta(\ell^\Delta(n) + 1) = \mathcal{A}_0\cdot\mathrm{Add}_{j,\varphi} = \mathcal{A}_r\cdot\mathrm{Add}_{j,\varphi}$.

This completes the inductive step of the construction. Notice that given

$$s \in \Delta(\ell^\Delta(n - 1) + 1) \quad \text{and} \quad q \in I_{s^\frown 0},$$

there is a unique $v \in \mathrm{Succ}_\Delta(s, \ell^\Delta(n) + 1)$ with $v(\ell^\Delta(n)) = q$; we denote this v by $s{*}q$.

We show that $\mathbf{Str}(\Delta) \cong \mathbf{K}$ by showing that it satisfies the extension property for one-element extensions. Fix $n < \omega$, and suppose $j < \mathsf{U}$ and that $\gamma_n := (\mathbf{Str}^\#(\Delta)|_n, \iota_j, \eta)$ is a rank 1 gluing with $\mathcal{K}\cdot\gamma_n \neq \emptyset$, thus representing an instance of an extension problem. Given $m < n$, set $\gamma_m = (\mathbf{Str}^\#(\Delta)|_m, \rho, \eta|_{1\times m})$. Thus $\max(P_j) = \mathcal{K}\cdot\gamma_0 \supseteq \cdots \supseteq \mathcal{K}\cdot\gamma_n \in P_j$, so pick some $\mathfrak{p} \in \mathsf{MP}_j$ with $\mathcal{K}\cdot\gamma_m \in \mathfrak{p}$ for every $m \leq n$.

Set $s_0 = (\mathfrak{p}, \emptyset) \in \Delta$. Suppose that $m < n$ and that s_m has been determined with $\mathrm{Age}_\Delta(s_m) = \mathcal{K}\cdot\gamma_m$ and such that if $m > 0$, we have $s_m \in \Delta(\ell^\Delta(m - 1) + 1)$. Write $t = c^\Delta(m)|_{\ell^\Delta(m-1)+2}$, and note that $t(\ell^\Delta(m - 1) + 1) = 1$; in particular, $s_m^\frown 0 \neq t$. Then $q_m := \eta(1, m) < k$ is a valid passing number for $(s_m^\frown 0, t)$ since $\mathcal{K}\cdot\gamma_{m+1} \in \mathfrak{p}$. We then set $s_{m+1} = s_m{*}q_m$.

Upon defining s_n, any $t \in \mathrm{CdNd}(\Delta) \cap \mathrm{Succ}_\Delta(s_n)$ will witness the desired instance of the extension property. ∎

A major theorem of this paper is that any two diaries which code $\mathbf{K}$ are bi-embeddable.

Theorem 3.4.12. *Let Θ be any diary, and let Δ be any diary with $\mathbf{Str}(\Delta) \cong \mathbf{K}$. Then $\mathrm{DEmb}(\Theta, \Delta) \neq \emptyset$. In particular, any two diaries coding $\mathbf{K}$ are bi-embeddable.*

We postpone the proof of Theorem 3.4.12 until Section 4. Now is a good time to mention various aspects of why our definition of diary contains all of the features it has. Namely, much of what motivates Definition 3.4.1 is that we want Theorem 3.4.12 to be true. One could imagine, for instance, trying to relax item (5) of the definition to only demand that $\mathrm{Age}_\Delta(m + 1) = \mathrm{Age}_\Delta(m)\cdot\mathrm{Add}_{j,\varphi}$. But if we did this and Δ was a diary coding $\mathbf{K}$, we could find $X \subseteq \mathrm{CdNd}(\Delta)$ with $\mathbf{Str}(\Delta\|_X) \cong \mathbf{K}$ and so that all coding levels of $\Delta\|_X$ had controlled coding triples. Similarly, one could imagine trying to simplify the definition of $\mathrm{Sp}(\mathcal{A}, i)$ to always be as in case (1) of Definition 3.3.5. But again, if we did this and Δ was a diary coding $\mathbf{K}$, we could find $X \subseteq \mathrm{CdNd}(\Delta)$ with $\mathbf{Str}(\Delta\|_X) \cong \mathbf{K}$ and so that for all splitting levels m corresponding to case (2) of Definition 3.3.5, then level $m + 1$ would be an age-change level such that $\mathrm{Age}_\Delta(m + 2) = \mathrm{Sp}(\mathrm{Age}_\Delta(m), i)$ as we currently have it defined.

We end the section by showing how Theorem 3.4.12 yields lower bounds for big Ramsey degrees.

Definition 3.4.13. Given $\mathbf{A} \in \mathcal{K}$, we set

$$D_{\mathbf{A}} := \{(\Theta, g) : \Theta \text{ is a diary and } g\colon \mathbf{A} \to \mathbf{Str}(\Theta) \text{ is an isomorphism}\}.$$

Note that by item (2) of Definition 3.4.1 and Proposition 2.1.4, $D_{\mathbf{A}}$ is finite.

If Δ is a diary and $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{Str}(\Delta))$, the Δ-*shape of* f is defined via

$$\mathrm{Shp}_\Delta(f) := (\Delta\|_{\mathrm{Im}(f)}, \pi_{\Delta,\mathrm{Im}(f)} \circ f) \in D_{\mathbf{A}}.$$

Viewing Shp_Δ as a class function, we write $\mathrm{Shp}_{\Delta,\mathbf{A}}\colon \mathrm{Emb}(\mathbf{A}, \mathbf{Str}(\Delta)) \to D_{\mathbf{A}}$ for the restriction of Shp_Δ to $\mathrm{Emb}(\mathbf{A}, \mathbf{Str}(\Delta))$.

Theorem 3.4.14. *Let* $\mathbf{A} \in \mathcal{K}$. *Then* $\mathrm{BRD}(\mathbf{A}, \mathcal{K}) \geq |D_{\mathbf{A}}|$.

Proof. Fix a diary Δ with $\mathbf{K} \cong \mathbf{Str}(\Delta)$. Towards showing that $\mathrm{Shp}_{\Delta,\mathbf{A}}$ is unavoidable, fix $(\Theta, g) \in D_{\mathbf{A}}$, and let $\eta \in \mathrm{Emb}(\mathbf{K}, \mathbf{Str}(\Delta))$. Write $\Delta' = \Delta\|_{\mathrm{Im}(\eta)}$ and $\varphi = \varphi_{\Delta,\mathrm{Im}(\eta)}$. Then $\mathbf{Str}(\Delta') \cong \mathbf{K}$, so in particular by Theorem 3.4.12, there is some $\sigma \in \mathrm{DEmb}(\Theta, \Delta')$. Then $\varphi \circ \sigma \circ g \in \mathrm{Emb}(\mathbf{A}, \mathbf{Str}(\Delta))$ satisfies $\mathrm{Im}(\varphi \circ \sigma \circ g) \subseteq \mathrm{Im}(\eta)$ and $\mathrm{Shp}_{\Delta,\mathbf{A}}(\varphi \circ \sigma \circ g) = (\Theta, g)$. ∎

Lastly, we show how diaries can be encoded via expansions of $\mathbf{K}$ in a finite relational language.

Definition 3.4.15. Define a relational language $\mathcal{L}^* \sqsupseteq \mathcal{L}$ as follows. Let $\mathbf{B}_0, \dots, \mathbf{B}_{N-1} \in \mathcal{K}$ list each enumerated structure in $\mathcal{K}$ with size at most $r := \max(2(\|\mathcal{F}\| - 1), 4)$. For each $M < N$ and $y \in D_{\mathbf{B}_M}$, introduce a new $|B_M|$-ary relation R_y into $\mathcal{L}^*$.

Given a diary Δ, define the $\mathcal{L}^*$-expansion $\mathbf{Str}^*(\Delta)$ of $\mathbf{Str}(\Delta)$ as follows. Fix $M < N$, and write $|B_M| = d$. Now given $x_0, \dots, x_{d-1} \in \mathrm{CdNd}(\Delta)$, we set $R_y^{\mathbf{Str}^*(\Delta)}(x_0, \dots, x_{d-1})$ if and only if the x_i are distinct and, writing $X = \{x_0, \dots, x_{d-1}\}$ and $y = (\Theta, g)$, we have $\Delta\|_X = \Theta$ and $g(i) = \pi_{\Delta,X}(x_i)$ for each $i < d$.

Theorem 3.4.16. *There are a finite relational language* $\mathcal{L}^* \supseteq \mathcal{L}$ *and an* $\mathcal{L}^*$*-expansion* $\mathbf{K}^*$ *of* $\mathbf{K}$ *such that given any* $\mathbf{A} \in \mathcal{K}$, $|\mathbf{K}^*(\mathbf{A})| = |D_{\mathbf{A}}|$ *and the map from* $\mathrm{Emb}(\mathbf{A}, \mathbf{K}) \to \mathbf{K}^*(\mathbf{A})$ *sending* f *to* $\mathbf{K}^* \cdot f$ *witnesses that* $\mathrm{BRD}(\mathbf{A}, \mathcal{K}) \geq |D_{\mathbf{A}}|$.

Proof. Fix a diary Δ with $\mathbf{K} \cong \mathbf{Str}(\Delta)$, and set $\mathbf{S}^* := \mathbf{Str}^*(\Delta)$. Now fix $\mathbf{A} \in \mathcal{K}$; we show that if $f_0, f_1 \in \mathrm{Emb}(\mathbf{A}, \mathbf{Str}(\Delta))$, then $\mathbf{S}^* \cdot g_0 = \mathbf{S}^* \cdot g_1$ if and only if $\mathrm{Shp}_{\Delta,\mathbf{A}}(f_0) = \mathrm{Shp}_{\Delta,\mathbf{A}}(f_1)$. Certainly, the right-to-left implication holds. For the left-to-right, this amounts to saying that a finite diary Θ (in fact any diary) is completely determined by the subdiaries $\Theta|_X$ for $X \subseteq \mathrm{CdNd}(\Theta)$ of size at most r. Since $r \geq 4$, we accurately recover the relative levels of all coding and splitting events. Since $r \geq (\|\mathcal{F}\| - 1) + 2$, we accurately recover the relative levels between coding/splitting and age-change events. And since $r \geq 2(\|\mathcal{F}\| - 1)$, we accurately record the relative levels between any pair of age-change events. Taken together, this suffices to completely reconstruct Θ. ∎

4. Proof of Theorem 3.4.12

Our proof of Theorem 3.4.12 is adapted from the characterization of the exact big Ramsey degrees for the class of finite graphs given by Laflamme, Sauer, and Vuksanovic [33]. In their proof, they more or less treat all nodes as coding nodes, and while the resulting graph is not the Rado graph, it is bi-embeddable with it. Upon defining a suitable notion of "diary", the authors then fix an embedding ψ from the universal graph coded by $2^{<\omega}$ and the diary. This leads to an important collection of pairs of nodes, namely those pairs (u, v) where $\psi^{-1}(\mathrm{Succ}(v))$ is dense above u.

Our strategy is similar, but we require several notions of largeness which are more sophisticated than density above u, as we will need to keep track of the relevant age-set structure.

For this section, we let Θ be any diary, Δ a diary with $\mathbf{Str}(\Delta) \cong \mathbf{K}$, and without loss of generality, we take $\mathbf{K} = \mathbf{Str}^{\#}(\Delta)$.

4.1. Large subsets of coding nodes

We begin by developing the notions of largeness we will need. This subsection only refers to subsets of $\mathrm{CT}^{\mathbf{K}}$; the diary Δ will not feature until the next subsection.

Definition 4.1.1. Suppose that $S \subseteq \mathrm{Im}(c^{\mathbf{K}})$. Fix $i < \mathsf{U}$, $\mathfrak{p} \in \mathsf{Full}_i$, and $u \in \mathrm{CT}^{\mathbf{K}}_i$ with $\mathrm{Age}_{\mathbf{K}}(u) = \max(\mathfrak{p})$. We define the property that S is $(u, \mathfrak{p})$-*large* by induction on $|\mathfrak{p}|$.

- If $|\mathfrak{p}| = 1$, then S is $(u, \mathfrak{p})$-large if it is dense in $\mathrm{CT}^{\mathbf{K}}$ over u.
- Otherwise, we say that S is $(u, \mathfrak{p})$-large if for every $v \sqsupseteq u$ with $\mathrm{Age}_{\mathbf{K}}(v) = \mathrm{Age}_{\mathbf{K}}(u) = \max(\mathfrak{p})$, there is $w \sqsupseteq v$ with $\mathrm{Age}_{\mathbf{K}}(w) = \max(\mathfrak{p}')$ such that S is $(w, \mathfrak{p}')$-large.

As a convention, if we refer to anything being $(u, \mathfrak{p})$-large for some $\mathfrak{p} \in \mathsf{Full}_i$, it is assumed that $u \in \mathrm{CT}^{\mathbf{K}}_i$ and $\mathrm{Age}_{\mathbf{K}}(u) = \max(\mathfrak{p})$.

Lemma 4.1.2. *Suppose* $i < \mathsf{U}$, $u \in \mathrm{CT}^{\mathbf{K}}_i$, *and* $S \subseteq \mathrm{Im}(c^{\mathbf{K}})$ *is dense in* $\mathrm{CT}^{\mathbf{K}}$ *over* u. *Then* S *is* $(u, \mathfrak{p})$-*large for any* $\mathfrak{p} \in \mathsf{Full}_i$ *with* $\max(\mathfrak{p}) = \mathrm{Age}_{\mathbf{K}}(u)$.

Proof. We induct on $|\mathfrak{p}|$. When $|\mathfrak{p}| = 1$, then we have $\mathfrak{p} = \{\min(P_i)\}$, and this is just the definition of $(u, \mathfrak{p})$-large. Now suppose $|\mathfrak{p}| \geq 2$. Consider $v \sqsupseteq u$ with $\mathrm{Age}_{\mathbf{K}}(v) = \mathrm{Age}_{\mathbf{K}}(u) = \max(\mathfrak{p})$. Find $w \sqsupseteq v$ with $\mathrm{Age}_{\mathbf{K}}(w) = \max(\mathfrak{p}')$. Since S is dense in $\mathrm{CT}^{\mathbf{K}}$ over w, our inductive assumption implies that S is $(w, \mathfrak{p}')$-large. Hence S is $(u, \mathfrak{p})$-large as desired. ∎

Lemma 4.1.3. *Suppose that* $i < \mathsf{U}$, $\mathfrak{p} \in \mathsf{Full}_i$, $u \in \mathrm{CT}^{\mathbf{K}}_i$, *and that* $S \subseteq \mathrm{Im}(c^{\mathbf{K}})$ *is* $(u, \mathfrak{p})$-*large. If* $n < \omega$ *and we write* $S = \bigcup_{j<n} S_j$, *then for some* $j < n$ *and some* $v \sqsupseteq u$ *with* $\mathrm{Age}_{\mathbf{K}}(v) = \mathrm{Age}_{\mathbf{K}}(u) = \max(\mathfrak{p})$, *we have that* S_j *is* $(v, \mathfrak{p})$-*large.*

Proof. First assume that $\mathfrak{p} = \{\min(P_i)\}$. Towards a contradiction, suppose we could find $u \in \mathrm{CT}^{\mathbf{K}}$ with $\mathrm{Age}_{\mathbf{K}}(u) = \min(P_i)$, $S \subseteq \mathrm{Im}(c^{\mathbf{K}})$ dense in $\mathrm{CT}^{\mathbf{K}}$ over u, and a partition $S = S_0 \cup \cdots \cup S_{n-1}$ so that the conclusion of the lemma fails. Set $u = u_0$. Suppose

$u_j \sqsupseteq \cdots \sqsupseteq u_0$ have been defined with $\mathrm{Age}_{\mathbf{K}}(u_j) = \min(P_i)$. As S_j is not dense in $\mathrm{CT}^{\mathbf{K}}$ over u_j, find $u_{j+1} \sqsupseteq u_j$ with $\mathrm{Age}_{\mathbf{K}}(u_{j+1}) = \min(P_i)$ and such that $S_j \cap \mathrm{Succ}(u_{j+1}) = \emptyset$. Continue until u_n has been defined. Then S is dense in $\mathrm{CT}^{\mathbf{K}}$ over u_n, but also $S_j \cap \mathrm{Succ}(u_n) = \emptyset$ for every $j < n$. This is a contradiction.

Now we proceed by induction on $|\mathfrak{p}|$. Set $u = u_0$. Suppose $u_j \sqsupseteq \cdots \sqsupseteq u_0$ have been defined for some $j < n$ with $\mathrm{Age}_{\mathbf{K}}(u_j) = \max(\mathfrak{p})$. Since S_j is not $(u_j, \mathfrak{p})$-large, we can find $u_{j+1} \sqsupseteq u_j$ with $\mathrm{Age}_{\mathbf{K}}(u_{j+1}) = \max(\mathfrak{p})$ such that for any $v \sqsupseteq u_{j+1}$ with $\mathrm{Age}_{\mathbf{K}}(v) = \max(\mathfrak{p}')$, we have that S_j is not $(v, \mathfrak{p}')$-large. However, since S is (u, Γ)-large, we can find $v \sqsupseteq u_n$ with $\mathrm{Age}_{\mathbf{K}}(v) = \max(\mathfrak{p}')$ and such that S is $(v, \mathfrak{p}')$-large. By our induction assumption, we can find $j < n$ and $w \sqsupseteq v$ with $\mathrm{Age}_{\mathbf{K}}(w) = \max(\mathfrak{p}')$ and such that S_j is $(w, \mathfrak{p}')$-large. But since $w \sqsupseteq u_{j+1}$, this is a contradiction. ■

The next proposition says that large sets of coding nodes code suitably rich substructures of $\mathbf{K}$. The proof is straightforward, but a bit long, and can be skipped on a first reading.

Proposition 4.1.4. *Let $m < \omega$, let $X = \{x_0, \ldots, x_{d-1}\} \subseteq \mathrm{CT}^{\mathbf{K}}(m)$, and write $\rho = \mathrm{Sort}(X)$. For each $i < d$, let $\mathfrak{p}_i \in \mathsf{Full}_{\rho(i)}$, and let $S_i \subseteq \mathrm{Im}(c^{\mathbf{K}}) \cap \mathrm{Succ}(x_i)$ be such that S_i is $(x_i, \mathfrak{p}_i)$-large for each $i < d$. Setting $S = \bigcup_{i<d} S_i$ and viewing $(c^{\mathbf{K}})^{-1}(S)$ as an $\mathcal{L}_d$-structure in the natural way, we have $\mathrm{Age}((c^{\mathbf{K}})^{-1}(S)) = \mathrm{Age}_{\mathbf{K}}(X)$.*

Proof. If $|\mathfrak{p}_i| = 1$ for each $i < d$, the assumption on the S_i just says that S is dense in $\mathrm{CT}^{\mathbf{K}}$ above each x_i. For each $\mathbf{B} \in \mathrm{Age}_{\mathbf{K}}(X)$, we show that $\mathbf{B} \in \mathrm{Age}((c^{\mathbf{K}})^{-1}(S))$ by induction on the size of $\mathbf{B}$ (for all d simultaneously). If $\mathbf{B}$ contains a single point b with $V^{\mathbf{B}}(b) = i$, the result is clear. Now suppose $\mathbf{B}$ is an $\mathcal{L}_d$-structure and that $|B| \geq 2$. Pick some $b \in B$, and write $V^{\mathbf{B}}(b) = i$. Let $w_i \sqsupseteq x_i$ be any member of S_i, and for each $j \in d \setminus \{i\}$, we set $w_j = \mathrm{Left}(x_j, \ell(w_i))$. Write $W = \{w_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} w_{d-1}\}$, and write $Y = \mathrm{IS}_{\mathbf{K}}(W) := \{y_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} y_{\alpha-1}\}$. Note that since $\mathrm{Age}_{\mathbf{K}}(x_i) = \max(\mathfrak{p}_i) = \min(P_{\rho(i)})$ (since $|\mathfrak{p}_i| = 1$), we have $\mathrm{Age}_{\mathbf{K}}(X) = \mathrm{Age}_{\mathbf{K}}(W)$. Let $\mathbf{C}$ be the $\mathcal{L}_\alpha$-structure on underlying set $B \setminus \{b\}$ defined as follows. The $\mathcal{L}^{\mathsf{b}}$-part of $\mathbf{C}$ is induced from $\mathbf{B}$. For the unary part, fix $a \in B \setminus \{b\}$, and suppose that $V^{\mathbf{B}}(a) = j$ and $R^{\mathbf{B}}(a, b) = q$. Then $w_j{}^\frown q \in \mathrm{CT}^{\mathbf{K}}$. Now if $\beta < \alpha$ is such that $y_\beta = w_j{}^\frown q$, we set $V^{\mathbf{C}}(a) = \beta$. Since S is dense above each member of Y, our inductive hypothesis shows that viewing $(c^{\mathbf{K}})^{-1}(S \cap \mathrm{Succ}(Y))$ as an $\mathcal{L}_\alpha$-structure, we have $\mathbf{C} \in \mathrm{Age}((c^{\mathbf{K}})^{-1}(S \cap \mathrm{Succ}(Y)))$. It follows that $\mathbf{B} \in \mathrm{Age}((c^{\mathbf{K}})^{-1}(S))$.

Now suppose we are given $\{\mathfrak{p}_j : j < d\}$, where $|\mathfrak{p}_j| \leq n$ for each $j < d$. Suppose we know the result (for all d) whenever the paths have size less than n. Write $I = \{i < d : |\mathfrak{p}_i| = n\}$. Let $\mathbf{B} \in \mathrm{Age}_{\mathbf{K}}(X)$, and write $\mathbf{B}_i$ for the induced substructure on $\{b \in B : V^{\mathbf{B}}(b) = i\}$. Fix some $N \geq \max\{|\mathbf{B}_i| : i \in I\}$, and fix some $i \in I$. Since S_i is $(x_i, \mathfrak{p}_i)$-large, there is $x'_{i,0} \sqsupseteq x_i$ so that S_i is $(x'_{i,0}, \mathfrak{p}'_i)$-large. If $x'_{i,r}$ has been defined, then since S_i is $(x_i, \mathfrak{p}_i)$-large and $\mathrm{Age}_{\mathbf{K}}(\mathrm{Left}(x_i, \ell(x'_{i,r}))) = \mathrm{Age}_{\mathbf{K}}(x_i) = \max(\mathfrak{p}_i)$, there is $x'_{i,r+1} \sqsupseteq \mathrm{Left}(x_i, \ell(x'_{i,r}))$ such that S_i is $(x'_{i,r+1}, \mathfrak{p}'_i)$-large. Continue until $x'_{i,N-1}$ is defined, and write $X_i = \{x'_{i,N-1} \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} x'_{i,0}\}$.

Do the same procedure for every $i \in I$, at each stage moving relevant points up leftmost far enough to be above anything previously considered, producing a set X_i as above.

If $i \in d\setminus I$, set $X_i = \{x_i\}$. Pick some level L above everything we have considered, and set $Y = \bigcup_{i<d} \mathrm{Left}(X_i, L) = \{y_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} y_{\alpha-1}\}$. Let $\pi\colon \alpha \to d$ be the map such that $y_\beta \sqsupseteq x_{\pi(\beta)}$ for each $\beta < \alpha$. Then by the construction of Y, we have

$$\mathrm{Age}_{\mathbf{K}}(Y) = \mathrm{Age}_{\mathbf{K}}(X)\cdot\pi \cap \bigcap_{i\in I} \bigcap_{\beta\in\pi^{-1}(\{i\})} \max(\mathfrak{p}_i')\cdot\iota_\beta^{-1}.$$

Using $\mathbf{B}$, we form an $\mathcal{L}_\alpha$-structure $\mathbf{C}$ on underlying set B as follows. The $\mathcal{L}^{\mathsf{b}}$-part is the same as $\mathbf{B}$. For each $i \in d\setminus I$, let $\beta_i < \alpha$ be the unique member of $\pi^{-1}(\{i\})$, and set $V^{\mathbf{C}}(b) = \beta_i$ for each $b \in B_i$. If $i \in I$, then for each $b \in B_i$, choose a distinct $\beta_b \in \pi^{-1}(\{i\})$, and set $V^{\mathbf{C}}(b) = \beta_b$. Now $\mathbf{C} \in \mathrm{Age}_{\mathbf{K}}(Y)$, so by our inductive hypothesis, $\mathbf{C} \in \mathrm{Age}((c^{\mathbf{K}})^{-1}(S \cap \mathrm{Succ}(Y)))$. So also $\mathbf{B} \in \mathrm{Age}((c^{\mathbf{K}})^{-1}(S))$ as desired. ∎

4.2. Compatible pairs

Our proof of Theorem 3.4.12 will proceed by building $\varphi \in \mathrm{DEmb}(\Theta, \Delta)$ by induction on the levels of Θ. To get started, we must have $\varphi(\mathfrak{p}, \emptyset) \in \Delta_{\mathfrak{p}}$ whenever $\Theta_{\mathfrak{p}} \neq \emptyset$. However, this implies $\Delta_{\mathfrak{p}} \neq \emptyset$; we need to prove this just from knowing that Δ codes $\mathbf{K}$. This is Proposition 4.2.5, which provides a lower bound for the big Ramsey degrees of singleton structures and sets the stage for the inductive construction of φ in the next subsection.

For the rest of the section, write $\psi\colon \mathrm{Im}(c^{\mathbf{K}}) \to \Delta$ for the map with $\psi(c^{\mathbf{K}}(n)) = c^{\Delta}(n)$.

Definition 4.2.1. Suppose $\mathfrak{p} \in \mathsf{Full}_i$. We say that a pair $(u, v) \in \mathrm{CT}^{\mathbf{K}} \times \Delta$ is $\mathfrak{p}$-*compatible* if $\mathrm{Age}_\Delta(v) = \max(\mathfrak{p})$ and $\psi^{-1}(\mathrm{Succ}(v))$ is $(u, \mathfrak{p})$-large. In particular, note that $\mathrm{Age}_{\mathbf{K}}(u) = \mathrm{Age}_\Delta(v) = \max(\mathfrak{p})$. Write $\mathrm{Pair}(\mathfrak{p})$ for the set of $\mathfrak{p}$-compatible pairs.

Given $v \in \Delta$, the *path continuation* of v, denoted by $\mathrm{PCon}(v)$, is the unique $\mathfrak{p} \in \mathsf{Full}$ such that $\mathrm{Age}_\Delta(v) = \max(\mathfrak{p})$ and $v^{\mathsf{p}} = \mathrm{Path}_\Delta(v) \cup \mathfrak{p}$. Corollary 4.2.4 will tell us that if $(u, v) \in \mathrm{Pair}(\mathfrak{p})$, then $\mathfrak{p} = \mathrm{PCon}(v)$. We say that $(u, v) \in \mathrm{CT}^{\mathbf{K}} \times \Delta$ is a *compatible pair* and write $(u, v) \in \mathrm{Pair}$ if $(u, v) \in \mathrm{Pair}(\mathrm{PCon}(v))$.

Given $d < \omega$, we say that functions $\sigma\colon d \to \mathrm{CT}^{\mathbf{K}}$ and $\xi\colon d \to \Delta$ are *compatible* if $(\sigma(i), \xi(i))$ is compatible for every $i < d$. In a mild abuse of notation, we simply write $(\sigma, \xi) \in \mathrm{Pair}$ when this happens. As a warning, we do not necessarily have $\mathrm{Age}_{\mathbf{K}}(\sigma) = \mathrm{Age}_\Delta(\xi)$; indeed, if ξ is not level, then $\mathrm{Age}_\Delta(\xi)$ is not even defined. If ξ is level, Propositions 4.1.4 and 3.4.3 (1) give us that $\mathrm{Age}_\Delta(\xi) \supseteq \mathrm{Age}_{\mathbf{K}}(\sigma)$.

Notice that if $(u, v) \in \mathrm{Pair}(\mathfrak{p})$ and $u_0 \sqsupseteq u$ satisfies $\mathrm{Age}_{\mathbf{K}}(u_0) = \mathrm{Age}_{\mathbf{K}}(u)$, then also $(u_0, v) \in \mathrm{Pair}(\mathfrak{p})$.

We collect some straightforward "pair-extension" properties. Before proving these, let us rephrase Proposition 3.4.3 in a way that is more suitable for our setting.

Lemma 4.2.2. *Suppose* $\sigma\colon d \to \mathrm{CT}^{\mathbf{K}}$ *is a function, and suppose* $N > \max\{\ell^\Delta(n) : n < \max\{\ell(\sigma(i)) : i < d\}\}$. *For each* $i < d$, *pick some* $s_i \in \mathrm{Im}(c^{\mathbf{K}}) \cap \mathrm{Succ}(\sigma(i))$ *such that* $\ell(\psi(s_i)) \geq N$, *and define* $\xi\colon d \to \Delta$ *via* $\xi(i) = \pi_N \circ \psi(s_i)$. *Then* $\mathrm{Age}_\Delta(\xi) \subseteq \mathrm{Age}_{\mathbf{K}}(\sigma)$.

Proof. We have for each $n < \max\{\ell(\sigma(j)) : j < d\}$ that $\psi(s_j)(\ell^\Delta(n)) = s_j(n)$. The result now follows from Proposition 3.4.3. ∎

Proposition 4.2.3. *Suppose* $i < \mathsf{U}$, $\mathfrak{p} \in \mathsf{Full}_i$, *and* $(u, v) \in \mathrm{Pair}(\mathfrak{p})$.

(1) *If* $N > \ell(v)$, *then there are* $u_0 \sqsupseteq u$ *and* $v_0 \sqsupseteq v$ *with* $\ell(v_0) = N$ *and* $(u_0, v_0) \in \mathrm{Pair}(\mathfrak{p})$.

(2) *Suppose* $|\mathfrak{p}| \geq 2$. *Then there is* $(u', v') \in \mathrm{Pair}(\mathfrak{p}')$ *with* $u' \sqsupseteq u$ *and* $v' \sqsupseteq v$.

Proof. (1) Write $\mathrm{Succ}(v) \cap \Delta(N) = \{t_0, \dots, t_{d-1}\}$. There is a finite set F for which we have

$$\psi^{-1}(\mathrm{Succ}(v)) \setminus F = \bigcup_{j<d} \psi^{-1}(\mathrm{Succ}(t_j)).$$

By Lemma 4.1.3, there is $u_0 \sqsupseteq u$ and $j < d$ such that $\psi^{-1}(\mathrm{Succ}(t_j))$ is $(u_0, \mathfrak{p})$-large. It follows from Proposition 4.1.4 that $t_j := v_0$ must have $\mathrm{Age}_\Delta(v_0) = \max(\mathfrak{p})$, i.e., $(u_0, v_0) \in \mathrm{Pair}(\mathfrak{p})$.

(2) Because $\psi^{-1}(\mathrm{Succ}(v))$ is $(u, \mathfrak{p})$-large, find $u_0 \sqsupseteq u$ with $\mathrm{Age}_\Delta(u_0) = \max(\mathfrak{p}')$ and such that $\psi^{-1}(\mathrm{Succ}(v))$ is $(u_0, \mathfrak{p}')$-large. Using Lemma 4.2.2, find N so that whenever $s \in \psi^{-1}(\mathrm{Succ}(v)) \cap \mathrm{Succ}(u_0)$ and $\ell(\psi(s)) > N$, we have $\mathrm{Age}_\Delta(\pi_N \circ \psi(s)) \subseteq \max(\mathfrak{p}')$. Let us write

$$S = \{s \in \psi^{-1}(\mathrm{Succ}(v)) \cap \mathrm{Succ}(u) : \ell(\psi(s)) > N\}.$$

Then S is a cofinite subset of $\psi^{-1}(\mathrm{Succ}(v)) \cap \mathrm{Succ}(u)$, a $(u_0, \mathfrak{p}')$-large set; hence S is also $(u_0, \mathfrak{p}')$-large. Writing $\pi_N \circ \psi[S] = \{t_0, \dots, t_{d-1}\}$, let $S_j = \{s \in S : \pi_N \circ \psi(s) = t_j\}$. We can find $u' \sqsupseteq u_0$ such that some S_j is $(u', \mathfrak{p}')$-large. Set $v' = t_j$. It remains to check that $\mathrm{Age}_\Delta(v') = \max(\mathfrak{p}')$. By choice of v', we must have $\mathrm{Age}_\Delta(v') \subseteq \max(\mathfrak{p}')$. The reverse inclusion follows from Propositions 4.1.4 and 3.4.3. ∎

Corollary 4.2.4. *Suppose* $i < \mathsf{U}$, $\mathfrak{p} \in \mathsf{Full}_i$, *and* $(u, v) \in \mathrm{Pair}(\mathfrak{p})$. *Then* $\mathfrak{p} = \mathrm{PCon}(v)$.

Proof. First note that $\mathfrak{p} \cap \mathrm{Path}_\Delta(v) = \{\max(\mathfrak{p})\}$. Our proof proceeds by induction on $|\mathfrak{p}|$. When $|\mathfrak{p}| = 1$, then $\mathrm{Path}_\Delta(v) = v^{\mathfrak{p}} \in \mathsf{MP}_i$, and the result is clear. Now suppose $|\mathfrak{p}| \geq 2$. Use Proposition 4.2.3 to find $u' \sqsupseteq u$ and $v' \sqsupseteq v$ with $(u', v') \in \mathrm{Pair}(\mathfrak{p}')$. By our inductive assumption, $\mathfrak{p}' \cup \mathrm{Path}_\Delta(v') = v^{\mathfrak{p}}$. But also $\mathfrak{p}' \cup \mathrm{Path}_\Delta(v') = \mathfrak{p} \cup \mathrm{Path}_\Delta(v)$. ∎

Proposition 4.2.5. $\Delta(0) = \mathsf{MP} \times \{\emptyset\}$. *Furthermore, if* $\xi\colon |\mathsf{MP}| \to \Delta(0)$ *lists* $\Delta(0)$ *in* $\preceq_{\mathrm{lex}}$-*order, then there is* $\sigma\colon |\mathsf{MP}| \to \mathrm{CT}^{\mathbf{K}}$ *with* $\mathrm{Age}_{\mathbf{K}}(\sigma) = \mathrm{Age}_\Delta(\xi) = \mathrm{Age}_\Delta(0)$ *and* $(\sigma, \xi) \in$ Pair.

Proof. Write $\mathsf{MP} = \{\mathfrak{p}_0 \leq_{\mathsf{MP}} \cdots \leq_{\mathsf{MP}} \mathfrak{p}_{d-1}\}$, set $n_0 = 0$, and let $\sigma_0\colon d \to \mathrm{CT}^{\mathbf{K}}$ be given by $\sigma_0(i) = (\mathsf{u}(\mathfrak{p}_i), \emptyset)$.

Assume for some $i < d$ that $0 = n_0 < \cdots < n_i$ and $\sigma_i\colon d \to \mathrm{CT}^{\mathbf{K}}$ have been determined, and $\ell(\sigma_i(j)) = n_{j+1}$ for $j < i$ and $\sigma_i(j) = \sigma_0(j)$ for $j \geq i$. Write $h = \mathsf{u}(\mathfrak{p}_i)$, and first observe that $\mathrm{Im}(c^{\mathbf{K}}) \cap (\{h\} \times k^{<\omega})$ is $((h, 0^{n_i}), \mathfrak{p})$-large. It follows from Lemma 4.1.3 that for some $u \sqsupseteq (h, 0^{n_i})$ with $\mathrm{Age}_{\mathbf{K}}(u) = \max(P_h)$ and some $\mathfrak{q} \in \mathsf{MP}_h$, we have $(u, (\mathfrak{q}, \emptyset)) \in \mathrm{Pair}(\mathfrak{p}_i)$. By Fact 3.4.2 (2) and Corollary 4.2.4, we must have $\mathfrak{q} = \mathfrak{p}_i$. In particular, $(\mathfrak{p}_i, \emptyset) \in \Delta(0)$. We set $n_{i+1} = \ell(u)$, $\sigma_{i+1}(i) = u$, and $\sigma_{i+1}(j) = \sigma_i(j)$ for $j \in d \setminus \{i\}$.

Set $\sigma = \sigma_d$; since at each stage, we moved up and leftmost in $\mathrm{CT}^{\mathbf{K}}$ above all relevant nodes, we have $\mathrm{Age}_{\mathbf{K}}(\sigma) = \mathrm{Age}_{\mathbf{K}}(\sigma_0) = \mathrm{Age}_\Delta(0)$. ∎

Remark. If $\mathbf{A} \in \mathcal{K}$ is a singleton with unary i, then Proposition 4.2.5 recovers the result of Sauer [46] that $\mathrm{BRD}(\mathbf{A}, \mathcal{K}) \geq |\mathsf{MP}_i|$.

We end the subsection with the following strengthening of Proposition 4.2.3, which allows us to run the proposition on pairs of functions while preserving the ages of said functions.

Proposition 4.2.6. *Suppose $\sigma\colon d \to \mathrm{CT}^{\mathbf{K}}$ and $\xi\colon d \to \Delta$ are functions with $(\sigma, \xi) \in \mathrm{Pair}$. Let $N < \omega$ satisfy $N \geq \max\{\ell(\xi(i)) : i < d\}$ and $N > \{\ell^{\Delta}(n) : n < \max\{\ell(\sigma(i)) : i < d\}\}$. Then there are functions $\sigma' \sqsupseteq \sigma$ and $\xi' \sqsupseteq \xi$ such that $\ell(\xi'(i)) = N$ for each $i < d$, $(\sigma', \xi') \in \mathrm{Pair}$, and $\mathrm{Age}_{\mathbf{K}}(\sigma) = \mathrm{Age}_{\mathbf{K}}(\sigma') = \mathrm{Age}_{\Delta}(\xi')$.*

Remark. Suppose $i < d$ and $\mathfrak{p} = \mathrm{PCon}(\xi(i))$, so that $(\sigma(i), \xi(i)) \in \mathfrak{p}$. Then since $\xi'(i) \sqsupseteq \xi(i)$ and $\mathrm{Age}_{\mathbf{K}}(\sigma) = \mathrm{Age}_{\Delta}(\xi')$, we will also have $\mathrm{PCon}(\xi'(i)) = \mathfrak{p}$.

Proof. Suppose that $i < d$ and that we have chosen $\sigma'(j)$ and $\xi'(j)$ for every $j < i$. If $\xi(i) = (h, 0^r)$ for some $r < \omega$ and $h \in \mathsf{U}_{\mathrm{non}}$, we simply set $\xi'(i) = (h, 0^N)$ and $\sigma'(i) = \sigma(i)$. If not, write $\mathfrak{p} = \mathrm{PCon}(\xi(i))$, pick some $M > \max(\{\ell(\sigma'(j)) : j < i\} \cup \{\ell(\sigma_j) : j < d\})$ and run Proposition 4.2.3 on $(\mathrm{Left}(\sigma(i), M), \xi(i)) \in \mathrm{Pair}(\mathfrak{p})$ to obtain $\sigma'(i) \sqsupseteq \mathrm{Left}(\sigma(i), M)$ and $\xi'(i) \sqsupseteq \xi(i)$ with $\ell(\xi'(i)) = N$ and $(\sigma'(i), \xi'(i)) \in \mathrm{Pair}(\mathfrak{p})$. Then $\mathrm{Age}_{\mathbf{K}}(\sigma') = \mathrm{Age}_{\mathbf{K}}(\sigma)$ by construction, the key here being that at each stage, we moved up and left beyond anything built so far before applying Proposition 4.2.3. As $(\sigma', \xi') \in \mathrm{Pair}$, we have $\mathrm{Age}_{\Delta}(\xi') \supseteq \mathrm{Age}_{\mathbf{K}}(\sigma')$. If N is suitably large, Lemma 4.2.2 implies $\mathrm{Age}_{\mathbf{K}}(\xi') = \mathrm{Age}_{\mathbf{K}}(\sigma)$. ∎

4.3. The construction

We now prove Theorem 3.4.12, constructing $\varphi \in \mathrm{DEmb}(\Theta, \Delta)$ level by level. We start by setting $\varphi^+(\mathfrak{p}, \emptyset) = (\mathfrak{p}, \emptyset)$ for each $(\mathfrak{p}, \emptyset) \in \Theta(0)$. Now fix $m < \mathrm{ht}(\Theta)$, and inductively assume we have defined $\varphi\colon \Theta(<m) \to \Delta$ and $\varphi^+\colon \Theta(\leq m) \to \Delta$. Write $\Theta(m) = \{t_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} t_{d-1}\}$ and $\varphi^+[\Theta(m)] = \{v_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} v_{d-1}\}$. We assume that our construction to this point satisfies all of the following:

1. If $m \geq 1$, then the map $\varphi^+\colon \Theta(m) \to \Delta$ satisfies $\varphi^+(v^\frown i) = \varphi(v)^\frown i$ for every $v \in \Theta(m-1)$ and $i < k$ with $v^\frown i \in \Theta$, and furthermore, $\mathrm{Age}_{\Theta}(m) = \mathrm{Age}_{\Delta}(\varphi^+[\Theta(m)])$.
2. There are $\sigma_m\colon d \to \mathrm{CT}^{\mathbf{K}}$ and $\xi_m\colon d \to \Delta$ with $\xi_m(i) \sqsupseteq v_i$ for each $i < d$, with $\mathrm{Im}(\xi_m)$ a level set, with $\mathrm{Age}_{\Delta}(\varphi^+[\Theta(m)]) = \mathrm{Age}_{\Delta}(\xi_m) = \mathrm{Age}_{\mathbf{K}}(\sigma_m)$, and with $(\sigma_m, \xi_m) \in \mathrm{Pair}$.

When $m = 0$, we let $\xi_0(i) = v_i \in \mathsf{MP} \times \{\emptyset\}$ and $\sigma_0(i)$ be given by Proposition 4.2.5.

There are now three cases depending on whether $m \in \mathrm{Sp}(\Theta)$, $\mathrm{AC}(\Theta)$, or $\mathrm{Cd}(\Theta)$.

Case 1: $m \in \mathrm{Sp}(\Theta)$. Suppose $t_i \in \Theta(m)$ is the splitting node. Write $t_i^{\mathsf{p}} = \mathfrak{p} \in \mathsf{MP}$ and $t_i^{\mathsf{u}} = h < \mathsf{U}$. This case splits into three further subcases. Two of these cases are quite similar. In the first subcase, we have $h \in \mathsf{U}_{\mathrm{fr}}$. Let $1 \leq q < k$ and $j < \mathsf{U}$ witness that $h \in \mathsf{U}_{\mathrm{fr}}$, i.e., that $\mathcal{K}\cdot\gamma_{j,h,q} = \mathcal{K}\cdot\tilde{\iota}_i$. In the second subcase, we have $h \in \mathsf{U}_{\mathrm{non}}$, but with $t_i \neq$

$(\mathfrak{p}, 0^m)$. We note that $\mathrm{Age}_\Theta(t_i) = \mathrm{Age}_\Delta(\xi_m(i)) = \mathrm{Age}_{\mathbf{K}}(\sigma_m(i)) \subsetneq \max(P_h)$. It follows that $\sigma_m(i)^{\mathsf{seq}} \neq 0^{\ell(\sigma_m(i))}$. If $n < \ell(\sigma_m(i))$ is least with $\sigma_m(i)(n) \neq 0$, write $q = \sigma_m(i)(n)$, and write $c^{\mathbf{K}}(n)^{\mathsf{u}} = j$. For these two cases, the proof is now identical with these choices of $q < k$ and $j < \mathsf{U}$. Find some $r > \max\{\ell(u) : u \in \mathrm{Im}(\sigma_m)\}$ such that $c^{\mathbf{K}}(r) = (j, 0^r)$ and $\ell^\Delta(r) > \max\{\ell(\xi_m(\alpha)) : \alpha < d\}$. It follows that we have

$$\mathrm{Age}_{\mathbf{K}}(\sigma_m(i)) = \mathrm{Age}_{\mathbf{K}}(\mathrm{Left}(\sigma_m(i), r+1)) = \mathrm{Age}_{\mathbf{K}}(\mathrm{Left}(\sigma_m(i), r)^\frown q).$$

In particular, writing $\mathfrak{q} = \mathrm{PCon}(\xi_m(i))$, we have

$$(\mathrm{Left}(\sigma_m(i), r+1), \xi_m(i)),\ (\mathrm{Left}(\sigma_m(i), r)^\frown q, \xi_m(i)) \in \mathrm{Pair}(\mathfrak{q}).$$

Using Proposition 4.2.6, find $w_0 \sqsupseteq \mathrm{Left}(\sigma_m(i), r+1)$, $w_q \sqsupseteq \mathrm{Left}(\sigma_m(i), r)^\frown q$, and $x_0, x_q \sqsupseteq \xi_m(i)$ satisfying:

- $\ell(x_0) = \ell(x_q) = \ell^\Delta(r) + 1$,
- $(w_a, x_a) \in \mathrm{Pair}(\mathfrak{q})$ for each $a \in \{0, q\}$,
- $\mathrm{Age}_{\mathbf{K}}(\{w_0, w_q\}) = \mathrm{Sp}(\max(\mathfrak{q}), 0)$.

Note that $w_0 \preceq_{\mathrm{lex}} w_q$ or $w_q \preceq_{\mathrm{lex}} w_0$ are both possible. For $a \in \{0, q\}$, note that since $w_a(r) = a$ and since $(w_a, x_a) \in \mathrm{Pair}(\mathfrak{q})$, we must have $x_a(\ell^\Delta(r)) = a$ and $x_q(\ell^\Delta(r)) = q$. In particular, $x_0 \neq x_q$. We set

$$\varphi(t_i) = x_0 \wedge x_q,$$

and hence $\widetilde{\varphi}(m) = \ell(x_0 \wedge x_q)$. Put $y_i = \varphi(t_i)^\frown 0$ and $y_{i+1} = \varphi(t_i)^\frown 1$. For $a \in \{0, q\}$, write $w_a = \overline{w}_i$ or $\overline{w}_{i+1}$ depending on if $\pi_{\widetilde{\varphi}(m)+1}(x_a) = y_i$ or y_{i+1}. Define $\sigma' \colon (d+1) \to \mathrm{CT}^{\mathbf{K}}$ and $\xi' \colon (d+1) \to \Delta$ be given by

$$\sigma'(b) = \begin{cases} \mathrm{Sp}(\sigma_m, i)(b) & \text{if } b \notin \{i, i+1\}, \\ \overline{w}_b & \text{if } b \in \{i, i+1\}, \end{cases}$$

$$\xi'(b) = \begin{cases} \mathrm{Sp}(\xi_m, i)(b) & \text{if } b \notin \{i, i+1\}, \\ y_b & \text{if } b \in \{i, i+1\}. \end{cases}$$

We note that $\mathrm{Age}_{\mathbf{K}}(\sigma') = \mathrm{Sp}(\mathrm{Age}_{\mathbf{K}}(\sigma_m), i) = \mathrm{Sp}(\mathrm{Age}_\Theta(m), i)$ and that $(\sigma', \xi') \in \mathrm{Pair}$. Now use Proposition 4.2.6 on (σ', ξ') using a suitably large $N > \widetilde{\varphi}(m) + 1$. Doing this, we obtain functions $\sigma_{m+1} \colon (d+1) \to \mathrm{CT}^{\mathbf{K}}$ and $\xi_{m+1} \colon (d+1) \to \Delta(N)$ with $\sigma_{m+1} \sqsupseteq \sigma'$, $\xi_{m+1} \sqsupseteq \xi'$, $(\sigma_{m+1}, \xi_{m+1}) \in \mathrm{Pair}$, and $\mathrm{Age}_{\mathbf{K}}(\sigma') = \mathrm{Age}_{\mathbf{K}}(\sigma_{m+1}) = \mathrm{Age}_\Delta(\xi_{m+1})$. For each $b \in d \backslash \{i\}$, set

$$\varphi(t_b) = \pi_{\widetilde{\varphi}(m)} \circ \xi_{m+1} \circ ((d+1) \backslash \{i\})(b).$$

We need to check that $\mathrm{Age}_\Theta(m) = \mathrm{Age}_\Delta(\varphi[\Theta(m)])$. To do this, we define

$$\varphi^+ \colon \Theta(m+1) \to \Delta$$

from $\varphi|_{\Theta(m)}$ in the only way we can and verify that

$$\mathrm{Age}_\Theta(m+1) = \mathrm{Age}_\Delta(\varphi^+[\Theta(m+1)]).$$

Note that if we write $\varphi^+[\Theta(m+1)] = \{z_0 \preceq_{\text{lex}} \cdots \preceq_{\text{lex}} z_d\}$, we have $\xi_{m+1}(b) \sqsupseteq z_b$ for each $b < d+1$. Hence, we have

$$\begin{aligned}\text{Age}_\Theta(m+1) &= \text{Sp}(\text{Age}_\Theta(m), i) = \text{Age}_{\mathbf{K}}(\sigma') = \text{Age}_\Delta(\xi_{m+1})\\ &\subseteq \text{Age}_\Delta(\varphi^+[\Theta(m+1)]).\end{aligned}$$

The reverse inequality follows from the observation that

$$\begin{aligned}\text{Age}_\Delta(\varphi^+[\Theta(m+1)]) &\subseteq \text{Sp}(\text{Age}_\Delta(\varphi[\Theta(m)]), i) = \text{Sp}(\text{Age}_\Theta(m), i)\\ &= \text{Age}_\Theta(m+1).\end{aligned}$$

In the third subcase, we have $h \in \mathsf{U}_{\text{non}}$ and $t_i = (\mathfrak{p}, 0^m)$. Since we have $\text{Age}_\Delta(\xi_m) = \text{Age}_\Delta(\varphi^+[\Theta(m)])$, we must also have $\xi_m(i)^{\mathsf{seq}} = 0^{\ell(\xi_m(i))}$. We apply Proposition 4.2.3 to find $u \sqsupseteq \sigma_m(i)$ and $v \sqsupseteq \xi_m(i)$ with $(u, v) \in \text{Pair}(\mathfrak{p}')$. As $\text{Age}_\Delta(v) = \max(\mathfrak{p}') \subsetneq \max(\mathfrak{p})$, we have $v^{\mathsf{seq}} \neq 0^\ell(v)$. If $n < \ell(v)$ is least with $v(n) \neq 0$, then $n > \ell(\xi_m(i))$, and we set

$$\varphi(t_i) = v|_n,$$

and hence $\widetilde{\varphi}(m) = n$. Form the functions $\sigma' \colon (d+1) \to \text{CT}^{\mathbf{K}}$ and $\xi' \colon (d+1) \to \Delta$ as follows:

$$\sigma'(b) = \begin{cases} \text{Sp}(\sigma_m, i)(b) & \text{if } b \neq i+1,\\ u & \text{if } b = i+1,\end{cases}$$

$$\xi'(b) = \begin{cases} \text{Sp}(\xi_m, i)(b) & \text{if } b \neq i+1,\\ v & \text{if } b = i+1.\end{cases}$$

We note that $(\sigma', \xi') \in \text{Pair}$. From here, the definitions of σ_{m+1}, ξ_{m+1}, and $\varphi(t_b)$ for $b \in d\backslash\{i\}$ as well as the verification that $\text{Age}_\Theta(m+1) = \text{Age}_\Delta(\varphi^+[\Theta(m+1)])$ are identical to the first two cases.

Case 2: $m \in \text{AC}(\Theta)$. Write $\rho = \text{Sort}(\Theta(m))$, and write $e \colon d_0 \to d$ for the unique increasing injection with $(\text{Age}_\Theta(m)\cdot e, \text{Age}_\Theta(m+1)\cdot e) \in \text{ECon}(\rho \circ e)$ (Proposition 2.2.6). There are two cases. If $d_0 > 1$, we first find a function $\sigma' \colon d \to \text{CT}^{\mathbf{K}}$ with $\sigma' \sqsupseteq \sigma_m$ so that $\text{Age}_{\mathbf{K}}(\sigma') = \text{Age}_\Theta(m+1)$. When $d_0 > 1$, we still have $(\sigma', \xi_m) \in \text{Pair}$, so we run Proposition 4.2.6 on (σ', ξ_m) with some suitably large N, obtaining functions $\sigma_{m+1} \colon (d+1) \to \text{CT}^{\mathbf{K}}$ and $\xi_{m+1} \colon (d+1) \to \Delta(N)$ with $\sigma_{m+1} \sqsupseteq \sigma'$, $\xi_{m+1} \sqsupseteq \xi_m$, $(\sigma_{m+1}, \xi_{m+1}) \in \text{Pair}$, and $\text{Age}_{\mathbf{K}}(\sigma') = \text{Age}_{\mathbf{K}}(\sigma_{m+1}) = \text{Age}_\Delta(\xi_{m+1})$. To determine φ, we search between the levels of ξ_m and ξ_{m+1} to find the level of the age-change, and call this level $\widetilde{\varphi}(m)$, and then set $\varphi(t_b) = \pi_{\widetilde{\varphi}(m)} \circ \xi_{m+1}(b)$. Define $\varphi^+|_{\Theta(m+1)}$ in the only way possible, and by construction $\text{Age}_\Theta(m+1) = \text{Age}_\Delta(\varphi^+[\Theta(m+1)])$.

If $d_0 = 1$, suppose $e = \iota_i$ for some $i < d$. Set $\mathfrak{p} = \text{PCon}(\xi_m(i))$. We use Proposition 4.2.3 to find $(u, v) \in \text{Pair}(\mathfrak{p}')$ with $u \sqsupseteq \sigma_m(i)$ and $v \sqsupseteq \xi(i)$. We then consider the functions σ' and ξ', which are identical to σ_m and ξ, except we set $\sigma'(i) = u$ and $\xi'(i) = v$, and then apply Proposition 4.2.6. The rest of the proof is similar to the $d > 1$ case.

Case 3: $m \in \mathrm{Cd}(\Delta)$. Let $j < d$ be such that $t_j \in \mathrm{CdNd}(\Theta)$, and set $t_j^{\mathsf{p}} = \mathfrak{p} \in \mathsf{MP}$, $t_j^{\mathsf{u}} = h < \mathsf{U}$. Write $\mathcal{A} = \mathrm{Age}_\Theta(m)$, so also $\mathrm{Age}_{\mathbf{K}}(\sigma_m) = \mathrm{Age}_\Delta(\xi_m) = \mathcal{A}$. We also write $\chi\colon (d-1) \to k$ for the function with $(t_{(d\setminus\{j\})(i)})^\frown \chi(i) \in \Theta(m+1)$ for each $i < d-1$. Since $(\mathcal{A}, j, \chi)$ is a controlled coding triple, item (2) of Definition 3.2.2 gives us that $\mathrm{Age}_\Theta(\xi_m(j)) = \min(P_h)$, implying that $(\sigma_m(j), \xi_m(j)) \in \mathrm{Pair}(\min(P_h))$. So find $r < \omega$ such that $c^{\mathbf{K}}(r) \sqsupseteq \sigma_m(j)$ and $c^\Delta(r) \sqsupseteq \xi_m(j)$. Define $\sigma'\colon (d-1) \to \mathrm{CT}^{\mathbf{K}}$ via $\sigma'(i) = \mathrm{Left}(\sigma_m \circ (d\setminus\{j\})(i), r)^\frown \chi(i)$. Since $\mathrm{Age}_{\mathbf{K}}(\sigma') = \mathcal{A}\cdot\mathrm{Add}_{j,\chi}$, item (1) of Definition 3.2.2 yields $\mathrm{Age}_{\mathbf{K}}(\sigma') = \mathcal{A}\cdot(d\setminus\{j\})$.

We now run Proposition 4.2.6 on $(\sigma', \xi_m \circ (d\setminus\{j\}))$ using some suitably large $N > r$. Doing this, we obtain functions $\sigma_{m+1}\colon (d-1) \to \mathrm{CT}^{\mathbf{K}}$ and $\xi_{m+1}\colon (d-1) \to \Delta(N)$ with $\sigma_{m+1} \sqsupseteq \sigma'$, $\xi_{m+1} \sqsupseteq \xi_m \circ (d\setminus\{j\})$, $(\sigma_{m+1}, \xi_{m+1}) \in \mathrm{Pair}$, and $\mathrm{Age}_{\mathbf{K}}(\sigma') = \mathrm{Age}_{\mathbf{K}}(\sigma_{m+1}) = \mathrm{Age}_\Delta(\xi_{m+1})$. As $\sigma_{m+1}(i) \sqsupseteq \sigma'(i)$ for each $i < d-1$, we must have $\xi_{m+1}(i)(r) = \chi(i)$. We now set $\varphi(t_j) = c^\Delta(r)$ (so $\widetilde{\varphi}(m) = \ell^\Delta(r)$), and for $b \in d\setminus\{j\}$, we set

$$\varphi(t_b) = \begin{cases} \pi_{\widetilde{\varphi}(m)} \circ \mathrm{Sp}(\xi_{m+1}, j)(b) & \text{if } j < d-1, \\ \pi_{\widetilde{\varphi}(m)} \circ \xi_{m+1}(b) & \text{if } j = d-1. \end{cases}$$

With this definition of $\varphi|_{\Theta(m)}$, the function $\varphi^+|_{\Theta(m+1)}$ is well defined. To show that $\mathrm{Age}_\Delta(\varphi[\Theta(m)]) = \mathcal{A}$, write $\mathcal{B} = \mathrm{Age}_\Delta(\varphi[\Theta(m)])$. Then

$$\mathcal{B} \subseteq \mathcal{A}, \quad \mathcal{B}\cdot(d\setminus\{j\}) = \mathcal{A}\cdot(d\setminus\{j\}) \quad \text{and} \quad \mathcal{B}\cdot\mathrm{Add}_{j,\chi} = \mathcal{A}\cdot\mathrm{Add}_{j,\chi}.$$

As $(\mathcal{A}, j, \chi)$ is a controlled coding triple, item (3) of Definition 3.2.2 gives us that $\mathcal{A} = \mathcal{B}$. Lastly, $\mathrm{Age}_\Theta(m+1) = \mathrm{Age}_\Delta(\varphi^+[\Theta(m+1)])$ by the properties of controlled coding triples.

4.4. Degenerate unaries

Up until now, we have been assuming that every $i < \mathsf{U}$ is a non-degenerate unary predicate (Definition 3.3.1). We now briefly discuss the modifications that need to be made when there are degenerate unary predicates; luckily, these modifications are all very straightforward.

We now assume that for some $\mathsf{U}_{\mathrm{deg}} \subseteq \mathsf{U}$, every $i \in \mathsf{U}_{\mathrm{deg}}$ is degenerate. We have for each $i \in \mathsf{U}_{\mathrm{deg}}$ that $\{a \in \mathbf{K} : U^{\mathbf{K}}(a) = i\}$ is an infinite set of isolated points. Letting $\mathbf{K}_{nd} = \{a \in \mathbf{K} : U^{\mathbf{K}}(a) \notin \mathsf{U}_{\mathrm{deg}}\} \subseteq \mathbf{K}$ denote the induced substructure on the points with non-degenerate unary predicate, we have $\mathrm{Aut}(\mathbf{K}) \cong \mathrm{Aut}(\mathbf{K}_{nd}) \times (S_\infty)^{|\mathsf{U}_{\mathrm{deg}}|}$. If $\mathbf{A} \leq \mathbf{K}$ is enumerated, then for every $a < A$ with $U^{\mathbf{A}}(a) \in \mathsf{U}_{\mathrm{deg}}$, we have $c^{\mathbf{A}}(a)^{\mathsf{seq}} = 0^a$.

We now collect all of the new conventions and definition modifications needed to define diaries and prove Theorem 3.4.12 in this slightly more general setting.

(1) Degenerate unary predicates are not free; however, we reserve the notation $\mathsf{U}_{\mathrm{non}}$ for the set of non-free *and non-degenerate* unaries. Hence $\mathsf{U} = \mathsf{U}_{\mathrm{fr}} \sqcup \mathsf{U}_{\mathrm{non}} \sqcup \mathsf{U}_{\mathrm{deg}}$, and similarly we write $\mathsf{MP} = \mathsf{MP}_{\mathrm{fr}} \sqcup \mathsf{MP}_{\mathrm{non}} \sqcup \mathsf{MP}_{\mathrm{deg}}$. If $i \in \mathsf{U}_{\mathrm{deg}}$, then $P_i = \{\mathcal{K}\cdot\tilde{\imath}_i\}$ and $|\mathsf{MP}_i| = 1$.

(2) In Definition 3.4.1, the levels $m < \mathrm{ht}(\Delta)$ with $|\mathrm{IS}_\Delta(s)| = 1$ for every $s \in \Delta(m)$ now can be either age-change levels or a "degenerate coding" level. Writing $\mathrm{Dcd}(\Delta)$ for the set of degenerate coding levels, we have $\mathrm{ht}(\Delta) = \mathrm{Cd}(\Delta) \sqcup \mathrm{DCd}(\Delta) \cup \mathrm{Sp}(\Delta) \cup \mathrm{AC}(\Delta)$.

If $m \in \mathrm{Sp}(\Delta) \cup \mathrm{Cd}(\Delta)$ and $t \in \Delta(m)$ is the splitting or coding node, we demand $t^{\mathsf{u}} \notin \mathsf{U}_{\mathrm{deg}}$. Hence if $s \in \Delta(m)$ has $s^{\mathsf{u}} \in \mathsf{U}_{\mathrm{deg}}$, then $s^{\mathsf{seq}} = 0^m$.

If $m \in \mathrm{DCd}(\Delta)$, then we designate exactly one $t \in \Delta(m)$ with $t^{\mathsf{u}} \in \mathsf{U}_{\mathrm{deg}}$ to be a coding node. We can choose whether or not t is terminal. If t is terminal, then writing $S = \Delta(m) \setminus \{t\}$, we set $\Delta(m+1) = \{s^\frown 0 : s \in S\}$ and $\mathrm{Age}_\Delta(m+1) = \mathrm{Age}_\Delta(S)$. If t is non-terminal, we set $\Delta(m+1) = \{s^\frown 0 : s \in \Delta(m)\}$ and $\mathrm{Age}_\Delta(m+1) = \mathrm{Age}_\Delta(m)$. Hence $\mathrm{CdNd}(\Delta)$ now contains every terminal node of Δ (in particular, all non-degenerate coding nodes) as well as possibly some non-terminal degenerate coding nodes.

(3) In the proof of Proposition 3.4.8, we explicitly define $\mathrm{CdNd}(\Delta\|_X) = \mathrm{Im}(\pi_{\Delta,X})$.

(4) The construction of the previous subsection now has a case (4), namely $m \in \mathrm{DCd}(\Theta)$. If $t_j \in \Theta(m)$ is the coding node, find a suitably large N with

$$\mathrm{Left}(\xi_m(j), N-1) \in \mathrm{CdNd}(\Delta).$$

Apply Proposition 4.2.6 on (σ_m, ξ_m) with level N to obtain $(\sigma_{m+1}, \xi_{m+1})$, and for each $i < d$, set $\varphi(t_i) = \pi_{N-1} \circ \xi_{m+1}(i)$. Define $\varphi^+|_{\Theta(m+1)}$ as we must, depending on whether or not t_j is terminal.

5. Upper bounds

This section uses the "coding tree Milliken theorem" [51, Theorem 3.5] to show that the lower bounds we produced in Sections 3 and 4 are sharp.

For this section, we take $\mathbf{K}$ to be an enumerated structure. Furthermore, we choose this enumeration to be *left dense* (see [51] for the definition). For us, this amounts to saying that for every $t \in \mathrm{CT}^{\mathbf{K}}$, there is $n \geq \mathrm{ht}(t)$ with $\mathrm{Left}(t,n) = c^{\mathbf{K}}(n)$. Every Fraïssé free amalgamation class admits an enumerated left-dense Fraïssé limit.

Theorem ([51, Theorem 3.5]). *For any enumerated* $\mathbf{A} \in \mathcal{K}$, $r < \omega$, *and coloring*

$$\chi\colon \mathrm{AEmb}(\mathrm{CT}^{\mathbf{A}}, \mathrm{CT}^{\mathbf{K}}) \to r,$$

there is $\varphi \in \mathrm{AEmb}(\mathrm{CT}^{\mathbf{K}}, \mathrm{CT}^{\mathbf{K}})$ *such that* χ *is constant on* $\varphi \circ \mathrm{AEmb}(\mathrm{CT}^{\mathbf{A}}, \mathrm{CT}^{\mathbf{K}})$.

Remark. Coding trees of enumerated structures are defined slightly differently in [51] (see Section 3 for a discussion); however, the proof of the theorem for the version we use here is almost identical.

5.1. *Shapes of embeddings into enumerated structures*

Notation 5.1.1. Let $\mathcal{L}' = \mathcal{L} \cup \{\leq_\ell\}$. We can view an $\mathcal{L}$-structure $\mathbf{A}$ with $A \subseteq \omega$ as an $\mathcal{L}'$-structure by letting $\leq_\ell$ be the usual order. If Θ is a diary, we can view $\mathbf{Str}(\Theta)$ as an $\mathcal{L}'$-

structure where $\leq_\ell$ is the order of relative levels. We will refer to embeddings between these instances of $\mathcal{L}'$-structures as *ordered* embeddings; any reference to embeddings without the word "ordered" refers to $\mathcal{L}$-structures. If $\mathbf{A}$ and $\mathbf{B}$ are $\mathcal{L}'$-structures as above, we write $\mathrm{OEmb}(\mathbf{A}, \mathbf{B})$ for the $\mathcal{L}'$-embeddings ("O" for "ordered") and $\mathrm{Emb}(\mathbf{A}, \mathbf{B})$ for the $\mathcal{L}$-embeddings.

We recall some facts from [51] that we will need going forward.

Fact 5.1.2. (1) By [51, Theorem 2.10], $\mathrm{AEmb}(\mathrm{CT}^{\mathbf{Y}}, \mathrm{CT}^{\mathbf{K}}) \neq \emptyset$ for any enumerated structure $\mathbf{Y} \leq \mathbf{K}$. If $\mathbf{E}, \mathbf{Y} \leq \mathbf{K}$ are enumerated structures and $\varphi \in \mathrm{AEmb}(\mathrm{CT}^{\mathbf{E}}, \mathrm{CT}^{\mathbf{Y}})$, note that $\widetilde{\varphi} \in \mathrm{OEmb}(\mathbf{E}, \mathbf{Y})$.

(2) Conversely, fix enumerated structures $\mathbf{E}, \mathbf{Y} \leq \mathbf{K}$. Given $S \subseteq Y$, we call S $\mathbf{Y}$*-closed* if $\mathrm{Crit}^{\mathbf{Y}}(c^{\mathbf{Y}}[S]) = S$, and the *closure of* S *in* $\mathbf{Y}$, denoted by $\overline{S}^{\mathbf{Y}}$, is the smallest $\mathbf{Y}$-closed subset of Y containing S. If $\sigma \in \mathrm{OEmb}(\mathbf{E}, \mathbf{Y})$, then $\sigma = \widetilde{\varphi}$ for some $\varphi \in \mathrm{AEmb}(\mathrm{CT}^{\mathbf{E}}, \mathrm{CT}^{\mathbf{Y}})$ if and only if $\mathrm{Im}(\sigma) = \overline{\mathrm{Im}(\sigma)}^{\mathbf{Y}}$; this is more or less [51, Proposition 4.2 and Definition 5.1]. Furthermore, *because in this paper we define* $\mathrm{CT}^{\mathbf{E}} = \mathrm{Im}(c^{\mathbf{E}})\!\downarrow$, such a φ is unique.

Because of this fact, we can work interchangeably with either aged embeddings of $\mathrm{CT}^{\mathbf{E}}$ into $\mathrm{CT}^{\mathbf{Y}}$ or ordered embeddings of $\mathbf{E}$ into $\mathbf{Y}$ with $\mathbf{Y}$-closed image.

Definition 5.1.3. Fix an enumerated $\mathbf{Y} \leq \mathbf{K}$, $\mathbf{A} \leq \mathbf{Y}$, and $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{Y})$. The $\mathbf{Y}$*-shape* of f, denoted by $\mathrm{Shp}_{\mathbf{Y}}(f)$, is the unique pair $(\mathbf{E}, g)$ such that

(1) $\mathbf{E} \leq \mathbf{Y}$ is an enumerated structure and $g \in \mathrm{Emb}(\mathbf{A}, \mathbf{E})$.

(2) There is (a unique) $\alpha \in \mathrm{OEmb}(\mathbf{E}, \mathbf{Y})$ with $\alpha \circ g = f$ and $\mathrm{Im}(\alpha) = \overline{\mathrm{Im}(f)}^{\mathbf{Y}}$.

In particular, $\mathbf{E} \cong \mathbf{Y}|_{\overline{\mathrm{Im}(f)}^{\mathbf{Y}}}$.

Fact 5.1.4. Reasoning as in the proof of [51, Theorem 4.6], we have for any $\mathbf{A} \in \mathcal{K}$ and for any $\eta \in \mathrm{Emb}(\mathbf{K}, \mathbf{K})$ that

$$\mathrm{BRD}(\mathbf{A}, \mathcal{K}) \leq |\{\mathrm{Shp}_{\mathbf{K}}(\eta \circ f) : f \in \mathrm{Emb}(\mathbf{A}, \mathbf{K})\}|.$$

The main theorem of this section is an application of Fact 5.1.4. Recall, given $\mathbf{A} \in \mathcal{K}$, the set $D_{\mathbf{A}}$ and the coloring $\mathrm{Shp}_{\Delta,\mathbf{A}} \colon \mathrm{Emb}(\mathbf{A}, \mathbf{Str}(\Delta)) \to D_{\mathbf{A}}$ from Definition 3.4.13.

Theorem 5.1.5. *Fix a diary* Δ *with* $\mathbf{Str}(\Delta) \cong \mathbf{K}$. *There is* $\eta \in \mathrm{Emb}(\mathbf{Str}(\Delta), \mathbf{K})$ *such that for any* $\mathbf{A} \in \mathcal{K}$ *and* $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{Str}(\Delta))$, $\mathrm{Shp}_{\mathbf{K}}(\eta \circ f)$ *depends only on* $\mathrm{Shp}_{\Delta}(f)$. *Thus* $\mathrm{BRD}(\mathbf{A}, \mathcal{K}) \leq |D_{\mathbf{A}}|$.

We prove Theorem 5.1.5 in the next subsection. Combined with Theorem 3.4.12, we have $\mathrm{BRD}(\mathbf{A}, \mathcal{K}) = |D_{\mathbf{A}}|$, proving Theorem 4. Note that Theorems 3.4.16 and 5.1.5 together imply that whenever Δ is a diary with $\mathbf{Str}(\Delta) \cong \mathbf{K}$, then $\mathbf{Str}^*(\Delta)$ (Definition 3.4.15) is a big Ramsey structure. The following proves Theorem 5.

Theorem 5.1.6. *Whenever* Δ *is a diary with* $\mathbf{Str}(\Delta) \cong \mathbf{K}$, *then* $\mathbf{Str}^*(\Delta)$ *is a big Ramsey structure for* $\mathcal{K}$ *satisfying IRT.*

Proof. Write $\mathbf{S}^* = \mathbf{Str}^*(\Delta)$. Fix $\mathbf{A}^* \in \mathrm{Age}(\mathbf{S}^*)$ and a coloring $\gamma\colon \mathrm{Emb}(\mathbf{A}^*, \mathbf{S}^*) \to 2$. Writing $\mathbf{A} = \mathbf{A}^*|_{\mathcal{L}}$, we obtain a coloring $\overline{\gamma}\colon \mathrm{Emb}(\mathbf{A}, \mathbf{Str}(\Delta)) \to 2 \cup \mathbf{S}^*(\mathbf{A})\backslash\{\mathbf{A}^*\}$ by setting

$$\overline{\gamma}(f) = \begin{cases} \mathbf{S}^*\cdot f & \text{if } \mathbf{S}^*\cdot f \neq \mathbf{A}^*, \\ \gamma(f) & \text{if } \mathbf{S}^*\cdot f = \mathbf{A}^*. \end{cases}$$

By Theorem 5.1.5, there is $\eta \in \mathrm{Emb}(\mathbf{Str}(\Delta), \mathbf{Str}(\Delta))$ such that $|\mathrm{Im}(\overline{\gamma} \circ \eta)| \leq |\mathbf{S}^*(\mathbf{A})|$, i.e., so that we can avoid one of the colors of $\overline{\gamma}$. Write $\Theta = \Delta\|_{\mathrm{Im}(\eta)}$. As $\mathbf{Str}(\Theta) \cong \mathbf{K}$, Theorem 3.4.12 implies that the avoided color is one of the two colors from the $\mathbf{S}^*\cdot f = \mathbf{A}^*$ case. Write $\varphi = \varphi_{\Delta,\mathrm{Im}(\eta)}$, and using Theorem 3.4.12, fix some $\psi \in \mathrm{DEmb}(\Delta, \Theta)$. Then $\varphi \circ \psi|_{\mathbf{Str}(\Delta)} := \sigma \in \mathrm{Emb}(\mathbf{S}^*, \mathbf{S}^*)$ satisfies that $\gamma \circ \sigma$ is constant. ■

Remark. Fix a Fraïssé class $\mathcal{M}$ with $\mathrm{Flim}(\mathcal{M}) = \mathbf{M}$. If $\mathbf{M}^*$ is an expansion of $\mathbf{M}$, call $\mathbf{M}^*$ *recurrent* if for every $\eta \in \mathrm{Emb}(\mathbf{M}, \mathbf{M})$, there is $\theta \in \mathrm{Emb}(\mathbf{M}, \mathbf{M})$ with $\mathbf{M}^*\cdot(\eta \circ \theta) = \mathbf{M}^*$. Hence Theorem 3.4.12 asserts that $\mathbf{Str}^*(\Delta)$ is a recurrent expansion of $\mathbf{Str}(\Delta)$. The argument in Theorem 5.1.6 shows generally that if $\mathbf{M}^*$ is a recurrent big Ramsey structure for $\mathcal{M}$, then $\mathbf{M}^*$ satisfies IRT. In a partial converse, one can show that if $\mathbf{M}^*$ is a big Ramsey structure in a finite relational language which satisfies IRT, then $\mathbf{M}^*$ is recurrent.

5.2. *Proof of Theorem* 5.1.5

We fix a diary Δ which codes $\mathbf{K}$. *To ease notation*, we assume $\mathsf{U}_{\mathrm{deg}} = \emptyset$. The proof is almost identical when $\mathsf{U}_{\mathrm{deg}} \neq \emptyset$.

Fix a function $\mathsf{f}\colon \mathsf{MP} \to \mathsf{U} \times (k\backslash\{0\})$ such that if $\mathfrak{p} \in \mathsf{MP}$, $\mathsf{u}(\mathfrak{p}) = i < \mathsf{U}$, and $\mathsf{f}(\mathfrak{p}) = (j, q)$, then if $i \in \mathsf{U}_{\mathrm{fr}}$, we have $\mathcal{K}\cdot\gamma_{j,i,q} = \mathcal{K}\cdot\tilde{\iota}_i$ (Definition 3.3.2), and if $i \in \mathsf{U}_{\mathrm{non}}$, then $\mathcal{K}\cdot\gamma_{j,i,q} = \max(\mathfrak{p}')$ (Fact 3.3.4). We can arrange for $i \in \mathsf{U}_{\mathrm{fr}}$ that $|\mathsf{f}[\mathsf{MP}_i]| = 1$, so simply write $\mathsf{f}(i)$.

Write $\mathrm{ECon} = \bigsqcup_\rho \mathrm{ECon}(\rho)$, where the union is taken over all path sorts. By Fact 2.2.5 and Definition 2.3.3, there are only finitely many path sorts ρ with $\mathrm{ECon}(\rho) \neq \emptyset$, so in particular, ECon is finite. For each $(\mathcal{A}, \mathcal{B}) \in \mathrm{ECon}(\rho)$, fix a gluing $\gamma_{\mathcal{A},\mathcal{B}} = (\mathbf{X}_{\mathcal{A},\mathcal{B}}, \rho, \eta_{\mathcal{A},\mathcal{B}})$ such that the following both hold:

(1) For any gluing $\gamma_{\mathcal{A}}$ with $\mathcal{K}\cdot\gamma_{\mathcal{A}} = \mathcal{A}$, the gluing $\delta = \gamma_{\mathcal{A}} \sqcup \gamma_{\mathcal{A},\mathcal{B}}$ satisfies $\mathcal{K}\cdot\delta = \mathcal{B}$.

(2) $X_{\mathcal{A},\mathcal{B}}$ has minimum possible cardinality.

To see that a gluing as in item (1) exists, one can take any $\gamma_{\mathcal{B}}$ with $\mathcal{K}\cdot\gamma_{\mathcal{B}} = \mathcal{B}$. In the case when $\mathrm{dom}(\rho) = 1$, $\rho(0) = \mathfrak{p} \in \mathsf{MP}_i$ for $i \in \mathsf{U}_{\mathrm{non}}$, and $\mathcal{A} = \max(\mathfrak{p})$, then writing $\mathsf{f}(\mathfrak{p}) = (j, q)$, we arrange that $\gamma_{\mathcal{A},\mathcal{B}} = \gamma_{j,i,q}$.

We will use f and the gluings $\{\gamma_{\mathcal{A},\mathcal{B}} : (\mathcal{A}, \mathcal{B}) \in \bigsqcup_\rho \mathrm{ECon}(\rho)\}$ to systematically build for every diary Θ an enumerated structure $\mathbf{Y}^\Theta \leq \mathbf{K}$ and $\sigma^\Theta \in \mathrm{OEmb}(\mathbf{Str}(\Theta), \mathbf{Y}^\Theta)$. When $\Theta = \Delta$, we omit the Δ-superscript, and we will show that for any $\mathbf{A} \in \mathcal{K}$ and $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{Str}(\Delta))$, $\mathrm{Shp}_{\mathbf{Y}}(\sigma \circ f)$ depends only on $\mathrm{Shp}_\Delta(f)$, which will suffice to prove Theorem 5.1.5 (pick any $\varphi \in \mathrm{AEmb}(\mathrm{CT}^{\mathbf{Y}}, \mathrm{CT}^{\mathbf{K}})$ and set $\eta = \widetilde{\varphi} \circ \sigma$).

To build $\mathbf{Y}^\Theta$, we first build an $\mathcal{L}'$-structure $\mathbf{Z}^\Theta \supseteq \mathbf{Str}(\Theta)$ such that $\leq_\ell$ has order type $|Z_\Theta|$. We will let $\mathbf{Y}^\Theta$ be the enumerated $\mathcal{L}$-structure which is $\mathcal{L}'$-isomorphic to $\mathbf{Z}^\Theta$

and $\sigma^\Theta\colon \mathbf{Str}(\Theta) \to \mathbf{Y}^\Theta$ be induced by the inclusion $\mathbf{Str}(\Theta) \subseteq \mathbf{Z}^\Theta$. To build $\mathbf{Z}^\Theta$, we start with $\mathbf{Str}(\Theta)$ and for each $m < \mathrm{ht}(\Theta)$, we attach a new finite $\mathcal{L}'$-structure $\mathbf{Z}^\Theta_m$ to $\mathbf{Str}(\Theta)$. So suppose $m < \mathrm{ht}(\Theta)$ and we have constructed an $\mathcal{L}'$-structure $\mathbf{Str}(\Theta) \cup \bigcup_{i<m} \mathbf{Z}^\Theta_i$ embeddable into $\mathbf{K}$ as an $\mathcal{L}$-structure and so that $\leq_\ell$ is either a finite order or an ω-order. If $m \in \mathrm{Cd}(\Theta)$, we set $\mathbf{Z}^\Theta_m = \emptyset$. The other two cases are the following.

Case 1: $m \in \mathrm{Sp}(\Theta)$. Let $Z^\Theta_m = \{z^\Theta_m\}$ for some new point z^Θ_m. Suppose $t \in \Theta(m)$ is the splitting node with $t^{\mathsf{u}} = i$ and $t^{\mathsf{p}} = \mathfrak{p}$. Write $\mathsf{f}(\mathfrak{p}) = (j, q)$. We set $U^{\mathbf{Z}^\Theta_m}(z^\Theta_m) = j$ and for each $y \in \mathrm{CdNd}(\Theta)$ with $y \sqsupseteq s^\frown 1$, we set $R^{\mathbf{Z}^\Theta}(y, z^\Theta_m) = q$. All other binary relations with z^Θ_m are zero. Note by the definition of f, case (2) of Definition 3.3.5, and Proposition 3.4.3 that $\mathbf{Str}(\Theta) \cup \bigcup_{-1\leq i\leq m} \mathbf{Z}^\Theta_i \leq \mathbf{K}$. We put z^Θ_m as $\leq_\ell$-small as possible while being $\leq_\ell$-above all members of $\mathrm{CdNd}(\Theta) \cap \Theta(<m)$ and $\bigcup_{i<m} Z^\Theta_i$.

Case 2: $m \in \mathrm{AC}(\Theta)$. Suppose the age change is essential on $S = \{s_0 \preceq_{\mathrm{lex}} \cdots \preceq_{\mathrm{lex}} s_{d-1}\} \subseteq \Theta(m)$, and write $\rho = \mathrm{Sort}(S)$. Set $\mathrm{Age}_\Theta(S) = \mathcal{A}$ and $\mathrm{Age}_\Theta(\{s_i^\frown 0 : i < d\}) = \mathcal{B}$ so that $(\mathcal{A}, \mathcal{B}) \in \mathrm{ECon}(\rho)$. Let $Z^\Theta_m = X_{\mathcal{A},\mathcal{B}} \times \{z^\Theta_m\}$ for some new point z^Θ_m, and let $\mathbf{Z}^\Theta_m$ be isomorphic to $\mathbf{X}_{\mathcal{A},\mathcal{B}}$ in the obvious way, with the $\leq_\ell$-order induced from the usual $\leq$-order on $X_{\mathcal{A},\mathcal{B}} \subseteq \omega$. For each $x \in X_{\mathcal{A},\mathcal{B}}$ and $y \in \mathrm{CdNd}(\Theta)$ with $s_i \sqsubseteq y$ for some $i < d$, we set $R^{\mathbf{Z}^\Theta}(y, (x, z^\Theta_n)) = \eta_{\mathcal{A},\mathcal{B}}(i, x)$. There are no other new non-zero binary relations. By our choice of $\gamma_{\mathcal{A},\mathcal{B}}$ and Proposition 3.4.3, we have $\mathbf{Str}(\Theta) \cup \bigcup_{i\leq m} \mathbf{Z}^\Theta_m \leq \mathbf{K}$. We put Z^Θ_m as an $\leq_\ell$-consecutive interval as $\leq_\ell$-small as possible while being $\leq_\ell$-above all members of $\mathrm{CdNd}(\Theta) \cap \Theta(<m)$ and $\bigcup_{i<m} Z^\Theta_i$.

To finish defining $\mathbf{Z}^\Theta$, we attach one last finite $\mathcal{L}'$-structure $\mathbf{B}^\Theta$. Given $i \in \mathsf{U}_{\mathrm{fr}}$, write $\mathsf{P}^\Theta_i = \{s^{\mathsf{p}} : s \in \Theta_i\}$, and write $\mathsf{Q}^\Theta_i = \mathsf{P}^\Theta_i \backslash \{\min_{\leq_{\mathsf{MP}}}(\mathsf{P}^\Theta_i)\}$. We let

$$\mathsf{R}^\Theta_i = \begin{cases} \mathsf{P}^\Theta_i & \text{if } \exists m < \mathrm{ht}(\Theta) \text{ and } \exists x \in Z^\Theta_m \text{ such that } U^{\mathbf{Z}^\Theta_m}(x) = i, \\ \mathsf{Q}^\Theta_i & \text{else.} \end{cases}$$

We set $B^\Theta = (\bigcup_{j\in \mathsf{U}_{\mathrm{fr}}} \mathsf{R}^\Theta_j) \times \{b^\Theta\}$ for some new point b^Θ. In $\mathbf{B}^\Theta$, there are no non-zero binary relations. If $\mathfrak{p} \in \mathsf{R}^\Theta_i$ and $\mathsf{f}(i) = (j, q)$, we set $U^{\mathbf{B}^\Theta}((\mathfrak{p}, b^\Theta)) = j$. If $y \in \mathrm{CdNd}(\Theta) \cap \Theta_{\mathfrak{p}}$, we set $R^{\mathbf{Z}^\Theta}(y, (\mathfrak{p}, b^\Theta)) = q$. There are no other new non-zero binary relations; by the definition of f, the resulting structure embeds into $\mathbf{K}$. On $\mathbf{B}^\Theta$, the $\leq_\ell$-order is induced from $\leq_{\mathsf{MP}}$, and in $\mathbf{Z}^\Theta$, we put B^Θ $\leq_\ell$-below everything else.

This finishes the construction of $\mathbf{Z}^\Theta$, so also of $\mathbf{Y}^\Theta$ and $\sigma^\Theta\colon \mathbf{Str}(\Theta) \to \mathbf{Y}^\Theta$. We mildly abuse notation and also write $\sigma^\Theta\colon \mathbf{Z}^\Theta \to \mathbf{Y}^\Theta$ for the $\mathcal{L}'$-isomorphism. We will primarily make use of $\mathbf{Y}^\Delta$, and simply write $\mathbf{Y}$, $\mathbf{Z}$, $\mathbf{Z}_m$, z_m, b, σ, R_i, etc. when referring to Δ.

The intuition behind the construction of $\mathbf{Y}$ is to as much as possible build Δ into a coding tree of an enumerated structure. With this idea in mind, let us investigate $\mathbf{Y}$ more closely. We start by defining $h\colon \omega \to \omega$ by setting

$$h(m) = \Big| B \cup (\mathrm{CdNd}(\Delta) \cap \Delta(<m)) \cup \bigcup_{i<m} Z_i \Big|.$$

We also write $\xi := c^{\mathbf{Y}} \circ \sigma\colon Z \to \mathrm{CT}^{\mathbf{Y}}$. We now collect facts about $\mathbf{Y}$, h, and ξ we will need going forward.

Fact 5.2.1. (1) If $s \in \mathrm{CdNd}(\Delta)$, then $\ell(\xi(s)) = h(\ell(s))$. If $x \in Z_n$ is $\leq_\ell$-least, then $\ell(\xi(x)) = h(n)$.

(2) If $n < \omega$ and $x \in Z_n$, then $(\xi(x)|_{h(n)})^{\mathsf{seq}} = 0^{h(n)}$.

(3) Fix $i < \mathsf{U}$ and $s \in \mathrm{CdNd}(\Delta_i)$.

- If $i \in \mathsf{U}_{\mathrm{fr}}$ and $\mathsf{R}_i = \mathsf{P}_i$, then for every $s \in \mathrm{CdNd}(\Delta_i)$, we have $\xi(s)^{\mathsf{seq}} \not\sqsupseteq 0^{h(0)}$.
- If $i \in \mathsf{U}_{\mathrm{non}}$ and $s^{\mathsf{seq}} \neq 0^\ell(s)$, then if $m < \ell(s)$ is largest with $\pi_m(s)^{\mathsf{seq}} = 0^m$, then $m \in \sigma[Z_n]$ for some $n < \omega$.

(4) Fix $s, t \in \mathrm{CdNd}(\Delta)$.

- If $s, t \in \mathrm{CdNd}(\Delta)$ and $s^{\mathsf{p}} = t^{\mathsf{p}}$, then $\ell(\xi(s) \wedge \xi(t)) = h(\ell(s \wedge t))$.
- If $s^{\mathsf{u}} = t^{\mathsf{u}} \in \mathsf{U}_{\mathrm{fr}}$ and $s^{\mathsf{p}} \neq t^{\mathsf{p}}$, then $\ell(\xi(s) \wedge \xi(t)) = \sigma((\min_{\leq_{\mathsf{MP}}}(s^{\mathsf{p}}, t^{\mathsf{p}}), b)) < h(0)$.
- If $s^{\mathsf{u}} = t^{\mathsf{u}} \in \mathsf{U}_{\mathrm{non}}$ and $s^{\mathsf{p}} \neq t^{\mathsf{p}}$, then if $m < \omega$ is largest with $\mathrm{Age}_\Delta(s) = \mathrm{Age}_\Delta(t) = \mathcal{K}\cdot\tilde{\iota}_i$, then $\ell(\xi(s) \wedge \xi(t)) = h(m)$.

(5) For any $d < \omega$, function $e\colon d \to \mathrm{CdNd}(\Delta)$, and $m \leq \min\{\ell(e(i)) : i < d\}$, we have $\mathrm{Age}_\Delta(\pi_m \circ e) = \mathrm{Age}_{\mathbf{Y}}(\pi_{h(m)} \circ \xi \circ e)$.

Now fix $f \in \mathrm{Emb}(\mathbf{A}, \mathbf{Str}(\Delta))$, and write $\mathrm{Shp}_\Delta(f) = (\Theta, g) \in D_{\mathbf{A}}$. We will show that $\mathrm{Shp}_{\mathbf{Y}}(\sigma \circ f) = (\mathbf{Y}^\Theta, \sigma^\Theta \circ g)$, which will prove Theorem 5.1.5. First, we use f to build $\beta \in \mathrm{OEmb}(\mathbf{Z}^\Theta, \mathbf{Z})$ as follows. Start by setting $\beta((\mathfrak{p}, b^\Theta)) = (\mathfrak{p}, b)$ for each $(\mathfrak{p}, b^\Theta) \in B^\Theta$; we note that $\mathsf{R}_i^\Theta \subseteq \mathsf{R}_i$ for each $i \in \mathsf{U}_{\mathrm{fr}}$. Write $\varphi_f = \varphi_{\Delta,\mathrm{Im}(f)}$. For $m < \omega$, since $\varphi_f \in \mathrm{DEmb}(\Theta, \Delta)$, we have that $\mathbf{Z}_m^\Theta$ and $\mathbf{Z}_{\widetilde{\varphi}_f(m)}$ are $\mathcal{L}'$-isomorphic, and we let $\beta|_{\mathbf{Z}_m^\Theta}$ be the unique $\mathcal{L}'$-isomorphism. Given $x \in \mathrm{CdNd}(\Theta)$, we set $\beta(x) = \varphi_f(x)$. One needs to check that β respects the binary relations of $\mathbf{Z}^\Theta$, but this follows from the construction of $\mathbf{Z}^\Theta$ and $\mathbf{Z}$ and since $\varphi_f \in \mathrm{DEmb}(\Theta, \Delta)$. This concludes the definition of β, and we let $\alpha \in \mathrm{OEmb}(\mathbf{Y}^\Theta, \mathbf{Y})$ be induced from β in the obvious way. Note that $\alpha \circ \sigma^\Theta = \sigma \circ (\varphi_f|_{\mathrm{CdNd}(\Theta)})$ and that $\mathrm{Im}(\alpha) = \mathrm{Im}(\sigma \circ \beta)$.

We now show that $\mathrm{Im}(\alpha) = \overline{\mathrm{Im}(\sigma \circ f)}^{\mathbf{Y}}$. As $\mathrm{Im}(\sigma \circ f)$ is finite, we can compute its $\mathbf{Y}$-closure using the "top-down" procedure discussed in [51] after Definition 4.4, which goes as follows: Given an arbitrary $S \subseteq Y$, set $S_{\max(S)} = \{\max(S)\}$. If $m < \max(S)$ and S_{m+1} has been defined, we set $S_m = S_{m+1}$ or $S_m = S_{m+1} \cup \{m\}$. The latter happens if and only if $m \in S$ or $m \in \mathrm{Crit}^{\mathbf{Y}}(c^{\mathbf{Y}}[S_{m+1}])$. Upon reaching $m = 0$, we have $S_0 = \overline{S}^{\mathbf{Y}}$.

Now write $S = \mathrm{Im}(\sigma \circ f)$, and consider the above procedure. We prove by reverse induction on $m \leq \max(S)$ that $S_m = \mathrm{Im}(\alpha) \setminus m$. So fix $m \leq \max(S)$ and assume this holds for all n with $m < n \leq \max(S)$.

- If $m = \sigma(t)$ for some $t \in \mathrm{CdNd}(\Delta)$, then if $t \in \mathrm{Im}(f)$, we have $m \in S$ and $S_m = S_{m+1} \cup \{m\}$. If $t \notin \mathrm{Im}(f)$, it follows from our inductive hypothesis and Fact 5.2.1 (2) that any $x \in \pi_{m+1} \circ c^{\mathbf{Y}}[S_{m+1} \setminus S]$ satisfies $x^{\mathsf{seq}} = 0^{\ell(x)}$. It then follows from Fact 5.2.1 (3), (4), and (5) that $m \notin \mathrm{Crit}^{\mathbf{Y}}(c^{\mathbf{Y}}[S_{m+1}])$.
- If $m \in \sigma[Z_n]$ for some $n < \omega$, there are two cases to consider.
  - If $n \in \mathrm{Sp}(\Delta)$, then $m = h(n)$, and $m \in \mathrm{Im}(\alpha)$ if and only if there is $i \in \mathrm{Sp}(\Theta)$ with $\widetilde{\varphi}_f(i) = n$.

If $m \in \mathrm{Im}(\alpha)$, we can find $s, t \in \mathrm{CdNd}(\Theta)$ with $s^{\mathfrak{p}} = t^{\mathfrak{p}}$ and $\ell(s) \wedge \ell(t) = i$. Then by Fact 5.2.1 (4), we have $\ell(\xi \circ \varphi_f(s) \wedge \xi \circ \varphi_f(t)) = h(n)$, and as $\xi \circ (\varphi_f|_{\mathrm{CdNd}(\Theta)}) = c^{\mathbf{Y}} \circ \alpha \circ \sigma^{\Theta}$, this shows that $h(n) = m \in \mathrm{Crit}^{\mathbf{Y}}(c^{\mathbf{Y}}[S_{m+1}])$.
If $m \notin \mathrm{Im}(\alpha)$, then $n \notin \mathrm{Im}(\widetilde{\varphi}_f)$. In articular, by Fact 5.2.1 (4), there are no incomparable $s, t \in c^{\mathbf{Y}}[S]$ with $\ell(s) \wedge \ell(t) = m$, and by Fact 5.2.1 (5), $\mathrm{Age}_{\mathbf{Y}}(\pi_{m+1} \circ c^{\mathbf{Y}}[S_{m+1}]) = \mathrm{Age}_{\mathbf{Y}}(\pi_m \circ c^{\mathbf{Y}}[S_{m+1}])$. Hence $m \notin \mathrm{Crit}^{\mathbf{Y}}(c^{\mathbf{Y}}[S_{m+1}])$.

  - If $n \in \mathrm{AC}(\Delta)$, then we consider all $m \in \sigma[Z_n]$ at once. We have $\min(\sigma[Z_n]) = h(n) \leq m$. We have $m \in \mathrm{Im}(\alpha)$ if and only if there is $i \in \mathrm{AC}(\Theta)$ with $\widetilde{\varphi}_f(i) = n$. If $m \in \mathrm{Im}(\alpha)$, i.e., if $\sigma[Z_n] \subseteq \mathrm{Im}(\alpha)$, then as $i \in \mathrm{AC}(\Theta)$, there is by Fact 5.2.1 (5) an age change between $\pi_{h(n)} \circ c^{\mathbf{Y}}[S_{m+1} \cap S]$ and $\pi_{h(n+1)} \circ c^{\mathbf{Y}}[S_{m+1} \cap S]$. Hence $\mathrm{Crit}^{\mathbf{Y}}(c^{\mathbf{Y}}[S_{m+1}]) \cap \sigma[Z_n] \neq \emptyset$, and this intersection must be all of $\sigma[Z_n]$ since, if level i of Θ features an essential age change from $\mathcal{A}$ to $\mathcal{B}$, we chose $\mathbf{X}_{\mathcal{A},\mathcal{B}}$ with $|X_{\mathcal{A},\mathcal{B}}|$ as small as possible.
  If $m \notin \mathrm{Im}(\alpha)$, i.e., if $\sigma[Z_n] \cap \mathrm{Im}(\alpha) = \emptyset$, then by Fact 5.2.1 (2), (4), and (5), we see that $\mathrm{Crit}^{\mathbf{Y}}(c^{\mathbf{Y}}[S_{m+1}]) \cap \sigma[Z_n] = \emptyset$ and $S \cap \sigma[Z_n] = \emptyset$, so $S_{m+1} = S_{h(n)}$.

- If $m \in \sigma[B]$, then suppose $i \in \mathsf{U}_{\mathrm{fr}}$ is such that $m = \sigma((\mathfrak{p}, b))$ for $\mathfrak{p} \in \mathsf{R}_i$. If $m \in \mathrm{Im}(\alpha)$, then if $\mathfrak{p} \in \mathsf{Q}_i^{\Theta}$, then if $s \in \mathrm{CdNd}(\Theta_{\mathfrak{p}})$ and $t \in \mathrm{CdNd}(\Theta_{\mathfrak{q}})$ with $\mathfrak{q} \in \mathsf{MP}_i$ and $\mathfrak{q} >_{\mathsf{MP}} \mathfrak{p}$, we have $\ell(\xi \circ \varphi_f(s) \wedge \xi \circ \varphi_f(s)) = m$. If $\mathfrak{p} \in \mathsf{R}_i^{\Theta} \backslash \mathsf{Q}_i^{\Theta}$, then there is $n < \mathrm{ht}(\Theta)$ and $x \in Z_n^{\Theta}$ with $U^{\mathbf{Z}_m^{\Theta}}(x) = i$. If $s \in \mathrm{CdNd}(\Theta_{\mathfrak{p}})$, then $\ell(\xi \circ \varphi_f(s) \wedge \sigma \circ \beta(x)) = m$. In both cases, we have $m \in \mathrm{Crit}^{\mathbf{Y}}(c^{\mathbf{Y}}[S_{m+1}])$.
  For the converse, we note that at levels in $\sigma[B]$, age changes are not an issue, and by Fact 5.2.1 (2), (3), and (4), splitting can only occur due to the two situations outlined above. Hence if $m \notin \mathrm{Im}(\alpha)$, we cannot have $m \in \mathrm{Crit}^{\mathbf{Y}}(c^{\mathbf{Y}}[S_{m+1}])$.

This concludes the proof of Theorem 5.1.5.

6. Conclusion and future directions

This paper concludes the project of understanding big Ramsey degrees for finitely constrained binary free amalgamation classes. Naturally, this leads to many open questions and future research directions, some of which we shall state below.

6.1. Finiteness of the language

For the Fraïssé limit of the class of all finite complete graphs with edges colored by countably many colors, it is possible to construct unavoidable colorings using arbitrarily many colors. This shows that the assumption of a finite language is necessary in Theorem 4.

On the other hand, it is possible to prove finite big Ramsey degrees for the homogeneous unconstrained hypergraph which has hyperedges of each finite arity [8]. It suggests that ω-categoricity might be a relevant property in the study of big Ramsey degrees; see also [50, Section 7.3].

6.2. *Infinite sets of constraints*

Our result assumes that the sets of constraints are finite, as infinite sets in general lead to infinite posets of ages and, consequently, our strategy yields infinite embedding shapes in such cases. We believe that this is an inherent property of big Ramsey degrees, not just an artifact of our proof strategies, and state the following conjecture.

Conjecture 6.2.1. *Let $\mathcal{L}$ be a finite language, let $\mathcal{F}$ be an infinite collection of finite irreducible $\mathcal{L}$-structures such that no member of $\mathcal{F}$ embeds to any other member of $\mathcal{F}$, and let $\mathbf{K}$ be the Fraïssé limit of* $\mathrm{Forb}(\mathcal{F})$*. Then there is $\mathbf{A} \in \mathrm{Forb}(\mathcal{F})$ such that the big Ramsey degree of $\mathbf{A} \in \mathcal{K}$ is infinite. Furthermore, the size of such an $\mathbf{A} \in \mathcal{K}$ depends only on $\mathcal{L}$.*

Sauer in [45] considers free amalgamation classes of directed graphs and provides examples of infinite collections $\mathcal{F}$ of finite tournaments such that in the class $\mathrm{Forb}(\mathcal{F})$, the singleton graph still has finite big Ramsey degree. We remark that the statement of [45, Theorem 1.1] is not correct, as it should refer to maximal paths through $P(1)$ rather than just antichains; however, the arguments in [45] correctly show that if $\max(\vec{P}_0)$ is finite, then the singleton graph has big Ramsey degree at most $|\max(\vec{P}_0)|$ in $\mathrm{Forb}(\mathcal{F})$. Thus for a binary language $\mathcal{L}$ and an infinite set of finite irreducible structures $\mathcal{F}$, a natural guess is that some $\mathbf{A} \in \mathrm{Forb}(\mathcal{F})$ of size 2 has infinite big Ramsey degree.

6.3. *Languages of higher arity*

A natural generalization of the main result is to drop the requirement that all relations are of arity at most two. Unlike the binary case, there are not yet general upper bound results (and trying to obtain them is a very active area of research in which many of the authors are presently involved), which are a pre-requisite for obtaining characterizations of exact big Ramsey degrees.

Nevertheless, there exists an upper bound theorem for the generic 3-uniform hypergraph [7] and also, more generally, unrestricted structures in languages with finitely many relations of every arity [8]. There are two major differences between the 3-uniform hypergraph and binary structures. Firstly, instead of a single tree of types, a product of two trees naturally appears. The first tree is an analog of the tree of 1-types, while the second is a binary tree which can be thought of as a tree of "higher-order" types (see also [6] for the intuition behind the second tree). This makes the analysis of embedding shapes and construction of diagonal substructures more complicated, because one has to control the interplay between meets in both trees. Secondly, since the language is ternary, the tree of 1-types branches more and more at each successive level.

Problem 6.3.1. Characterize the big Ramsey degrees of the generic 3-uniform hypergraph. Does the generic 3-uniform hypergraph admit a big Ramsey structure?

Once Problem 6.3.1 is solved, the following two problems are natural generalizations.

Problem 6.3.2. Given a relational language $\mathcal{L}$ with finitely many relational symbols of each arity, characterize the big Ramsey degrees for the class $\mathcal{K}$ of all finite $\mathcal{L}$-structures. Letting $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$, does $\mathbf{K}$ admit a big Ramsey structure?

An upper bound for these kinds of structures appeared in [8]. We believe that a characterization is possible with the present tools and methodology and we expect that the result will naturally generalize the situation for the 3-uniform hypergraph.

Problem 6.3.3. Let $\mathcal{L}$ be a finite relational language, let $\mathcal{F}$ be a finite collection of finite irreducible $\mathcal{L}$-structures and let $\mathcal{K} = \mathrm{Forb}(\mathcal{F})$. Characterize the big Ramsey degrees of $\mathcal{K}$. Letting $\mathbf{K} = \mathrm{Flim}(\mathcal{K})$, does $\mathbf{K}$ admit a big Ramsey structure?

As has already been mentioned, a key ingredient for a solution to this problem is missing, namely a general upper bound. A sufficient condition (a stronger form of amalgamation) has been announced in [2], where it is also conjectured that it is also necessary for finiteness of big Ramsey degrees.

6.4. *Strong amalgamation classes*

Another obvious condition in the main theorem which can be relaxed is the requirement that our classes have free amalgamation. Instead, one could ask for *relational* classes which have strong amalgamation (every amalgamation class can be expanded to a strong amalgamation class by adding functions which represent closures).

For certain collections of strong amalgamation classes with finitely many relations each of arity at most two, exact big Ramsey degrees have been successfully characterized. Devlin [13] characterizes the exact big Ramsey degrees for finite linear orders, and Laflamme, Sauer, and Vuksanovic [33] characterize the exact big Ramsey degrees for *unconstrained* binary strong amalgamation classes, including graphs, digraphs, tournaments, and similar classes of structures with finitely many binary relations. Unconstrained classes have the property that there are no age changes in their coding trees, and hence, their exact big Ramsey degrees are characterized simply by weakly diagonal structures. Similar characterizations has been found for the rationals with an equivalence relation with finitely many dense equivalence classes and the circular directed graphs (see [11, 32]).

Recently, Coulson, Dobrinen, and Patel in [9, 10] formulated an amalgamation property called SDAP, a strengthening of strong amalgamation, and proved that its slightly stronger version, SDAP$^+$, implies that the exact big Ramsey degrees are characterized by weakly diagonal structures. The framework in [9, 10] encompasses the results in [13, 32, 33] (minus $\mathbf{S}(2)$) and provides exact big Ramsey degrees for new classes including unconstrained strong amalgamation classes with an additional linear order (for instance, ordered graphs and ordered tournaments), generic k-partite graphs (with or without an additional linear order), and finitely many nested convex equivalence relations. The methods in [9, 10] also show that structures satisfying SDAP$^+$ in finite languages with relations of *any* arity are indivisible. This includes hypergraphs omitting some finite set of 3-

irreducible substructures and their ordered versions. We point out that the classes $\mathrm{Forb}(\mathcal{F})$ for which the main results of this paper hold do not satisfy SDAP whenever $\mathcal{F}$ contains an irreducible substructure of size three or greater. More generally, SDAP implies that the natural generalizations of the posets $P(\rho)$ are singletons for every sort ρ.

For some strong amalgamation classes in a binary language not satisfying SDAP, upper bound results already exist. In particular, using parameter space methods developed by Hubička [5,27], upper bounds have been obtained for the generic poset, for metric spaces in a finite language and for certain free superpositions of these and other classes. Exact big Ramsey degrees for the generic poset have been characterized by the authors of this paper [4], see also [3] (we remark that [4] also contains a short self-contained description of the big Ramsey degrees of the triangle-free graph). It is quite interesting that in this case the upper bound from [27] was not flexible enough and a stronger upper bound theorem had to be proved as well.

A natural next step in this direction would be to characterize the exact big Ramsey degrees for metric spaces of finite diameter with integer distances.

Problem 6.4.1. Let $n \geq 3$ and let $\mathcal{M}_n$ denote the class of all finite metric spaces with distances $\{1, \ldots, n\}$. Characterize the big Ramsey degrees of $\mathcal{M}_n$. Letting $\mathbf{M}_n = \mathrm{Flim}(\mathcal{M}_n)$, does $\mathbf{M}_n$ admit a big Ramsey structure?

A subset of the authors has already done some work in this direction. In particular, besides splitting and coding, the age-change events come in three different flavors. For single types, each type has a *diameter*, i.e., the largest distance between two vertices of the given type. At the beginning, this diameter is equal to n, but if at some point a neighbor of the type in distance a is discovered, the diameter of the type cannot be larger than $2a$. Additionally, each pair of types has a *lower bound* and *upper bound* on the distances between realizations of these two types. For example, if a vertex with distance a to one type and b to the other is discovered, then we know that the minimum distance between realizations of these two types is at least $|a - b|$ and the maximum is at most $a + b$. We conjecture that the big Ramsey degrees will be characterized by precisely describing the interplay between these five types of interesting events.

6.5. Enumerating big Ramsey degrees and asymptotics

After characterizing the big Ramsey degrees of $\mathbf{A} \in \mathcal{K}$ by some combinatorial objects as is done in this paper, it remains to actually count these objects, which becomes an intriguing problem of enumerative combinatorics. One notable result is that in the class of finite linear orders, the big Ramsey degree of the k-element linear order is precisely $\tan^{(2k-1)}(0)$. Similar results for the class of finite linear orders with an equivalence relation with n many equivalence classes were obtained in [32] and explicit formulas for all circular directed graphs were obtained in [11], see also [34].

It seems likely that for most other cases, such nice explicit formulas do not exist. A more tractable problem would be to characterize the asymptotic growth of the big Ramsey degrees for certain distinguished sequences of structures from a given Fraïssé class.

For instance, the end of Example 3.4.4 suggests the problem of understanding the growth of the function which sends the number n to the big Ramsey degree of the n-element anti-clique in $\mathscr{G}_3$; the same problem makes sense for any finitely-constrained binary free amalgamation class.

In fact, calculating the asymptotic growth of big Ramsey degrees might serve as a measure of how "difficult" it is to show that a given Fraïssé class has finite big Ramsey degrees. For instance, we conjecture that among binary relational classes, those with SDAP will have slower big Ramsey degree growth than classes with non-trivial forbidden substructures. And among classes with non-trivial forbidden substructures, we conjecture that the sizes of the constraints influence the growth. For example, the asymptotic growth of big Ramsey degrees for the class of finite K_4-free graphs seems to be much larger than that of the class of finite posets or that of the class of K_3-free graphs. This may suggest which sorts of Ramsey theorems are needed to show that a given class has finite big Ramsey degrees. There are three main types of Ramsey theorems which have successfully been applied to big Ramsey degrees, each having different strengths and weaknesses: Milliken's tree theorem, which generalizes the ordinary infinite Ramsey theorem and is most suitable for unconstrained structures; various coding tree theorems (see [9, 10, 15, 19, 51]), which this paper makes use of (see also [17]); and the Carlson–Simpson theorem about parameter spaces, which has recently been applied in proving finite big Ramsey degrees for the generic poset and giving a new proof of finite big Ramsey degrees for the class of finite K_3-free graphs [27]. This naturally raises interest in knowing what the limitations of each proof method are. For instance, coding tree methods can handle any finitely-constrained binary free amalgamation class, but cannot handle the class of finite posets. On the other hand, the parameter space method gives simple proofs for the class of finite K_3-free graphs, the class of finite posets, and some other classes defined by (not necessarily irreducible) constraints of size at most 3, but it seems unable to handle even the class of K_4-free graphs. Perhaps the parameter space method necessarily constrains the asymptotic growth of big Ramsey degrees for those classes on which it is successful.

6.6. *Infinite-dimensional Ramsey theorems*

Ramsey theorems involving colorings of infinite objects are called *infinite-dimensional*. A subset $\mathcal{X}$ of the Baire space $[\omega]^\omega$ is called *Ramsey* if for each infinite set $X \subseteq \omega$, there is an infinite subset $Y \subseteq X$ such that $[Y]^\omega$ is either contained in $\mathcal{X}$ or disjoint from $\mathcal{X}$. While the Axiom of Choice can be used to construct an $\mathcal{X} \subseteq [\omega]^\omega$ which is not Ramsey, there are positive results upon restricting to suitably definable $\mathcal{X}$ In particular, the Galvin–Prikry theorem [26] shows that every Borel $\mathcal{X} \subseteq [\omega]^\omega$ is Ramsey, and Ellentuck [24] strengthens this to characterize those $\mathcal{X} \subseteq [\omega]^\omega$ which are *completely* Ramsey in terms of a topology refining the usual metric topology on Baire space.

In [29], Kechris, Pestov, and Todorčević suggested the following direction of research.

Problem 6.6.1. Which Fraïssé structures admit versions of the Galvin–Prikry and/or Ellentuck theorems?

They also ask for dynamical properties associated to such infinite-dimensional Ramsey theorems. Big Ramsey degrees provide constraints on the types of theorems which are possible: Given an enumerated Fraïssé limit **K**, subcopies of **K** may be given different colors depending on their diaries. Thus, for an exact analog of the Galvin–Prikry theorem, one must restrict to subcopies of **K** with the same diagonal diary. The first theorem of this sort appeared in [14], where Dobrinen proved that upon fixing a particular coding tree for the Rado graph, all subcopies of the Rado graph with similar induced coding subtrees satisfy a version of the Galvin–Prikry theorem. This was extended and sharpened in [18], where Dobrinen proved Galvin–Prikry theorems for the classes of structures in [9, 10]. In a recent preprint building on the present work, Dobrinen and Zucker [21] show that for any finitely-constrained binary free amalgamation class $\mathcal{K}$, every big Ramsey structure for $\mathcal{K}$ which satisfies IRT also satisfies a sharp version of the Galvin–Prikry theorem.

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