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Higher-dimensional Heegaard Floer homology and Hecke algebras

Abstract. Given a closed oriented surface Σ of genus greater than 0, we construct a map $\mathcal{F}$ from the wrapped higher-dimensional Heegaard Floer homology of the cotangent fibers of $T^*\Sigma$ to the Hecke algebra associated to Σ and show that $\mathcal{F}$ is an isomorphism of algebras. We also establish analogous results for punctured surfaces.

Keywords: higher-dimensional Heegaard Floer homology, Hecke algebra, skein relation, categorification.

1. Introduction

In [12], Colin, Honda and Tian developed the foundations of the higher-dimensional Heegaard Floer homology (HDHF). HDHF is supposed to model the Fukaya category of the Hilbert scheme (or flag Hilbert scheme) of points on a Liouville domain and was used to analyze symplectic fillability questions in higher-dimensional contact topology. As an application, the HDHF of the 4-dimensional Milnor fibration of type A provided an invariant of links in S^3. This invariant is a close cousin of symplectic Khovanov homology [26,31] and especially its cylindrical reformulation [25]. As a categorified quantum invariant, Khovanov homology is directly related to the categorification of quantum groups and to various Hecke algebras, including affine Hecke algebras and quiver Hecke algebras (also called KLR algebras) [10, 22, 30]. There are several approaches to these Hecke algebras from the point of view of geometric representation theory: The affine Hecke algebra can be realized as the equivariant K-theory of the ordinary Steinberg variety [9, 21, 24]; the double affine Hecke algebra admits a similar realization in terms of the loop Steinberg variety [34]. Moreover, the rational double affine Hecke algebra of type A

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is closely related to the Hilbert scheme of points on $\mathbb{C}^2$; see [19,20]. It is therefore natural to ask the following.

Question 1.1. Is there a symplectic geometry interpretation of the various Hecke algebras?

The goal of this paper is to use the HDHF of cotangent bundles of oriented surfaces to give an answer for the various Hecke algebras of type A, including the finite, affine, and double affine Hecke algebras (abbreviated DAHA and also called Cherednik algebras). We consider two cases:

(1) a closed oriented surface Σ of genus $g > 0$;

(2) a surface $\overset{\circ}{\Sigma}$ which is obtained from a closed oriented surface of genus $g \geq 0$ by removing a finite number (> 0) of punctures.

More precisely, we realize the various Hecke algebras as the HDHF of the disjoint cotangent fibers of $T^*\Sigma$ (or $T^*\overset{\circ}{\Sigma}$):

Type of Hecke algebra	Surface
finite Hecke algebra	open disk
affine Hecke algebra	cylinder
DAHA	torus

Remark 1.2. Note that we exclude the case of a sphere which is more complicated since the homology of its based loop space is not supported in degree zero. We hope to revisit this in a future paper.

Remark 1.3. Recently, Ben-Zvi, Chen, Helm and Nadler identified the affine Hecke algebra with the endomorphism algebra of the coherent Springer sheaf, for any reductive algebraic group [8, Theorem 1.7]. It would be interesting to study the connection between the algebro-geometric realization of the affine Hecke algebra of type A and our symplectic geometry one.

Remark 1.4. In this paper, we consider $T^*\Sigma$ as a symplectic manifold, not as a holomorphic symplectic manifold corresponding to the Riemann surface Σ, which is more natural in many contexts (e.g., [29, Chapter 7]).

In addition to the definition of HDHF, this work crucially depends on three key ingredients:

(1) the relationship between the wrapped Floer cochain complex of a cotangent fiber and chains on the based loop space of the base due to Abbondandolo and Schwarz [2] and Abouzaid [4];

(2) an interpretation of the HOMFLY skein relation in terms of holomorphic curve counting due to Ekholm and Shende [14];

(3) a topological description of DAHA of $\mathfrak{gl}_\kappa$ as a *braid skein algebra* due to Morton and Samuelson [28] (we have been informed that this was also known to Cherednik but unpublished).

Our first ingredient is the result of Abbondandolo–Schwarz [2] which states that the wrapped Floer cochain complex $\mathrm{CW}^*(T_q^*\Sigma)$ of a cotangent fiber $T_q^*\Sigma$ and the chain complex $C_{-*}(\Omega_q\Sigma)$ of the based loop space of Σ are isomorphic as graded algebras on the cohomology level. Abouzaid [4] further improved this to an A_∞-equivalence on the chain level.

In this paper we investigate its generalization to HDHF. More precisely, we consider $\mathrm{CW}^*(\bigsqcup_{i=1}^\kappa T_{q_i}^*\Sigma)$, the wrapped HDHF cochain complex of κ disjoint cotangent fibers of $T^*\Sigma$; it can be given the structure of an A_∞-algebra. The HDHF complex $\mathrm{CW}(\bigsqcup_i T_{q_i}^*\Sigma)$ is defined over $\mathbb{Z}[\![\hbar]\!]$, the ring of formal power series in $\hbar$, where $\hbar$ keeps track of the Euler characteristic of the holomorphic curves that are counted in the definition of the A_∞-operations. Since Σ is a surface $\neq S^2$, $\mathrm{CW}(\bigsqcup_i T_{q_i}^*\Sigma)$ is supported in degree zero and hence is an ordinary algebra.

Our generalization of the based loop space of Σ will be the based loop space of the unordered configuration space $\mathrm{UConf}_\kappa(\Sigma)$ of κ points on Σ. Generalizing Abouzaid's map $\mathrm{CW}^*(T_q^*\Sigma) \to C_{-*}(\Omega_q\Sigma)$, we define an evaluation map

$$\mathcal{E}\colon \mathrm{CW}\Big(\bigsqcup_i T_{q_i}^*\Sigma\Big) \to C_0(\Omega(\mathrm{UConf}_\kappa(\Sigma))) \otimes \mathbb{Z}[\![\hbar]\!].$$

Here $C_0(\Omega(\mathrm{UConf}_\kappa(\Sigma)))$ is the 0-th chain space of the based loop space of $\mathrm{UConf}_\kappa(\Sigma)$ and all tensor products are over $\mathbb{Z}$, unless indicated otherwise. The map $\mathcal{E}$ is given by counting curves of "Heegaard Floer type"; the precise definition will be given in Section 6.1. This map however fails to be a homomorphism of algebras due to an additional degeneration of curves: the nodal degeneration. This phenomenon was recently clarified by Ekholm and Shende [14] (building on unpublished work of Fukaya on the relationship of higher-genus Lagrangian Floer homology and string topology): It is the HOMFLY skein relation that controls the boundaries of 1-dimensional moduli spaces of varying Euler characteristics; see Figure 9.

Starting with the map $\mathcal{E}$, taking the homology of both sides and quotienting out by the HOMFLY skein relation, we obtain a map

$$\mathcal{F}\colon \mathrm{HW}\Big(\bigsqcup_i T_{q_i}^*\Sigma\Big) \to H_0(\Omega(\mathrm{UConf}_\kappa(\Sigma))) \otimes \mathbb{Z}[\![\hbar]\!]/\{\text{the skein relation}\}. \qquad (1.1)$$

Here $\mathrm{HW}(\bigsqcup_i T_{q_i}^*\Sigma)$ is the homology of the wrapped HDHF complex $\mathrm{CW}(\bigsqcup_i T_{q_i}^*\Sigma)$. The map $\mathcal{F}$ is an algebra homomorphism; see Proposition 6.5.

At this point we observe that $H_0(\Omega(\mathrm{UConf}_\kappa(\Sigma)))$ is isomorphic to the group algebra of the surface braid group of Σ over $\mathbb{Z}$. In [28], Morton and Samuelson defined the *braid skein algebra* $\mathrm{BSk}_\kappa(\Sigma)$, which is a quotient of the group algebra of the surface braid group over $\mathbb{Z}[s^{\pm 1}, c^{\pm 1}]$ by the skein relation and the marked point relation; see Definition 4.1. Here s and c are parameters that appear in the skein and marked point relations, respectively. We define the *surface Hecke algebra* $\mathrm{H}_\kappa(\Sigma)$ by reformulating the marked point relation as a c-deformed homotopy relation, and making a change of variables $\hbar = s - s^{-1}$; see Definition 4.4. We show that $\mathrm{H}_\kappa(\Sigma)$ and $\mathrm{BSk}_\kappa(\Sigma)$ are isomorphic, up to a change of variables.

Motivated by the c-deformed homotopy relation, we consider $\mathrm{CW}(\bigsqcup_i T^*_{q_i}\Sigma)_c$, the wrapped HDHF with a parameter c. Adding the parameter c to the map in (1.1), we obtain

$$\mathcal{F}\colon \mathrm{HW}\Big(\bigsqcup_i T^*_{q_i}\Sigma\Big)_c \to \mathrm{H}_\kappa(\Sigma)\otimes_{\mathbb{Z}[\hbar]}\mathbb{Z}[\![\hbar]\!], \qquad (1.2)$$

which is still denoted by $\mathcal{F}$ by abuse of notation.

We then apply the Abbondandolo–Schwarz result [1] to show that the restriction of $\mathcal{F}$ to $\hbar = 0$ is an isomorphism. The following is the main result of this paper and directly follows from the isomorphism of $\mathcal{F}|_{\hbar=0}$.

Theorem 1.5. *The map $\mathcal{F}$ in* (1.2) *is an isomorphism of algebras.*

The main result from Morton–Samuelson [28] is that the double affine Hecke algebra $\ddot{\mathrm{H}}_\kappa$ of $\mathfrak{gl}_\kappa$ is naturally isomorphic to $\mathrm{BSk}_\kappa(T^2)$. Hence we have the following.

Corollary 1.6. *As algebras,* $\mathrm{HW}(\bigsqcup_i T^*_{q_i}T^2)_c \simeq \ddot{\mathrm{H}}_\kappa|_{\hbar=s-s^{-1}}\otimes_{\mathbb{Z}[\hbar]}\mathbb{Z}[\![\hbar]\!]$.

When the base $\overset{\circ}{\Sigma}$ has punctures, one can similarly define the HDHF A_∞-algebra $\mathrm{CW}(\bigsqcup_i T^*_{q_i}\overset{\circ}{\Sigma})$. It is possible to formally include a c-parameter, but we expect that it does not yield any extra information. Isomorphism (1.2) still holds, and the corresponding surface Hecke algebra $\mathrm{H}_\kappa(\overset{\circ}{\Sigma})$ is isomorphic to the finite Hecke algebra H_κ and the affine Hecke algebra $\dot{\mathrm{H}}_\kappa$ of $\mathfrak{gl}_\kappa$ when $\overset{\circ}{\Sigma}$ is an open disk and a cylinder, respectively.

Corollary 1.7. *As algebras,* $\mathrm{HW}(\bigsqcup_i T^*_{q_i}\overset{\circ}{\Sigma}) \simeq \mathrm{H}_\kappa|_{\hbar=s-s^{-1}}\otimes_{\mathbb{Z}[\hbar]}\mathbb{Z}[\![\hbar]\!]$ *when $\overset{\circ}{\Sigma}$ is an open disk and* $\mathrm{HW}(\bigsqcup_i T^*_{q_i}\overset{\circ}{\Sigma}) \simeq \dot{\mathrm{H}}_\kappa|_{\hbar=s-s^{-1}}\otimes_{\mathbb{Z}[\hbar]}\mathbb{Z}[\![\hbar]\!]$ *when $\overset{\circ}{\Sigma}$ is a cylinder.*

Returning to the discussion of categorification, the affine Hecke algebra is related to categorified quantum groups of type A [6]. This can be explained symplectically by noting that $T^*\overset{\circ}{\Sigma}$ for a cylinder $\overset{\circ}{\Sigma}$ is symplectomorphic to $\mathbb{R}^2 \times T^*S^1$. The latter naturally appears in the 4-dimensional Milnor fibration and the HDHF approach to symplectic Khovanov homology.[1]

Isomorphism (1.2) holds only after tensoring with $\mathbb{Z}[\![\hbar]\!]$. Nevertheless, we believe that the coefficient ring could be taken to be $\mathbb{Z}[\hbar]$.

Conjecture 1.8. *The algebra* $\mathrm{HW}(\bigsqcup_i T^*_{q_i}\Sigma)_c$ *is well defined over $\mathbb{Z}[\hbar]$, and Theorem* 1.5 *still holds over $\mathbb{Z}[\hbar]$.*

There is some evidence for this conjecture. In particular, direct computations of the second and third authors [33] show the well-definedness over $\mathbb{Z}[\hbar]$ when $\Sigma = \mathbb{R}^2$ and the surface Hecke algebra is isomorphic to the finite Hecke algebra.

Question 1.9. What is the geometric meaning of the change of variables $\hbar = s - s^{-1}$?

Question 1.9 is important from the perspective of representation theory. For instance, the affine Hecke algebra with the parameter s has interesting modules, but the situation

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[1]Colin, V., Honda, K., Tian, Y.: Towards a definition of Khovanov homology for links in fibered 3-manifolds. In preparation.

is not clear for the affine Hecke algebra with the parameter $\hbar$. A possible explanation of $\hbar = s - s^{-1}$ may require additional data of flat bundles which is well studied in mirror symmetry and Fukaya categories.

On the HDHF side, other κ-tuples of Lagrangians in $T^*\Sigma$ give rise to modules over $\mathrm{HW}(\bigsqcup_i T^*_{q_i}\Sigma)_c$. It is interesting to look at the category of such modules and try to relate it to the Fukaya category of the Hilbert scheme $\mathrm{Hilb}^\kappa(T^*\Sigma)$.

Remark 1.10. For any pair (X, G), where X is a smooth complex algebraic variety with an action of a finite group G, Etingof introduced a global analog of the rational Cherednik algebra and a Hecke algebra [15]. The two algebras are related by the KZ functor.

In the case of $(X, G) = (C^{\times n}, S_n)$, where C is an algebraic curve, and the symmetric group S_n acts on the product by permutation, the associated Hecke algebra is likely to be isomorphic to the braid skein algebra. This case was further studied by Finkelberg and Ginzburg via quantum Hamiltonian reduction [16].

It is interesting to compare our symplectic geometry approach with their algebraic geometry one, and to look for a symplectic explanation of the category $\mathcal{O}$ of rational Cherednik algebras.

Organization. In Section 2, we give a brief review of HDHF. We then restrict to the special case of cotangent bundles of surfaces and define $\mathrm{CW}(\bigsqcup_i T^*_{q_i}\Sigma)$. In Section 3, we review the chain complex of the based loop space and the results of [1, 4]. In Section 4, we discuss the based loop space of $\mathrm{UConf}_\kappa(\Sigma)$, the braid skein algebra $\mathrm{BSk}_\kappa(\Sigma)$, and its reformulation, the surface Hecke algebra $\mathrm{H}_\kappa(\Sigma)$. In Section 5, we define $\mathrm{CW}(\bigsqcup_i T^*_{q_i}\Sigma)_c$, the HDHF with a parameter c. In Section 6, we construct the map $\mathcal{F}$ and prove that it is an isomorphism of algebras. In Section 7, we show that our results can be extended to surfaces with punctures.

2. Wrapped higher-dimensional Heegaard Floer cohomology

We briefly review HDHF in Section 2.1 and then turn to the special case of disjoint cotangent fibers of $T^*\Sigma$ in Section 2.2.

2.1. Review of higher-dimensional Heegaard Floer homology

We refer the reader to [12] for more details.

Let (X, α) be a $2n$-dimensional completed Liouville domain, i.e., there exists a Liouville domain $X^c \subset X$ such that X is obtained from X^c by attaching the symplectization end $([0, \infty)_s \times \partial X^c, e^s\alpha|_{\partial X^c})$. Let $\omega = d\alpha$ be the exact symplectic form on X.

Definition 2.1. The objects of the A_∞-category $\mathcal{F}_\kappa(X)$ are κ-tuples of disjoint exact Lagrangians which are equipped with relative spin structures and grading data, and, if not compact, are cylindrical at infinity.

Given two objects $L_i = L_{i1} \sqcup \cdots \sqcup L_{i\kappa}$, $i = 0, 1$, whose components are mutually transverse, the morphism $\mathrm{Hom}_{\mathcal{F}_\kappa(X)}(L_0, L_1) = \mathrm{CF}(L_0, L_1)$ is the free abelian group

generated by all $\mathbf{y} = \{y_1, \ldots, y_\kappa\}$, where $y_j \in L_{0j} \cap L_{1\sigma(j)}$ and σ is some permutation of $\{1, \ldots, \kappa\}$. The coefficient ring is set to be $\mathbb{Z}[\![\hbar]\!]$. The A_∞-operations μ^m, $m = 1, 2, \ldots$, will be defined by (2.1).

To define the A_∞-operation μ^m for $i = 1, \ldots, m$, let $L_i = \bigsqcup_{j=1}^\kappa L_{ij}$ be such that L_{i-1} and L_i are transverse. Let $\mathbf{y}_i = \{y_{i1}, \ldots, y_{i\kappa}\}$ be a κ-tuple of points such that $y_{ij} \in L_{(i-1)j} \cap L_{i\sigma_i(j)}$, where σ_i is some permutation of $\{1, \ldots, \kappa\}$.

As in the cylindrical reformulation of Heegaard Floer homology by Lipschitz [23], we also need an extra "cylindrical" direction to keep track of points in "$\mathrm{Sym}^\kappa(X)$". Specifically, as shown in Figure 1, let D be the unit disk in $\mathbb{C}$ and

$$D_m = D \setminus \{p_0, \ldots, p_m\},$$

where $p_i \in \partial D$ are boundary marked points arranged counterclockwise. Let $\partial_i D_m$ be the boundary arc from p_i to p_{i+1}. Let $\mathscr{A}_m$ be the moduli space of D_m modulo automorphisms; we choose representatives D_m of equivalence classes of $\mathscr{A}_m$ in a smooth manner (e.g., by setting $p_0 = -i$ and $p_1 = i$) and abuse notation by writing $D_m \in \mathscr{A}_m$. We call D_m the "A_∞-base direction".
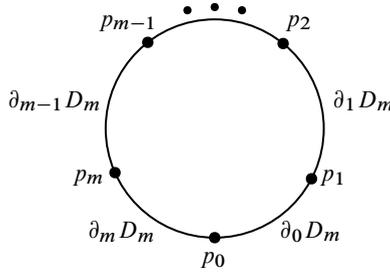

Fig. 1. The A_∞-base D_m.

The full ambient symplectic manifold is then $(D_m \times X, \Omega_m = \omega_m + \omega)$, where ω_m is an area form on D_m which restricts to $\mathrm{d}s_i \wedge \mathrm{d}t_i$ on the strip-like end $e_i \simeq [0, \infty)_{s_i} \times [0,1]_{t_i}$ around p_i for $i = 1, \ldots, m$ and $e_i \simeq (-\infty, 0]_{s_i} \times [0,1]_{t_i}$ around p_i for $i = 0$. (As we approach the puncture p_i, $s_i \to +\infty$ for $i = 1, \ldots, m$ and $s_i \to -\infty$ for $i = 0$.) Then we extend the Lagrangians to the A_∞-base direction: for $i = 0, \ldots, m$, let $\widetilde{L}_i = \partial_i D_m \times L_i$ and $\widetilde{L}_{ij} = \partial_i D_m \times L_{ij}$. Let $\pi_X : D_m \times X \to X$ be the projection to X and π_{D_m} be the symplectic fibration

$$\pi_{D_m} : (D_m \times X, \Omega_m) \to (D_m, \omega_m).$$

Let $\mathscr{J}_{X,\alpha}$ be the set of $\mathrm{d}\alpha$-compatible almost complex structures J_X on (X, ω) that are asymptotic to an almost complex structure on $[0, \infty)_s \times \partial X^c$ that takes ∂_s to the Reeb vector field of $\alpha|_{\partial X^c}$, takes $\ker \alpha|_{\partial X^c}$ to itself, and is compatible with $\mathrm{d}\alpha|_{\partial X^c}$.

There is a smooth assignment $D_m \mapsto J_{D_m}$, where $D_m \in \mathscr{A}_m$, such that

(J1) on each fiber $\pi_{D_m}^{-1}(p) = \{p\} \times X$, J_{D_m} restricts to an element of $\mathscr{J}_{X,\alpha}$;

(J2) J_{D_m} projects holomorphically onto D_m;

(J3) over each strip-like end $[0, \infty)_{s_i} \times [0, 1]_{t_i}$, for s_i sufficiently positive, or $(-\infty, 0]_{s_0} \times [0, 1]_{t_0}$, for s_0 sufficiently negative, J_{D_m} is invariant in the s_i-direction and takes ∂_{s_i} to ∂_{t_i}; when $m = 1$, J_{D_1} is invariant under $\mathbb{R}$-translation of the base and takes ∂_{s_i} to ∂_{t_i}.

One can inductively construct such an assignment for all $m \geq 1$ in a manner which is

(A) consistent with the boundary strata, and

(B) for which all the moduli spaces $\mathcal{M}(\mathbf{y}_1, \ldots, \mathbf{y}_m, \mathbf{y}_0)$, defined below, are transversely cut out.

A collection $\{ J_{D_m} \mid D_m \in \mathscr{A}_m, m \in \mathbb{Z}_{>0} \}$ satisfying (A) will be called a *consistent collection* of almost complex structures; if it satisfies (B) in addition, it is a *sufficiently generic consistent collection*.

Remark 2.2. To avoid cumbersome terminology, in what follows, when we say *sufficiently generic*, we mean that all the moduli spaces under consideration are transversely cut out.

Let $\mathcal{M}(\mathbf{y}_1, \ldots, \mathbf{y}_m, \mathbf{y}_0)$ be the moduli space of maps

$$u \colon (\dot{F}, j) \to (D_m \times X, J_{D_m}),$$

where (F, j) is a compact Riemann surface with boundary, $\mathbf{p}_0, \ldots, \mathbf{p}_m$ are disjoint κ-tuples of boundary marked points of F, $\dot{F} = F \setminus \bigcup_i \mathbf{p}_i$, and $D_m \in \mathscr{A}_m$, so that u satisfies the following conditions:

(1) $\mathrm{d}u \circ j = J_{D_m} \circ \mathrm{d}u$;

(2) each component of $\partial \dot{F}$ is mapped to a unique $\widetilde{L}_{ij}$;

(3) $\pi_X \circ u$ tends to $\mathbf{y}_i$ as $s_i \to +\infty$ for $i = 1, \ldots, m$;

(4) $\pi_X \circ u$ tends to $\mathbf{y}_0$ as $s_0 \to -\infty$;

(5) $\pi_{D_m} \circ u$ is a κ-fold branched cover of D_m.

With the identification of the strip-like end e_i, $i = 1, \ldots, m$, with $[0, \infty)_{s_i} \times [0, 1]_{t_i}$, condition (3) means that u maps the neighborhoods of the punctures of $\mathbf{p}_i$ asymptotically to the Reeb chords $[0, 1]_{t_i} \times \mathbf{y}_i$ as $s_i \to +\infty$. Condition (4) is similar.

The μ^m-composition map of $\mathcal{F}_\kappa(X)$ is then defined as

$$\mu^m(\mathbf{y}_1, \ldots, \mathbf{y}_m) = \sum_{\mathbf{y}_0, \chi \leq \kappa} \# \mathcal{M}^{\mathrm{ind}=0, \chi}(\mathbf{y}_1, \ldots, \mathbf{y}_m, \mathbf{y}_0) \cdot \hbar^{\kappa - \chi} \cdot \mathbf{y}_0, \qquad (2.1)$$

where the superscript χ denotes the Euler characteristic of F, ind denotes the Fredholm index (as given in Theorem 2.3 below), and the symbol # denotes the signed count of the corresponding moduli space.

Theorem 2.3. *The Fredholm index of* $\mathcal{M}^\chi(\mathbf{y}_1, \ldots, \mathbf{y}_m, \mathbf{y}_0)$ *is*

$$\mathrm{ind}(u) = (n - 2)\chi + \mu + 2\kappa - m\kappa n + m - 2,$$

where μ is the Maslov index of u, defined as in [12, Section 4].

If $2c_1(TX) = 0$ and the Maslov classes of all involved Lagrangians vanish, then there exists a well-defined $\mathbb{Z}$-grading. In this case, the dimension of $\mathcal{M}^{\chi}(\mathbf{y}_1, \ldots, \mathbf{y}_m, \mathbf{y}_0)$ can be rewritten as

$$\text{ind}(u) = (n-2)(\chi - \kappa) + |\mathbf{y}_0| - |\mathbf{y}_1| - \cdots - |\mathbf{y}_m| + m - 2, \qquad (2.2)$$

where $|\mathbf{y}_i| = |y_{i1}| + \cdots + |y_{i\kappa}|$. Note also that

$$|\hbar| = 2 - n. \qquad (2.3)$$

We omit the details about the orientation of $\mathcal{M}(\mathbf{y}_1, \ldots, \mathbf{y}_m, \mathbf{y}_0)$ and the A_∞-relation, and refer the reader to [12, Section 4].

2.2. Wrapped higher-dimensional Heegaard Floer homology of disjoint cotangent fibers

From now on, we restrict to the case $X = T^*M$, where M is a compact manifold of dimension n. Let $\pi_M : T^*M \to M$ be the standard projection; by abuse of notation we also denote the projection map $\pi_M \circ \pi_{T^*M}$ simply by π_M.

The wrapped HDHF category is denoted by $\mathcal{W}_\kappa(T^*M)$. Let $q_1, \ldots, q_\kappa$ be κ distinct points in a small disk $U \subset M$. We discuss the wrapped Heegaard Floer homology of the object $\bigsqcup_i T^*_{q_i} M$, where we assume i to range from 1 to κ in what follows.

Remark 2.4. When we take $\bigsqcup_i T^*_{q_i} M$, the points $q_1, \ldots, q_\kappa$ (as well as the Lagrangians $T^*_{q_1} M, \ldots, T^*_{q_\kappa} M$) are ordered.

Let g be a Riemannian metric on M and $|\cdot|$ be the induced norm on T^*M. Choose a time-dependent "quadratic at infinity" Hamiltonian

$$H_V : [0,1] \times T^*M \to \mathbb{R}, \quad H_V(t, q, p) = \frac{1}{2}|p|^2 + V(t, q), \qquad (2.4)$$

where $t \in [0,1]$, $q \in M$, $p \in T^*_q M$, and V is some perturbation term with small $W^{1,2}$-norm in the $([0,1] \times M)$-direction. The Hamiltonian vector field X_{H_V} with respect to the canonical symplectic form $\omega = dq \wedge dp$ is then given by $i_{X_{H_V}} \omega = dH_V$. Let $\phi^t_{H_V}$ be the time-t flow of X_{H_V}.

By choosing g and V generically, we can guarantee that all Hamiltonian chords of $\phi^t_{H_V}$ between the cotangent fibers $\{T^*_{q_1} M, \ldots, T^*_{q_\kappa} M\}$ are nondegenerate.

Definition 2.5. The *wrapped Heegaard Floer chain complex* of $\text{CW}(\bigsqcup_i T^*_{q_i} M)$ is $\text{CF}(\phi^1_{H_V}(\bigsqcup_i T^*_{q_i} M), \bigsqcup_i T^*_{q_i} M)$.

There is a subtlety when defining A_∞-operations for wrapped HDHF and we briefly review the usual rescaling argument of [3, Section 3].

Start with a consistent collection $\{J_{D_m} \mid D_m \in \mathcal{A}_m, m \in \mathbb{Z}_{>0}\}$ as before. We would like to impose Lagrangian boundary conditions $\tilde{L}_j := \phi^{m-j}_{H_V}(\bigsqcup_i T^*_{q_i} M)$ over the arc $\partial_j D_m$, but we need to make a modification near the end e_0 since the chain complex $\text{CF}(\phi^m_{H_V}(\bigsqcup_i T^*_{q_i} M), \bigsqcup_i T^*_{q_i} M)$ is not naturally isomorphic to $\text{CF}(\phi^1_{H_V}(\bigsqcup_i T^*_{q_i} M), \bigsqcup_i T^*_{q_i} M)$.

We will (slightly informally) explain how to modify the $\tilde{L}_j$ and the family J_{D_m} in two steps.

Step 1. Let $D_m \in \mathcal{A}_m \setminus N(\partial \mathcal{A}_m)$, where $N(\partial \mathcal{A}_m)$ is a small neighborhood of $\partial \mathcal{A}_m$ in $\bar{\mathcal{A}}_m$. Let ψ^ρ be the time-log ρ flow of the Liouville vector field $p \partial_p$ of $(T^*M, p\,dq)$. (Note that because of the log term, $(\psi^\rho)^{-1} = \psi^{1/\rho}$.) There is an isomorphism

$$\mathrm{CF}\Big(\psi^\rho\Big(\phi_{H_V}^1\Big(\bigsqcup_i T_{q_i}^* M\Big)\Big), \psi^\rho\Big(\bigsqcup_i T_{q_i}^* M\Big); \psi_*^\rho J_{D_m}\Big)$$

$$\cong \mathrm{CF}\Big(\phi_{H_V}^1\Big(\bigsqcup_i T_{q_i}^* M\Big), \bigsqcup_i T_{q_i}^* M; J_{D_m}\Big) \cong \mathrm{CW}\Big(\bigsqcup_i T_{q_i}^* M\Big).$$

Taking $\rho = \frac{1}{m}$, $\psi^{1/m}(\phi_{H_V}^1(\bigsqcup_i T_{q_i}^* M))$ limits to $\phi_{H_V}^m(\bigsqcup_i T_{q_i}^* M)$ as $|p| \to \infty$ and $\psi^{1/m}(\bigsqcup_i T_{q_i}^* M)$ remains the same as $\bigsqcup_i T_{q_i}^* M$.

We then modify $\tilde{L}_j$ only over the output end e_0 of D_m to the totally real boundary condition $\tilde{L}'_j(D_m)$ so that, for $s_0 \le -1$, $\tilde{L}'_0(D_m)$ is the trace of a fixed (but unspecified) exact, identity-at-infinity Lagrangian isotopy from $\phi_{H_V}^m(\bigsqcup_i T_{q_i}^* M)$ to $\psi^{1/m}(\phi_{H_V}^1(\bigsqcup_i T_{q_i}^* M))$ as $s_0 \to -\infty$, and $\tilde{L}'_m(D_m)$ is the identity trace from $\bigsqcup_i T_{q_i}^* M$ to $\psi^{1/m}(\bigsqcup_i T_{q_i}^* M)$ as $s_0 \to -\infty$.

We convert the totally real boundary condition into a Lagrangian one by changing the symplectic form as follows. First let $G(s_0, t_0, x)$ be an (s_0, t_0)-dependent, constant-at-infinity Hamiltonian function on T^*M such that the Hamiltonian vector field Y_{s_0,t_0} satisfying $i_{Y_{s_0,t_0}} \omega = d_{T^*M} G|_{(s_0,t_0)}$ (here $d_{T^*M} G$ is the component of d in the T^*M-direction) induces the exact Lagrangian isotopies along $t_0 = 0, 1$. Then we replace $ds_0 \wedge dt_0 + \omega$ by the symplectic form $\Omega = ds_0 \wedge (dt_0 - dG) + \omega$. Along $t_0 = 0, 1$, $\Omega = -ds_0 \wedge dG + \omega = -ds_0 \wedge d_{T^*M} G + \omega$ and the symplectic connection is given by $\partial_s + Y_{s_0,t_0}$. Hence $\tilde{L}'_0(D_m)$ and $\tilde{L}'_m(D_m)$ become Lagrangian with respect to Ω.

Next we modify J_{D_m} only over the output end e_0 of D_m to an almost complex structure J'_{D_m} so that for $s_0 \le -1$, $J'_{D_m} = \psi_*^{1/m} J_{D_m}$ and (J1)–(J3) still hold; this is possible since $\psi_*^\rho J_X \in \mathcal{J}_{X,\alpha}$ if $J_X \in \mathcal{J}_{X,\alpha}$.

Step 2. We will briefly indicate how to inductively apply the modifications from Step 1 to construct consistent collections of totally real boundary conditions $\tilde{L}''_j(D_m)$ and almost complex structures $D_m \mapsto J''_{D_m}$, $D_m \in \mathcal{A}_m$, so that (J1)–(J3) still hold; see Figure 2.

For $D_m \in \mathcal{A}_m \setminus N(\partial \mathcal{A}_m)$, we set $\tilde{L}''_j(D_m) := \psi^m(\tilde{L}'_j(D_m))$ and $J''_{D_m} := (\psi^m)_* J'_{D_m}$; this extra normalization is done for consistency with the evaluation map $\mathcal{F}$ in Section 6.

Suppose $D_m \in \mathcal{A}_m$ is close to breaking into D_a and D_b, $a + b = m + 1$, where the negative end of D_a is glued to the positive end of D_b corresponding to p_{b-i}, and J_{D_m} is close to breaking into J_{D_a} and J_{D_b}. We then apply Step 1 to obtain J'_{D_a} and J'_{D_b}, apply $\psi^b \phi_{H_V}^i \psi^a$ to J'_{D_a} (resp. to $\tilde{L}'_j(D_a)$ for all j) and ψ^b to J'_{D_b} (resp. to $\tilde{L}'_j(D_b)$ for all j) so the ends match, and glue them together.

We then choose an interpolation between the above choices of boundary conditions and almost complex structures as in Figure 2.
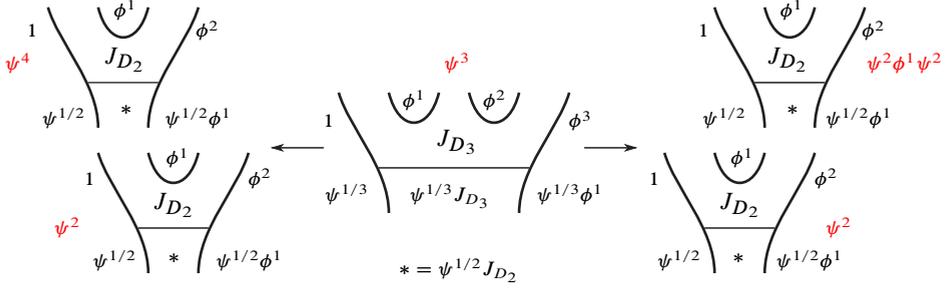
Fig. 2. Description of the totally real boundary conditions and almost complex structures for $a = b = 2$. The middle is D_3 and the left and right are degenerations into D_2 and D_2. For a diffeomorphism h of T^*M, the label h is shorthand for $h(\bigsqcup_i T^*_{q_i} M)$; we also abbreviate $\phi = \phi_{H_V}$. A diffeomorphism in red means "apply the diffeomorphism to all the labels on the closest disk".

By abuse of notation, we write J_{D_m} instead of J''_{D_m} from now on. Conditions (1)–(5) in the definition of the moduli space $\mathcal{M}(\mathbf{y}_1, \ldots, \mathbf{y}_m, \mathbf{y}_0)$ are modified as follows:

- $\widetilde{L}_{ij}$ is replaced by $\widetilde{L}''_{ij}(D_m)$, and
- the tuples $\mathbf{y}_i$ are replaced by the corresponding tuples of intersection points after applying the appropriate diffeomorphism to the two intersecting fiber Lagrangians; for example, in the case where D_m is close to breaking into D_a and D_b, we apply ψ^b to the intersection points corresponding to the positive ends of D_b.

Lemma 2.6. *Fix a sufficiently generic consistent choice of almost complex structures as above. Let $d = 0$ or 1. Given $\mathbf{y}_1, \ldots, \mathbf{y}_m \in \mathrm{CW}(\bigsqcup_i T^*_{q_i} M)$, $\mathcal{M}^{\mathrm{ind}=d,\chi}(\mathbf{y}_1, \ldots, \mathbf{y}_m, \mathbf{y}_0)$ is empty for all but finitely many $\mathbf{y}_0$. If it is nonempty, $\mathcal{M}^{\mathrm{ind}=d,\chi}(\mathbf{y}_1, \ldots, \mathbf{y}_m, \mathbf{y}_0)$ (and $\mathcal{M}^{\mathrm{ind}=d,\chi}(\mathbf{y}_1, \mathbf{y}_0)/\mathbb{R}$ if $m = 1$) admits a compactification for each Euler characteristic χ.*

Proof. This is similar to [3, Appendix B] or [5, Section 7]. ∎

Therefore, the A_∞-operation

$$\mu^m \colon \mathrm{CW}\Big(\bigsqcup_i T^*_{q_i} M\Big) \otimes \cdots \otimes \mathrm{CW}\Big(\bigsqcup_i T^*_{q_i} M\Big) \to \mathrm{CW}\Big(\bigsqcup_i T^*_{q_i} M\Big)$$

can be defined using a sufficiently generic consistent collection, making the chain complex $\mathrm{CW}(\bigsqcup_i T^*_{q_i} M)$ into an A_∞-algebra.

Remark 2.7. The same prescription gives us the wrapped HDHF category $\mathcal{W}_\kappa(X)$ of a Liouville domain, which generalizes the wrapped Fukaya category $\mathcal{W}_1(X)$ of X.

We now discuss the grading on $\mathrm{CW}(\bigsqcup_i T^*_{q_i} M)$. Since $c_1(T(T^*M)) = 0$ and the Maslov class of the Lagrangian vanishes, there is a well-defined $\mathbb{Z}$-grading. Following [7, Section 1.3], we choose a nonzero section μ of the trivial complex line bundle $\Lambda^2_{\mathbb{C}} T^*(T^*M)$: Let $\{U_\alpha\}$ be a cover of M. On any local chart U_α with coordinates $(x^1_\alpha, \ldots, x^n_\alpha)$, we define

$$\mu_\alpha = (\mathrm{d}x^1_\alpha - i \circ \mathrm{d}x^1_\alpha \circ J) \wedge \cdots \wedge (\mathrm{d}x^n_\alpha - i \circ \mathrm{d}x^n_\alpha \circ J), \tag{2.5}$$

where J is viewed as a bundle map which takes $v^* \in T^*M$ to its dual $v \in TM$ via g. Let $\mu = \sum_\alpha \varphi_\alpha \mu_\alpha$, where $\{\varphi_\alpha\}$ is some partition of unity with respect to $\{U_\alpha\}$. Then we consider the phase function

$$\varphi_\mu: \mathrm{LGr}(T(T^*M)) \to S^1, \quad A \mapsto \frac{\mu(v_1 \wedge \cdots \wedge v_n)^2}{\|\mu(v_1 \wedge \cdots \wedge v_n)\|^2},$$

where $\mathrm{LGr}(T(T^*M))$ is the Grassmannian of Lagrangian planes in $T(T^*M)$ and $\{v_1, \ldots, v_n\}$ are tangent vectors that span A. Note that φ_μ is independent of the choice of $\{v_1, \ldots, v_n\}$. For any loop l in $\mathrm{LGr}(T(T^*M))$, the Maslov index of l is then defined as the degree of φ_μ on l. Moreover, if the Maslov class of some Lagrangian L vanishes, we can lift φ_μ on L to a grading function $\widetilde{\varphi}_\mu: L \to \mathbb{R}$.

Given any $q \in M$, denote the tangent space of the zero section of T^*M at q by $T_q M$. One can check that $\varphi_\mu(T_q M) = 1 \in S^1$ for any choice of μ. Hence we can define a grading function on the zero section M which is identically 0. Similarly, given any $x \in T^*M$, denote the vertical space at x by V_x. We check that $\varphi_\mu(V_x) = (-1)^{2n} = 1 \in S^1$ for any choice of μ. Therefore, on any (unwrapped) cotangent fiber T_q^*M, we can lift φ_μ to a grading function $\widetilde{\varphi}_\mu: T_q^*M \to \mathbb{R}$ which is identically 0.

Now let $M = \Sigma$ be a closed oriented surface of genus greater than 0. In this case $|\hbar| = 2 - n = 0$ from (2.3). Let g be a Riemannian metric on Σ which is a small perturbation of the flat metric when Σ is a torus and of the hyperbolic metric when Σ has genus greater than 1. In this case we have the following.

Lemma 2.8. *Fix μ by any choice satisfying* (2.5). *Then the grading $|\mathbf{y}| = 0$ for every* $\mathbf{y} \in \mathrm{CF}(\phi^1_{H_V}(\bigsqcup_i T_{q_i}^* \Sigma), \bigsqcup_i T_{q_i}^* \Sigma)$.

Proof. Given $\mathbf{y} = \{y_1, \ldots, y_\kappa\} \in \mathrm{CF}(\phi^1_{H_V}(\bigsqcup_i T_{q_i}^* \Sigma), \bigsqcup_i T_{q_i}^* \Sigma)$, each $y_i \in \phi^1_{H_V}(L_i) \cap L_{i'}$ corresponds to a time-1 Hamiltonian chord from L_i to $L_{i'}$, parametrized by $(q(t), p(t))$, $t \in [0, 1]$. Its Legendre transform (see Definition 3.2) gives a perturbed geodesic γ on Σ. By classical results of Duistermaat [13, Theorem 4.3] and [1, Section 1.2], the Conley–Zehnder index of y_i with respect to μ is equal to the Morse index of γ with respect to its Lagrangian action. Lemma 3.6 then implies that $|y_i| = 0$. Hence $|\mathbf{y}| = \sum_{i=1}^\kappa |y_i| = 0$. ∎

Proposition 2.9. *The A_∞-algebra $\mathrm{CW}(\bigsqcup_i T_{q_i}^* \Sigma)$ is supported in degree zero, and hence is an ordinary algebra.*

Proof. The complex $\mathrm{CW}(\bigsqcup_i T_{q_i}^* \Sigma)$ is supported in degree zero by Lemma 2.8 and $|\hbar| = 0$. The A_∞-operation $\mu^m = 0$ for all $m \neq 2$ since the degree of μ^m is $2 - m$. Hence μ^2 is the only nontrivial A_∞-operation and $\mathrm{CW}(\bigsqcup_i T_{q_i}^* \Sigma)$ is an ordinary algebra. ∎

3. The relationship between the based loop space and the wrapped Floer homology of a cotangent fiber

Let M be a compact oriented manifold of dimension n. We review some basic properties of the based loop space on M and in particular the relationship to the wrapped Floer

homology of a single cotangent fiber of T^*M, i.e., the case when $\kappa = 1$. We refer the reader to [1, 4] for more details.

Let g be a generic Riemannian metric on M, and let ∇ be its associated Levi-Civita connection.

Consider the path space

$$\Omega(M, q_0, q_1) = \{\gamma \in C^0([0, 1], M) \mid \gamma(0) = q_0, \gamma(1) = q_1\}.$$

There is a composition map which is simply the concatenation of paths:

$$\Omega(M, q_0, q_1) \times \Omega(M, q_1, q_2) \to \Omega(M, q_0, q_2),$$

$$\gamma_1 \gamma_2(t) = \begin{cases} \gamma_1(2t), & 0 \le t \le \frac{1}{2}, \\ \gamma_2(2t - 1), & \frac{1}{2} \le t \le 1. \end{cases}$$

In order to do Morse theory on the path space, we use $\Omega^{1,2}(M, q_0, q_1)$, the subset of $\Omega(M, q_0, q_1)$ consisting of paths in the class $W^{1,2}$.

There is a natural action functional on $\Omega^{1,2}(M, q_0, q_1)$. Recall the Hamiltonian H_V, the Hamiltonian vector field X_{H_V} and the time-t Hamiltonian flow $\phi_{H_V}^t$ from Section 2.2. Consider the function $L_V \colon [0, 1] \times TM \to \mathbb{R}$ given by

$$L_V(t, q, v) = \frac{1}{2}|v|^2 - V(t, q), \tag{3.1}$$

where $t \in [0, 1]$, $q \in M$, and $v \in T_q M$. For each $\gamma \in \Omega^{1,2}(M, q_0, q_1)$, let

$$\mathcal{A}_V(\gamma) = \int_0^1 L_V(t, \gamma, \dot{\gamma}) \, \mathrm{d}t. \tag{3.2}$$

It is well known that $\mathcal{A}_V$ is a Morse function on $\Omega^{1,2}(M, q_0, q_1)$. Therefore, we can define $\mathrm{CM}_*(\Omega^{1,2}(M, q_0, q_1))$ as the Morse complex generated by the critical points of $\mathcal{A}_V$ and with differential induced by $\mathcal{A}_V$ and the metric g. We omit the details which can be found in [1, Section 2] and simply denote the Morse homology group of $\mathrm{CM}_*(\Omega^{1,2}(M, q_0, q_1))$ by $\mathrm{HM}_*(\Omega^{1,2}(M, q_0, q_1))$.

Definition 3.1. A *V-perturbed geodesic* γ is a map $[0, 1] \to M$ such that

$$\nabla_{\dot{\gamma}} \dot{\gamma} = -\nabla V, \tag{3.3}$$

where ∇V denotes the gradient of V with respect to g.

By a standard calculus of variations computation, the critical points of the functional $\mathcal{A}_V$ on $\Omega^{1,2}(M, q_0, q_1)$ are exactly V-perturbed geodesics.

We now recall some standard facts following [1, Section 2.1]. Let $\mathcal{C}_{H_V}$ be the set of time-1 integral curves of X_{H_V} on T^*M, and let $\mathcal{C}_{L_V}$ be the set of time-1 V-perturbed geodesics on M, i.e.,

$$\mathcal{C}_{H_V} := \{\zeta : [0, 1] \to T^*M \mid \zeta(t) = \phi_{H_V}^t \circ \zeta(0)\},$$
$$\mathcal{C}_{L_V} := \{\gamma : [0, 1] \to M \mid \nabla_{\dot{\gamma}} \dot{\gamma} = -\nabla V\}.$$

We define a map $\mathcal{L}\colon \mathcal{C}_{H_V} \to \mathcal{C}_{L_V}$ as follows: Given $\zeta \in \mathcal{C}_{H_V}$, let $\mathcal{L}(\zeta)$ be the path $[0,1] \to M$ given by

$$\mathcal{L}(\zeta)(t) := \pi_M \circ \zeta(t).$$

One can verify that $\mathcal{L}(\zeta)$ satisfies (3.3) and hence belongs to $\mathcal{C}_{L_V}$.

We define the inverse map $\mathcal{L}^{-1}\colon \mathcal{C}_{L_V} \to \mathcal{C}_{H_V}$ as follows: Given $\gamma \in \mathcal{C}_{L_V}$, we define $\mathcal{L}^{-1}(\gamma)\colon [0,1] \to T^*M$ as

$$\mathcal{L}^{-1}(\gamma) := (\gamma(t), \mathrm{d}L(t, \gamma(t), \dot{\gamma}(t))|_{T^v_{(\gamma(t),\dot{\gamma}(t))}TM}),$$

where $T^v_{(\gamma(t),\dot{\gamma}(t))}TM$ is the vertical fiber $\ker D\pi_M \cong T_{\gamma(t)}M$ at $(\gamma(t),\dot{\gamma}(t)) \in TM$ and $\pi_M\colon TM \to M$ is the projection.

Definition 3.2. We call $\mathcal{L}$ the *Legendre transform* and call $\mathcal{L}^{-1}$ the *inverse Legendre transform*.

Remark 3.3. Each generator $y \in \mathrm{CF}(\phi^1_{H_V}(T^*_{q_0}M), T^*_{q_1}M)$ corresponds to a time-1 integral curve of X_{H_V} from $T^*_{q_0}M$ to $T^*_{q_1}M$:

$$l_y\colon [0,1] \to T^*M, \quad l_y(t) = \phi^{t-1}_{H_V}(y).$$

We define $\mathcal{L}(y)$ to be $\mathcal{L}(l_y)$. Conversely, $\mathcal{L}^{-1}$ maps time-1 V-perturbed geodesics from q_0 to q_1 to a generator of $\mathrm{CF}(\phi^1_{H_V}(T^*_{q_0}M), T^*_{q_1}M)$.

Since the inclusion

$$\Omega^{1,2}(M, q_0, q_1) \hookrightarrow \Omega(M, q_0, q_1)$$

is a homotopy equivalence, we deduce that $\mathrm{HM}_*(\Omega^{1,2}(M, q_0, q_1))$ is isomorphic to the singular homology group $H_*(\Omega(M, q_0, q_1))$.

When $q_0 = q_1 = q$, we get the based loop space

$$\Omega(M, q) = \{\gamma \in C^0([0,1], M) \mid \gamma(0) = \gamma(1) = q\}.$$

Theorem 3.4 ([2, Theorem B]). *There is an isomorphism of graded algebras:*

$$H_{-*}(\Omega(M, q)) \to \mathrm{HW}^*(T^*_q M). \tag{3.4}$$

In the opposite direction, Abouzaid constructed a chain level evaluation map

$$\mathrm{CW}^*(T^*_q M) \to C_{-*}(\Omega(M, q)).$$

It induces an isomorphism on the level of homology:

$$\widetilde{\mathcal{F}}\colon \mathrm{HW}^*(T^*_q M) \to H_{-*}(\Omega(M, q)). \tag{3.5}$$

Remark 3.5. Theorem 3.4 also holds for path spaces by [1], i.e., there is an isomorphism

$$H_{-*}(\Omega(M, q_0, q_1)) \to \mathrm{HW}^*(T^*_{q_0}M, T^*_{q_1}M),$$

where the right-hand side is the wrapped Floer homology group whose generators are time-1 Hamiltonian flows of $\phi^t_{H_V}$ from $T^*_{q_0}M$ to $T^*_{q_1}M$.

Let us now specialize to $M = \Sigma$, a closed oriented surface of genus greater than 0. In this case, we further see the following.

Lemma 3.6. *The algebra* $H_*(\Omega(\Sigma, q_0, q_1))$ *is supported in degree* 0.

Proof. Recall that V is small in the $W^{1,2}$-norm. If Σ is a torus, then we can assume that g is the flat metric, where all V-perturbed geodesics with V sufficiently small are minimal and isolated. If the genus of Σ is greater than 1, then we can assume that g is the hyperbolic metric with constant curvature -1. It is well known that on a hyperbolic surface, there is a unique V-perturbed geodesic in each homotopy class of paths with fixed endpoints for V sufficiently small. For details the reader is referred to Milnor [27, Lemma 19.1]. Hence the Morse indices of all critical points of $\Omega^{1,2}(\Sigma, q_0, q_1)$ are 0. ∎

Note that when $M = \Sigma$, all terms of (3.4) vanish except for $* = 0$ by Lemma 3.6. We write $\mathrm{HW}(T_q^*\Sigma)$ for $\mathrm{HW}^0(T_q^*\Sigma)$. Moreover, $H_0(\Omega(\Sigma, q))$ is isomorphic to the group algebra $\mathbb{Z}[\pi_1(\Sigma, q)]$ of the fundamental group $\pi_1(\Sigma, q)$.

4. Hecke algebras

Theorem 3.4 and isomorphism (3.5) relate $H_0(\Omega(\Sigma, q))$ to the wrapped Floer homology $\mathrm{HW}(T_q^*\Sigma)$ of a single cotangent fiber, and represent the special case of $\kappa = 1$ in HDHF. In this section, we discuss the generalization of $H_0(\Omega(\Sigma, q))$ to $\kappa \geq 1$, which we show to be equivalent to $\mathrm{HW}(\bigsqcup_i T_{q_i}^*\Sigma)$ in later sections.

Summary. Consider the based loop space of the unordered configuration space of κ points on Σ. Its 0-th homology is isomorphic to the group algebra of the braid group of Σ. The *braid skein algebra* $\mathrm{BSk}_\kappa(\Sigma, \mathbf{q})$, due to Morton and Samuelson [28, Definition 3.1], is a quotient of the group algebra of the braid group by the HOMFLY skein relation and the marked point relation. Here $\mathbf{q}$ is a κ-tuple of distinct points on Σ. By reformulating the marked point relation, we obtain the *surface Hecke algebra* $\mathrm{H}_\kappa(\Sigma, \mathbf{q})$ in Definition 4.4. This surface Hecke algebra serves as an intermediary between the wrapped Floer homology and the braid skein algebra. On the one hand, we show that $\mathrm{H}_\kappa(\Sigma, \mathbf{q})$ and $\mathrm{BSk}_\kappa(\Sigma)$ are isomorphic up to a change of variables in Proposition 4.5. On the other hand, we will construct an evaluation map from the wrapped Floer homology to $\mathrm{H}_\kappa(\Sigma, \mathbf{q})$ in Section 6.1.

Note that the braid skein algebra for $\Sigma = T^2$ is isomorphic to the double affine Hecke algebra (DAHA) of $\mathfrak{gl}_\kappa$ [28, Theorem 3.7]. Hence $\mathrm{H}_\kappa(T^2, \mathbf{q})$ is also isomorphic to the DAHA.

Let $\mathrm{UConf}_\kappa(\Sigma) = \{\{q_1, \ldots, q_\kappa\} \mid q_i \in \Sigma, q_i \neq q_j \text{ for } i \neq j\}$ be the configuration space of κ unordered points on Σ. Fix a basepoint $\mathbf{q} \in \mathrm{UConf}_\kappa(\Sigma)$. The based loop space $\Omega(\mathrm{UConf}_\kappa(\Sigma), \mathbf{q})$ consists of κ-strand braids in Σ. Note that $H_0(\Omega(\mathrm{UConf}_\kappa(\Sigma), \mathbf{q}))$ is isomorphic to the group algebra $\mathbb{Z}[\mathrm{Br}_\kappa(\Sigma, \mathbf{q})]$, where $\mathrm{Br}_\kappa(\Sigma, \mathbf{q})$ denotes the braid group of Σ.

Fix a marked point $\star \in \Sigma$ which is disjoint from $\mathbf{q}$. Let $\mathrm{Br}_{\kappa,1}(\Sigma, \mathbf{q}, \star)$ be the subgroup of $\mathrm{Br}_{\kappa+1}(\Sigma, \mathbf{q} \sqcup \{\star\})$ consisting of braids whose last strand connects $\star$ to itself by a straight line in $[0,1] \times \Sigma$.

Definition 4.1 (Morton–Samuelson). The braid skein algebra $\mathrm{BSk}_\kappa(\Sigma, \mathbf{q})$ is the quotient of the group algebra $\mathbb{Z}[s^{\pm 1}, c^{\pm 1}][\mathrm{Br}_{\kappa,1}(\Sigma, \mathbf{q}, \star)]$ by two local relations:

(1) the HOMFLY skein relation

$$\text{} - \text{} = (s - s^{-1}) \text{}, \tag{4.1}$$

(2) the marked point relation $P = c^2$

$$P := \text{} = c^2 \text{}. \tag{4.2}$$

Here the black lines are strands between basepoints in $\mathbf{q}$ and the straight blue line connects the marked point $\star$ to itself.

The product is given by the concatenation of braids.

For the purposes of relating $\mathrm{BSk}_\kappa(\Sigma, \mathbf{q})$ to the wrapped Floer homology, it is more convenient to give another description of the marked point relation (4.2). Let

$$\Omega(\mathrm{UConf}_\kappa(\Sigma \setminus \{\star\}), \mathbf{q})$$
$$= \{\gamma \in C^0([0,1], \mathrm{UConf}_\kappa(\Sigma \setminus \{\star\})) \mid \gamma(0) = \gamma(1) = \mathbf{q}\}. \tag{4.3}$$

Given $\gamma_1, \gamma_2 \in \Omega(\mathrm{UConf}_\kappa(\Sigma \setminus \{\star\}), \mathbf{q})$ viewed as based loops on $\mathrm{UConf}_\kappa(\Sigma)$, let

$$H: [0,1]^2 \to \mathrm{UConf}_\kappa(\Sigma)$$

be a homotopy between γ_1 and γ_2 relative to the boundary, i.e.,

$$H(t,0) = \gamma_1(t), \quad H(t,1) = \gamma_2(t), \quad H(0,s) = H(1,s) = \mathbf{q}.$$

The homotopy H may intersect the marked point $\star$. We define $\langle H, \star \rangle := \langle H, Y \rangle$, the algebraic intersection number of H and Y, where

$$Y := \{\{p_1, \ldots, p_\kappa\} \in \mathrm{UConf}_\kappa(\Sigma) \mid p_i = \star \text{ for some } i\}$$

is a codimension two submanifold of $\mathrm{UConf}_\kappa(\Sigma)$. Here the orientation of Y is induced from that of Σ and the orientation of H is induced from $-\mathrm{d}t \wedge \mathrm{d}s$ on the square $[0,1]^2$ (note the coordinates on $[0,1]^2$ are t, s in that order). This is well defined since

$$H(\partial([0,1]^2)) \cap Y = \varnothing.$$

We identify $H_0(\Omega(\mathrm{UConf}_\kappa(\Sigma \setminus \{\star\}), \mathbf{q}))$ with the group algebra $\mathbb{Z}[\mathrm{Br}_\kappa(\Sigma \setminus \{\star\}, \mathbf{q})]$ of the braid group $\mathrm{Br}_\kappa(\Sigma \setminus \{\star\}, \mathbf{q})$.

Definition 4.2. The c-deformed braid group $\mathrm{Br}_\kappa(\Sigma, \mathbf{q})_c$ of Σ is generated by the group $\mathrm{Br}_\kappa(\Sigma \setminus \{\star\}, \mathbf{q})$ and a central element c, subject to the following c-deformed homotopy relation:

$$[\gamma_2] = c^{2\langle H, \star \rangle}[\gamma_1], \tag{4.4}$$

where $\gamma_i \in \Omega(\mathrm{UConf}_\kappa(\Sigma \setminus \{\star\}), \mathbf{q})$, H is the homotopy between them as above, and

$$[\gamma_i] \in H_0(\Omega(\mathrm{UConf}_\kappa(\Sigma \setminus \{\star\}), \mathbf{q})) \cong \mathbb{Z}[\mathrm{Br}_\kappa(\Sigma \setminus \{\star\}, \mathbf{q})].$$

Remark 4.3. The group algebra $\mathbb{Z}[\mathrm{Br}_\kappa(\Sigma, \mathbf{q})_c]$ is naturally isomorphic to the quotient of $C_0(\Omega(\mathrm{UConf}_\kappa(\Sigma \setminus \{\star\}), \mathbf{q})) \otimes \mathbb{Z}[c^{\pm 1}]$ by the c-deformed homotopy relation (4.4).

See Figure 3 for an example when $\kappa = 1$ and $\langle H, Y \rangle = 1$.
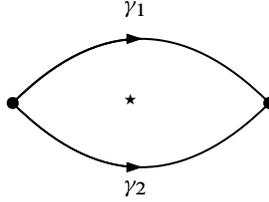
Fig. 3. The deformed homotopy relation $[\gamma_2] = c^2[\gamma_1]$. Both dotted points are $q \in \Sigma$.

Specializing to $c = 1$, $\mathrm{Br}_\kappa(\Sigma, \mathbf{q})_c$ recovers $\mathrm{Br}_\kappa(\Sigma, \mathbf{q})$. Hence there is a central extension of groups:

$$1 \to \langle c \rangle \to \mathrm{Br}_\kappa(\Sigma, \mathbf{q})_c \to \mathrm{Br}_\kappa(\Sigma, \mathbf{q}) \to 1, \tag{4.5}$$

where $\langle c \rangle$ is the free abelian group generated by c.

Let $\langle \mathrm{Br}_{\kappa,1}(\Sigma, \mathbf{q}, \star), c \rangle$ denote the group generated by $\mathrm{Br}_{\kappa,1}(\Sigma, \mathbf{q}, \star)$ and a central element c. Define the group $\mathrm{Br}_\kappa(\Sigma, \mathbf{q})'_c$ as the quotient of $\langle \mathrm{Br}_{\kappa,1}(\Sigma, \mathbf{q}, \star), c \rangle$ by the marked point relation $P = c^2$; see (4.2). There are two natural projections from $\langle \mathrm{Br}_{\kappa,1}(\Sigma, \mathbf{q}, \star), c \rangle$ to $\mathrm{Br}_\kappa(\Sigma, \mathbf{q})'_c$ and $\mathrm{Br}_\kappa(\Sigma, \mathbf{q})_c$, respectively,

The relation $P = c^2$ also holds in $\mathrm{Br}_\kappa(\Sigma, \mathbf{q})_c$. Therefore, there is a natural induced map $\phi \colon \mathrm{Br}_\kappa(\Sigma, \mathbf{q})'_c \to \mathrm{Br}_\kappa(\Sigma, \mathbf{q})_c$.

The algebra $\mathrm{Br}_\kappa(\Sigma, \mathbf{q})'_c$ also fits into a central extension similar to (4.5),

$$1 \to \langle c \rangle \to \mathrm{Br}_\kappa(\Sigma, \mathbf{q})'_c \to \mathrm{Br}_\kappa(\Sigma, \mathbf{q}) \to 1.$$

Moreover, the following diagram commutes:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \langle c \rangle & \longrightarrow & \mathrm{Br}_\kappa(\Sigma, \mathbf{q})_c & \longrightarrow & \mathrm{Br}_\kappa(\Sigma, \mathbf{q}) & \longrightarrow & 1 \\
& & \| & & \uparrow{\scriptstyle \phi} & & \| & & \\
1 & \longrightarrow & \langle c \rangle & \longrightarrow & \mathrm{Br}_\kappa(\Sigma, \mathbf{q})'_c & \longrightarrow & \mathrm{Br}_\kappa(\Sigma, \mathbf{q}) & \longrightarrow & 1.
\end{array}
$$

By the five lemma, ϕ is an isomorphism.

Recall from Definition 4.1 that the braid skein algebra $\mathrm{BSk}_\kappa(\Sigma, \mathbf{q})$ is the quotient of $\mathbb{Z}[s^{\pm 1}][\mathrm{Br}_\kappa(\Sigma, \mathbf{q})'_c]$ by the skein relation (4.1). We introduce the variable $\hbar = s - s^{-1}$ and add the corresponding skein relation to $\mathrm{Br}_\kappa(\Sigma, \mathbf{q})_c$.

Definition 4.4. The Hecke algebra $\mathrm{H}_\kappa(\Sigma, \mathbf{q})$ of Σ is the quotient of the group algebra $\mathbb{Z}[\hbar][\mathrm{Br}_\kappa(\Sigma, \mathbf{q})_c]$ by the local skein relation:

$$
\diagup\!\!\!\!\!\diagdown \;-\; \diagdown\!\!\!\!\!\diagup \;=\; \hbar \;\Big)\;\Big(\;. \tag{4.6}
$$

We now apply a change of variables $\hbar = s - s^{-1}$ to $\mathrm{BSk}_\kappa(\Sigma, \mathbf{q})$ and change the coefficient ring from $\mathbb{Z}[s^{\pm 1}, c^{\pm 1}]$ to $\mathbb{Z}[\hbar, c^{\pm 1}]$; the resulting algebra will be denoted by $\mathrm{BSk}_\kappa(\Sigma, \mathbf{q})|_\hbar$.

Proposition 4.5. *The algebra $\mathrm{H}_\kappa(\Sigma, \mathbf{q})$ is naturally isomorphic to $\mathrm{BSk}_\kappa(\Sigma, \mathbf{q})|_\hbar$.*

Finally, we consider the degeneration $\hbar = 0$ for later use. The skein relation (4.6) then reduces to the symmetric group relation. Further setting $c = 1$, the free abelian group $\mathrm{H}_\kappa(\Sigma, \mathbf{q})|_{\hbar=0, c=1}$ has a $\mathbb{Z}$-basis

$$\mathfrak{B} := \{\gamma_1 \cdots \gamma_\kappa \cdot \sigma \mid \gamma_i \in \pi_1(\Sigma, q_i), \sigma \in S_\kappa\}.$$

Here γ_i is the homotopy class of the i-th strand in the braid group and σ is the image of the natural quotient map from the braid group to the symmetric group. Hence we have a natural isomorphism of algebras

$$\mathrm{H}_\kappa(\Sigma, \mathbf{q})|_{\hbar=0, c=1} \cong \mathbb{Z}\Big[\prod_i \pi_1(\Sigma, q_i) \rtimes S_\kappa \Big],$$

where the symmetric group S_κ acts on $\prod_i \pi_1(\Sigma, q_i)$ by permuting the factors. Adding the parameter c, let $\pi_1(\Sigma, q)_c$ denote $\mathrm{Br}_1(\Sigma, q)_c$, the 1-strand c-deformed braid group. Its group algebra $\mathbb{Z}[\pi_1(\Sigma, q)_c]$ is a $\mathbb{Z}[c^{\pm 1}]$-algebra.

Lemma 4.6. *There is an isomorphism of $\mathbb{Z}[c^{\pm 1}]$-algebras*

$$\mathrm{H}_\kappa(\Sigma, \mathbf{q})|_{\hbar=0} \cong \Big(\bigotimes_i \mathbb{Z}[\pi_1(\Sigma, q_i)_c] \Big) \rtimes S_\kappa,$$

where the tensor product is over $\mathbb{Z}[c^{\pm 1}]$.

Proof. The free abelian group $H_\kappa(\Sigma, \mathbf{q})|_{\hbar=0}$ has $\mathbb{Z}$-basis

$$\mathfrak{B}_c := \{\gamma_1 \cdots \gamma_\kappa \cdot \sigma \mid \gamma_i \in \pi_1(\Sigma, q_i)_c, \sigma \in S_\kappa\}.$$

We have a natural surjective homomorphism

$$\left(\widetilde{\bigotimes_i} \mathbb{Z}[\pi_1(\Sigma, q_i)_c]\right) \rtimes S_\kappa \to H_\kappa(\Sigma, \mathbf{q})|_{\hbar=0},$$

where $\widetilde{\otimes}$ denotes the tensor product over $\mathbb{Z}$. It factors through

$$\left(\bigotimes_i \mathbb{Z}[\pi_1(\Sigma, q_i)_c]\right) \rtimes S_\kappa \to H_\kappa(\Sigma, \mathbf{q})|_{\hbar=0}$$

since c is central in $H_\kappa(\Sigma, \mathbf{q})|_{\hbar=0}$. The latter map is injective since the domain and target have the same $\mathbb{Z}[c^{\pm 1}]$-basis $\mathfrak{B}$. ∎

5. The parameter c in higher-dimensional Heegaard Floer homology

Motivated by the c-deformed homotopy relation (4.4), we will define $\mathrm{CW}(\bigsqcup_i T^*_{q_i}\Sigma)_c$, the wrapped HDHF homology of disjoint cotangent fibers with an additional parameter c. We inherit the notation from Section 2.2.

Let

$$\mathrm{CF}\left(\phi^1_{H_V}\left(\bigsqcup_i T^*_{q_i}\Sigma\right), \bigsqcup_i T^*_{q_i}\Sigma\right)_c := \mathrm{CF}\left(\phi^1_{H_V}\left(\bigsqcup_i T^*_{q_i}\Sigma\right), \bigsqcup_i T^*_{q_i}\Sigma\right) \otimes \mathbb{Z}[c^{\pm 1}]$$

as a $\mathbb{Z}$-module.

Given $u \in \mathcal{M}(\mathbf{y}_1, \ldots, \mathbf{y}_m, \mathbf{y}_0)$ of index 0 or 1, consider its projection to Σ and denote the image by $\pi_\Sigma(u)$. We enhance the A_∞-operations from Section 2.2 to include c-coefficients by keeping track of modified intersections of $\pi_\Sigma(u)$ and a fixed marked point $\star \in \Sigma$. Note that we cannot directly take the intersection number of $\pi_\Sigma(u)$ and $\star$ since the boundary of $\pi_\Sigma(u)$ could cross $\star$ in a generic 1-parameter family and the A_∞-relation would not be satisfied.

To remedy this issue, we carefully choose the marked point $\star \in \Sigma$ and modify $\pi_\Sigma(u)$.

Consider the set of V-perturbed geodesics (see Definition 3.1) with endpoints in $\{q_1, \ldots, q_\kappa\}$. Since there are only countably many such V-perturbed geodesics, we can choose a generic marked point $\star \in \Sigma$ in the complement of the images of these perturbed geodesics.

We then homotop the boundary of $\pi_\Sigma(u)$ on Σ to piecewise V-perturbed geodesics, which can be guaranteed to be disjoint from $\star$; see Figure 4. Specifically, suppose the domain of u is $\dot{F}$ and $\{p_{i-1}, p_i, p_{i+1}\}$ are three consecutive boundary marked points on ∂F, ordered according to the boundary orientation of ∂F. Write $\pi_{T^*\Sigma} \circ u(p_i)$ etc. for the values of the continuous extensions of $\pi_{T^*\Sigma} \circ u$ to the puncture p_i etc. Denote the boundary arc from p_i to p_{i+1} by $\partial_i \dot{F}$. Suppose the cotangent fibers $T^*_{q_{i-1}}\Sigma, T^*_{q_i}\Sigma$
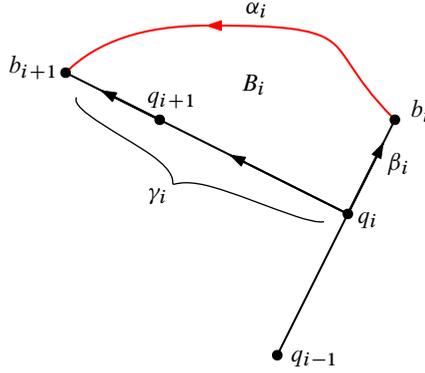
Fig. 4. The projection $\pi_\Sigma(u)$ of u to Σ. The path α_i is the projection of $u|_{\partial_i \dot{F}}$, where $\partial_i \dot{F}$ is a component of $\partial \dot{F}$. We choose any homotopy from α_i to $-\beta_i + \gamma_i$, where the relative homology class B_i that is swept out does not depend on the choice of homotopy. In the figure, we are assuming that V is independent of t and hence q_i, q_{i+1} and b_{i+1} lie on the same V-perturbed geodesic.

and $T^*_{q_{i+1}}\Sigma$ are wrapped using the wrapping functions ψ_{i-1}, ψ_i and ψ_{i+1}, respectively. For $r = 0, 1$, suppose $\pi_{T^*\Sigma} \circ u(p_{i+r}) \in \psi_{i+r-1}(T^*_{q_{i+r-1}}\Sigma) \cap \psi_{i+r}(T^*_{q_{i+r}}\Sigma)$, which corresponds to the intersection of two Hamiltonian chords that start from $T^*_{q_{i+r-1}}\Sigma$ (induced by ψ_{i+r-1}) and $T^*_{q_{i+r}}\Sigma$ (induced by ψ_{i+r}). The Legendre transforms of these Hamiltonian chords (see Definition 3.2) correspond to certain V-perturbed geodesics on Σ. Let β_{i+r-1} be that from q_{i+r-1} to b_{i+r-1}, and γ_{i+r-1} be that from q_{i+r-1} to b_{i+r}, where $b_i = \pi_\Sigma \circ u(p_i)$. (Observe that q_i, q_{i+1} and b_{i+1} lie on the same V-perturbed geodesic, if $V = V(t, x)$ is independent of t; this is because being able to intersect in T^*M means that the tangent vector of the two V-perturbed geodesics at b_i are the same.) Let α_i be the path $\pi_\Sigma \circ u(\partial_i \dot{F})$.

Fix parametrizations of α_i, β_i and γ_i by the interval $[0, 1]$. Note that α_i and its *piecewise V-geodesic replacement* $\beta_i^{-1} \cdot \gamma_i$ are homotopic as paths from b_i to b_{i+1}; this is due to the facts that α_i, β_i and γ_i are all projections of paths in $\psi_i(T^*_{q_i}\Sigma)$ and that $T^*_{q_i}\Sigma$ is contractible. Let B_i be a homotopy between α_i and $\beta_i^{-1} \cdot \gamma_i$ relative to boundary. We extend the image $\pi_\Sigma(u)$ on Σ by the homotopy B_i for all $\partial_i \dot{F} \subset \partial \dot{F}$ so that its new boundary lies in $C := \bigcup_i(\beta_i \cup \gamma_i)$; this defines a relative homology class $[\pi_\Sigma(u)]' \in H_2(\Sigma, C)$.

Given two homotopies B_i and B_i' from α_i to its piecewise V-geodesic replacement $\beta_i^{-1} \cdot \gamma_i$, their difference determines a map $S^2 \to \Sigma$, which induces the zero map on $H_2(S^2) \to H_2(\Sigma, C)$. Hence $[\pi_\Sigma(u)]'$ does not depend on the choice of $\{B_i\}$ and the algebraic intersection number $\langle u, \star \rangle := \langle [\pi_\Sigma(u)]', \star \rangle$ is well defined.

We modify the μ^m-composition map so that

$$\mu^m(\mathbf{y}_1, \ldots, \mathbf{y}_m) = \sum_{u \in \mathcal{M}^{\mathrm{ind}=0}(\mathbf{y}_1, \ldots, \mathbf{y}_m, \mathbf{y}_0)} (-1)^{\natural(u)} \cdot c^{2\langle u, \star \rangle} \cdot \hbar^{\kappa - \chi(u)} \cdot \mathbf{y}_0,$$

where u ranges over curves of index 0, and $\natural(u) \in \mathbb{Z}$.

Since C is disjoint from $\star$, $\langle u, \star \rangle$ is constant for any 1-parameter family of u. There-fore, by analyzing the degeneration of index-1 moduli spaces, we see that $\mathrm{CW}(\bigsqcup_i T^*_{q_i}\Sigma)_c$ is an A_∞-algebra. Proposition 2.9 can then be improved to the following.

Proposition 5.1. *The A_∞-algebra $\mathrm{CW}(\bigsqcup_i T^*_{q_i}\Sigma)_c$ is supported in degree zero and hence is an ordinary algebra.*

6. The evaluation map

Following Abouzaid [4], we construct the evaluation map

$$\mathcal{F} \colon \mathrm{CW}\left(\bigsqcup_i T^*_{q_i}\Sigma\right)_c \to \mathrm{H}_\kappa(\Sigma) \otimes_{\mathbb{Z}[\hbar]} \mathbb{Z}[\![\hbar]\!]$$

in Section 6.1. It is given by counting holomorphic curves between cotangent fibers and the zero section of $T^*\Sigma$ in the framework of HDHF. We then show that $\mathcal{F}$ is a homomorphism of algebras in Section 6.2. The key ingredient is the holomorphic curve inter-pretation of the HOMFLY skein relation due to Ekholm–Shende [14]. We finally prove Theorem 1.5 which states that the map $\mathcal{F}$ is an isomorphism.

6.1. The definition

At this point, we rename D_m as T_{m-1}, where $\partial_i T_{m-1} = \partial_i D_m$ for $i = 0, \dots, m$. The disk T_{m-1} will be the A_∞-base direction, where $p_1, \dots, p_{m-1} \in \partial D_m = \partial T_{m-1}$ corre-spond to inputs (κ-tuples of intersection points) and $\partial_m D_m = \partial_m T_{m-1}$ corresponds to the output (κ-tuples of arcs on Σ). We view all of the strip-like ends e_i, $i = 0, \dots, m$, of T_{m-1} corresponding to p_i as positive ends $[0, \infty)_{s_i} \times [0,1]_{t_i}$. Let $\mathcal{T}_{m-1}$ be the moduli space of T_{m-1} modulo automorphisms; again we choose representatives T_{m-1} of equiva-lence classes of $\mathcal{T}_{m-1}$ in a smooth manner.

Let $\pi_{T^*\Sigma}$ be the projection $T_{m-1} \times T^*\Sigma \to T^*\Sigma$. Choose a sufficiently generic con-sistent collection $T_{m-1} \mapsto J_{T_{m-1}}$ of compatible almost complex structures on $T_{m-1} \times T^*\Sigma$ for all $T_{m-1} \in \mathcal{T}_{m-1}$ and all $m \geq 2$ such that

(J1′) on each fiber $\pi^{-1}_{T_{m-1}}(p) = \{p\} \times T^*\Sigma$, $J_{T_{m-1}}$ restricts to an element of $\mathcal{J}_{T^*\Sigma, \alpha_{\mathrm{std}}}$ where α_{std} is the standard Liouville form on $T^*\Sigma$;

(J2′) $J_{T_{m-1}}$ projects holomorphically onto T_{m-1};

(J3′) over each strip-like end $[0, \infty)_{s_i} \times [0,1]_{t_i}$, for s_i sufficiently positive, $J_{T_{m-1}}$ is invariant in the s_i-direction and takes ∂_{s_i} to ∂_{t_i}.

Recall the time-t Hamiltonian flow of $\phi^t_{H_V}$ (2.4). We will refer to Σ as the zero section of $T^*\Sigma$ when it is clear from the context. Let $\mathbf{q}$ (resp. $\mathbf{q}'$) be the set of intersection points between $\bigsqcup_i T^*_{q_i}\Sigma$ (resp. $\phi^1_{H_V}(\bigsqcup_i T^*_{q_i}\Sigma)$) and Σ, and let $\mathbf{y} \in \mathrm{CF}(\phi^1_{H_V}(\bigsqcup_i T^*_{q_i}\Sigma), \bigsqcup_i T^*_{q_i}\Sigma)$. Note that there is a canonical identification between $\mathbf{q}$ and $\mathbf{q}'$.

We define $\mathcal{H}(\mathbf{q}', \mathbf{y}, \mathbf{q})$ as the moduli space of maps

$$u \colon (\dot{F}, j) \to (T_1 \times T^*\Sigma, J_{T_1}),$$

where (F, j) is a compact Riemann surface with boundary, $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$ are disjoint κ-tuples of boundary marked points of F, $\dot{F} = F \setminus \bigcup_i \mathbf{p}_i$, and u satisfies the following conditions:

- $du \circ j = J_{T_1} \circ du$;
- $\pi_{T^*\Sigma} \circ u(z) \in \phi^1_{H_V} \left(\bigsqcup_i T^*_{q_i} \Sigma \right)$ if $\pi_{T_1} \circ u(z) \subset \partial_0 T_1$;
- each component of $\partial \dot{F}$ that projects to $\partial_0 T_1$ maps to a distinct $\phi^1_{H_V}(T^*_{q_i}\Sigma)$;
- $\pi_{T^*\Sigma} \circ u(z) \in \bigsqcup_i T^*_{q_i}\Sigma$ if $\pi_{T_1} \circ u(z) \subset \partial_1 T_1$;
- each component of $\partial \dot{F}$ that projects to $\partial_1 T_1$ maps to a distinct $T^*_{q_i}\Sigma$;
- $\pi_{T^*\Sigma} \circ u(z) \in \Sigma$ if $\pi_{T_1} \circ u(z) \subset \partial_2 T_1$;
- $\pi_{T^*\Sigma} \circ u$ tends to $\mathbf{q}'$, $\mathbf{y}$, $\mathbf{q}$ as $s_0, s_1, s_2 \to +\infty$;
- $\pi_{T_1} \circ u$ is a κ-fold branched cover of a fixed $T_1 \in \mathcal{T}_1$.

See Figure 5.
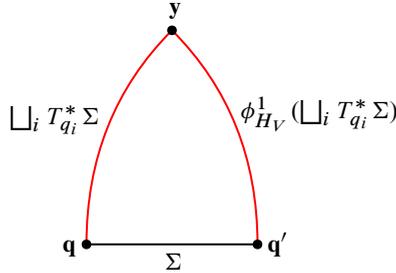
Fig. 5. The A_∞-base direction T_1 for $\mathcal{H}(\mathbf{q}', \mathbf{y}, \mathbf{q})$. The notation denotes the corresponding preimages of π_{T_1} in $T^*\Sigma$, e.g., p_0 is denoted by $\mathbf{q}'$ since $\pi_{T^*\Sigma} \circ u$ tends to $\mathbf{q}'$ as $\pi_{T_1} \circ u \to p_0$.

Lemma 6.1. *Fixing generic J_{T_1}, $\mathcal{H}(\mathbf{q}', \mathbf{y}, \mathbf{q})$ is of dimension 0 and consists of discrete regular curves for all $\mathbf{q}$, $\mathbf{y}$ and $\mathbf{q}'$.*

Proof. By the discussion before Lemma 2.8, $|q_1| = \cdots = |q_\kappa| = |q'_1| = \cdots = |q'_\kappa| = 0$. Hence $|\mathbf{q}| = |\mathbf{q}'| = 0$. By Lemma 2.8, $|\mathbf{y}| = 0$. We then see that the virtual dimension of $\mathcal{H}(\mathbf{q}', \mathbf{y}, \mathbf{q})$ is 0 by the index formula (2.2); note that the same index formula holds even when we do not assume that the copies of the zero section Σ are disjoint. The lemma then follows from standard transversality arguments. ∎

Let $\mathcal{H}^\chi(\mathbf{q}', \mathbf{y}, \mathbf{q})$ be the subset of $\mathcal{H}(\mathbf{q}', \mathbf{y}, \mathbf{q})$ such that $\chi(\dot{F}) = \chi$.

Lemma 6.2. *Given $\mathbf{q}$, $\mathbf{y}$ and $\mathbf{q}'$, the moduli space $\mathcal{H}^\chi(\mathbf{q}', \mathbf{y}, \mathbf{q})$ consists of finitely many curves for each Euler characteristic χ.*

Proof. Each $\mathbf{y}$ determines a unique $\mathbf{q}$ and $\mathbf{q}'$. Since there is an energy bound for curves in $\mathcal{H}(\mathbf{q}', \mathbf{y}, \mathbf{q})$, by Gromov compactness, $\mathcal{H}(\mathbf{q}', \mathbf{y}, \mathbf{q})$ contains a finite number of curves for each χ. ∎

Fix a parametrization of the arc $\partial_2 T_1$ from p_0 to p_2 by $\tau\colon [0,1] \to \partial_2 T_1$. There exists a sufficiently generic consistent collection $\{T_m \mapsto J_{T_m}\}$ such that for all $u \in \mathcal{H}(\mathbf{q}', \mathbf{y}, \mathbf{q})$, $(\pi_{T^*\Sigma} \circ u) \circ (\pi_{T_1} \circ u)^{-1} \circ \tau(t)$ consists of κ distinct points on $\Sigma \setminus \{\star\}$ for each $t \in [0,1]$ and hence gives a path in $\mathrm{UConf}_\kappa(\Sigma \setminus \{\star\})$:

$$\gamma(u)\colon [0,1] \to \mathrm{UConf}_\kappa(\Sigma \setminus \{\star\}),$$
$$t \mapsto (\pi_{T^*\Sigma} \circ u) \circ (\pi_{T_1} \circ u)^{-1} \circ \tau(t).$$

This is possible since by the usual transversality argument the evaluation map at any point on the domain of u is a submersion subject to the given Lagrangian boundary constraints. Define

$$\Omega(\mathrm{UConf}_\kappa(\Sigma \setminus \{\star\}), \mathbf{q}', \mathbf{q}) = \{\gamma\colon [0,1] \to \mathrm{UConf}_\kappa(\Sigma \setminus \{\star\}) \mid \gamma(0) = \mathbf{q}', \gamma(1) = \mathbf{q}\}.$$

Then

$$\gamma(u) \in \Omega(\mathrm{UConf}_\kappa(\Sigma \setminus \{\star\}), \mathbf{q}', \mathbf{q}).$$

For $u \in \mathcal{H}(\mathbf{q}', \mathbf{y}, \mathbf{q})$, we can define $[\pi_\Sigma(u)]'$ as in Section 5. We extend the image $\pi_\Sigma(u)$ by the homotopies B_i from Section 5, where $\partial_i \dot{F}$ ranges over all boundary arcs of $\partial \dot{F}$ which are not of "output type". Here $\partial_i \dot{F}$ is of "output type" if $\pi_{T_1} \circ u(\partial_i \dot{F}) \subset \partial_2 T_1$. We then define the algebraic intersection number

$$\langle u, \star \rangle := \langle [\pi_\Sigma(u)]', \star \rangle.$$

We now define the evaluation map

$$\mathcal{E}\colon \mathrm{CW}\Big(\bigsqcup_i T^*_{q_i}\Sigma\Big)_c \to C_0(\Omega(\mathrm{UConf}_\kappa(\Sigma \setminus \{\star\}), \mathbf{q}', \mathbf{q})) \otimes \mathbb{Z}[c^{\pm 1}] \otimes \mathbb{Z}[\![\hbar]\!],$$

$$\mathbf{y} \mapsto \sum_{u \in \mathcal{H}(\mathbf{q}', \mathbf{y}, \mathbf{q})} (-1)^{\natural(u)} \cdot c^{2\langle u, \star \rangle} \cdot \hbar^{\kappa - \chi(u)} \cdot \gamma(u).$$

Since the perturbation term V of (2.4) has small $W^{1,2}$-norm, the Hamiltonian vector field X_{H_V} has small norm near the zero section Σ and hence $\mathbf{q}'$ is close to $\mathbf{q}$. We then choose nonintersecting short paths γ_i on Σ from q_i to q_i' for $i = 1, \ldots, \kappa$. By pre-concatenating with $\{\gamma_i\}$, we can identify $\Omega(\mathrm{UConf}_\kappa(\Sigma \setminus \{\star\}), \mathbf{q}', \mathbf{q})$ with $\Omega(\mathrm{UConf}_\kappa(\Sigma \setminus \{\star\}), \mathbf{q})$ as in (4.3). By Definitions 4.2 and 4.4, there is a natural projection map

$$\mathcal{P}\colon C_0(\Omega(\mathrm{UConf}_\kappa(\Sigma \setminus \{\star\}), \mathbf{q})) \otimes \mathbb{Z}[c^{\pm 1}] \otimes \mathbb{Z}[\![\hbar]\!] \to \mathrm{H}_\kappa(\Sigma, \mathbf{q}) \otimes_{\mathbb{Z}[\hbar]} \mathbb{Z}[\![\hbar]\!], \quad (6.1)$$

by first taking the c-deformed homotopy class and then quotienting by the skein relation. We finally define the evaluation map

$$\mathcal{F} = \mathcal{P} \circ \mathcal{E}\colon \mathrm{CW}\Big(\bigsqcup_i T^*_{q_i}\Sigma\Big)_c \to \mathrm{H}_\kappa(\Sigma, \mathbf{q}) \otimes_{\mathbb{Z}[\hbar]} \mathbb{Z}[\![\hbar]\!]. \quad (6.2)$$

Note that $\mathcal{E}$ depends on the choice of the parametrization τ but $\mathcal{F}$ does not.

6.2. *The isomorphism*

Both the domain and target of the map $\mathcal{F}$ in (6.2) are ordinary algebras. We will show that $\mathcal{F}$ is an isomorphism of algebras in this subsection.

We consider a moduli space of curves whose A_∞-base direction is $T_2 \in \mathcal{T}_2$; see Figure 6. Let $\mathbf{q}''$ be the set of intersection points between $\phi^2_{H_V}(\bigsqcup_i T^*_{q_i}\Sigma)$ and Σ; there are canonical identifications $\mathbf{q} \simeq \mathbf{q}' \simeq \mathbf{q}''$. In this case, a 1-parameter family of $T_2 \in \mathcal{T}_2$ may degenerate into broken curves in $\partial \mathcal{T}_2$ as shown in Figure 7. As in Section 2.2, we need to construct the consistent collection carefully near the point T_2' of $\partial \mathcal{T}_2$ corresponding to the left-hand side of Figure 7.
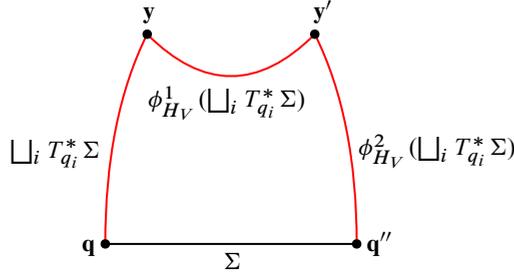

Fig. 6. The A_∞-base direction T_2 for $\mathcal{H}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$.
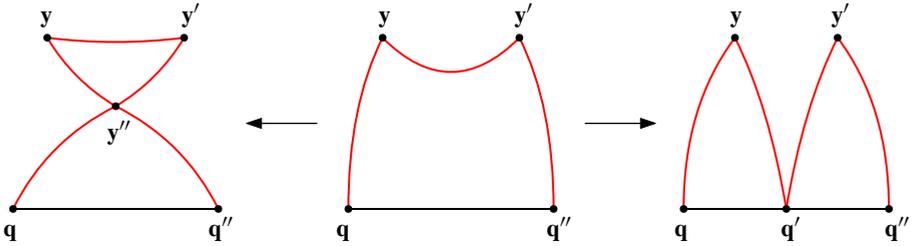

Fig. 7. Degeneration of $\mathcal{H}^{\mathrm{ind}=1}$, $\chi(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ in the T_2-direction. Left: concatenation of curves in $\mathcal{M}^{\mathrm{ind}=0,\chi'}(\mathbf{y}', \mathbf{y}, \mathbf{y}'')$ and $\mathcal{H}^{\mathrm{ind}=0,\chi''}(\mathbf{q}', \mathbf{y}'', \mathbf{q})$; right: concatenation of curves in $\mathcal{H}^{\mathrm{ind}=0,\chi'}(\mathbf{q}'', \mathbf{y}', \mathbf{q}')$ and $\mathcal{H}^{\mathrm{ind}=0,\chi''}(\mathbf{q}', \mathbf{y}, \mathbf{q})$.

We have already constructed consistent collections J''_{D_m} and $\widetilde{L}''_j(D_m)$ for $\mathcal{A}_m$. Let us also assume that we have constructed J_{T_m} for all T_m on large compact subsets of $\mathcal{T}_m$ such that J_{T_m} are consistent with J''_{D_1} at the ends e_i, $i = 1, \ldots, m$. Denote by $\mathcal{U}$ the set of all $T_2 \in \mathcal{T}_2$ that are sufficiently close to breaking into D_2 and T_1. Given $T_2 \in \mathcal{U}$, we choose the neck region $N(T_2)$ (chosen to be smoothly dependent on T_2) and write T_2^+ and T_2^- for the two components of $T_2 \setminus N(T_2)$ that can be viewed as subsets of D_2 and T_1, respectively. We then define J_{T_2} to equal J_{T_1} on $T_2^- \cup N(T_2)$ and J''_{D_2} on $T_2^+ \cup N(T_2)$, and note that J''_{D_2} and J_{T_1} agree on $N(T_2)$ by definition.

We also need to choose a way of wrapping the Lagrangians corresponding to the boundaries of T_2 so that it is consistent with the breaking into $D_2 \cup T_1$ as in the left-hand side of Figure 7. Note that as we degenerate to $D_2 \cup T_1$, the boundary condition

$\phi_{H_V}^2 (\bigsqcup_i T_{q_i}^* \Sigma)$ approaches $\phi_{H_V} (\bigsqcup_i T_{q_i}^* \Sigma)$. To keep track of this degeneration, we define $\Psi_j : \partial_j T_2 \to \text{Symp}(T^* \Sigma, \omega)$, $j = 0, 1, 2$, which depend smoothly on $T_2 \in \mathcal{T}_2$ and such that

$$\Psi_j = \begin{cases} \phi_{H_V}^{2-j} & \text{for } T_2 \in \mathcal{T}_2 \setminus \mathcal{U} \text{ and all } j, \\ \psi^2 \circ \phi_{H_V}^{2-j} \text{ on } \partial_j T_2 \cap T_2^+ & \text{for } T_2 \in \mathcal{U}' \text{ and all } j, \\ \phi_{H_V}^1 \text{ on } \partial_0 T_2 \cap T_2^- \text{ and id on } \partial_2 T_2 \cap T_2^- & \text{for } T_2 \in \mathcal{U}', \end{cases}$$

where $\mathcal{U}'$ is a slight retraction of $\mathcal{U}$.

Given generators

$$\mathbf{y}' \in \text{CF}\Big(\phi_{H_V}^2 \Big(\bigsqcup_i T_{q_i}^* \Sigma\Big), \phi_{H_V}^1 \Big(\bigsqcup_i T_{q_i}^* \Sigma\Big)\Big)_c, \quad \mathbf{y} \in \text{CF}\Big(\phi_{H_V}^1 \Big(\bigsqcup_i T_{q_i}^* \Sigma\Big), \bigsqcup_i T_{q_i}^* \Sigma\Big)_c,$$

let $\mathcal{H}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ be the moduli space of maps $u : (\dot{F}, j) \to (T_2 \times T^* \Sigma, J_{T_2})$, where (F, j) is a compact Riemann surface with boundary, $T_2 \in \mathcal{T}_2$, $\mathbf{p}_0, \dots, \mathbf{p}_3$ are disjoint κ-tuples of boundary marked points of F and $\dot{F} = F \setminus \bigcup_i \mathbf{p}_i$ so that u satisfies the following conditions:

- $\mathrm{d}u \circ j = J_{T_2} \circ \mathrm{d}u$;
- $\pi_{T^* \Sigma} \circ u(z) \in \Psi_j (\pi_{T_2} \circ u(z)) (\bigsqcup_i T_{q_i}^* \Sigma)$ if $\pi_{T_2} \circ u(z) \subset \partial_j T_2$ for $j = 0, 1, 2$;
- each component of $\partial \dot{F}$ that projects to $\partial_j T_2$ maps to a distinct $\Psi_j (\pi_{T_2} \circ u(z))(T_{q_i}^* \Sigma)$ for $j = 0, 1, 2$;
- $\pi_{T^* \Sigma} \circ u(z) \in \Sigma$ if $\pi_{T_2} \circ u(z) \subset \partial_3 T_2$;
- $\pi_{T^* \Sigma} \circ u$ tends to $\mathbf{y}', \mathbf{y}, \mathbf{q}$ as $s_1, s_2, s_3 \to +\infty$;
- $\pi_{T^* \Sigma} \circ u$ tends to a κ-tuple $\mathbf{q}''(u)$ of intersection points of $\Psi_0 (\pi_{T_2} \circ u(z))(\bigsqcup_i T_{q_i}^* \Sigma)$ for $\pi_{T_2} \circ u(z) \subset \partial_0 T_2$, and Σ, in bijection with $\mathbf{q}''$ as $s_0 \to +\infty$;
- $\pi_{T_2} \circ u$ is a κ-fold branched cover of some $T_2 \in \mathcal{T}_2$.

Remark 6.3. All the κ-tuples $\mathbf{q}''(u)$ are close to $\mathbf{q}''$ since $V(t, q)$ is small, and hence there is a family of diffeomorphisms that takes $\mathbf{q}''(u)$ to $\mathbf{q}''$ and is smoothly varying with $\mathbf{q}''(u)$.

Lemma 6.4. *There exists a sufficiently generic consistent collection of almost complex structures such that $\mathcal{H}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ is of dimension* 1 *and is transversely cut out for all $\mathbf{q}$, $\mathbf{y}$, $\mathbf{y}'$ and $\mathbf{q}''$. Moreover, $\mathcal{H}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ admits a compactification $\bar{\mathcal{H}}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ such that its boundary $\partial \bar{\mathcal{H}}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ is of dimension* 0 *and contains discrete broken or nodal curves.*

Proof. Similar to Lemma 6.1, since $|\mathbf{y}| = |\mathbf{y}'| = 0$ and $|\mathbf{q}| = |\mathbf{q}''|$, $\mathcal{H}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ has virtual dimension 0 by (2.2). By standard transversality arguments and Gromov compactness, a 1-parameter family of curves in $\mathcal{H}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ may limit to broken curves by pinching boundaries of T_2 or to nodal curves by letting branch points approach ∂T_2. ∎

We now define a map from $\mathcal{H}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q}) \times [0, 1]$ to the set of subsets of Σ by

$$\gamma(u)(t) = (\pi_{T^* \Sigma} \circ u) \circ (\pi_{T_2} \circ u)^{-1} \circ \tau(t), \tag{6.3}$$

where $\tau\colon [0,1] \to \partial_3 T_2$ parametrizes the boundary arc $\partial_3 T_2$ from p_0 to p_3. Here we are identifying all the $\mathbf{q}''(u)$ with $\mathbf{q}''$ using Remark 6.3.

Let

$$\mathcal{H}_0(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q}) = \{u \in \mathcal{H}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q}) \mid \gamma(u)(t) \text{ contains } \kappa \text{ distinct points of } \Sigma$$
$$\text{and } \gamma(u)(t) \cap \{\star\} = \varnothing \text{ for all } t\}.$$

In a manner similar to the definition of $\mathcal{F}$ in (6.2), we define the evaluation map

$$\mathcal{G}\colon \mathcal{H}_0(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q}) \to \mathrm{H}_\kappa(\Sigma, \mathbf{q}) \otimes_{\mathbb{Z}[\hbar]} \mathbb{Z}[\![\hbar]\!],$$
$$u \mapsto (-1)^{\natural(u)} \cdot c^{2\langle u, \star \rangle} \cdot \hbar^{\kappa - \chi(u)} \cdot [\gamma(u)].$$

Here we are writing $[\gamma] = \mathcal{P}(\gamma)$, where $\mathcal{P}(\gamma)$ is defined by (6.1). As before, $\langle u, \star \rangle :=$ $\langle [\pi_\Sigma(u)]', \star \rangle$, where $[\pi_\Sigma(u)]'$ is as in Section 6.1 with one difference: we replace all the $\alpha_i = \pi_\Sigma \circ u(\partial_i \dot{F})$, *except those that have boundary condition* $u(\partial_i \dot{F}) \subset \Sigma$, by their piecewise V-geodesic replacements. The algebraic intersection number $\langle u, \star \rangle$ is well defined since the boundary of $[\pi_\Sigma(u)]'$ is disjoint from $\star$ when $u \in \mathcal{H}_0(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$.

Proposition 6.5. *The map $\mathcal{F}$ in* (6.2) *is a homomorphism of algebras.*

Proof. It suffices to show that

$$\mathcal{F}(\mu^2(\mathbf{y}, \mathbf{y}')) = \mathcal{F}(\mathbf{y})\mathcal{F}(\mathbf{y}')$$

for any $\mathbf{y}, \mathbf{y}' \in \mathrm{CW}(\bigsqcup_i T^*_{q_i}\Sigma)_c$. We analyze the boundary of the index 1 moduli space $\overline{\mathcal{H}}^\chi(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ for varying χ.

For each $u \in \mathcal{H}^\chi(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$, consider the path $\gamma(u)$ defined by (6.3) as a κ-tuple of paths in Σ. A single generic u belongs to $\mathcal{H}_0{}^\chi(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ and $\mathcal{G}(u)$ is well defined but for a generic 1-parameter family $u_t \in \mathcal{H}^\chi(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$, $t \in [0, 1]$, $\gamma(u_t)$ may intersect $\star$ at some $t \in (0, 1)$. At any rate, $\mathcal{G}(u_0) = \mathcal{G}(u_1)$ since we are taking values in $\mathrm{H}_\kappa(\Sigma, \mathbf{q})$ and the marked point relation compensates for each extra intersection with $\star$. Hence we may ignore the intersections with $\star$ in this proof.

Note that codimension-1 degenerations only occur in the A_∞-base direction, i.e., the projection of curves to the A_∞-base direction leads to codimension-1 degenerations of D_m. The schematic picture is shown in Figures 7 and 8.

There are three types of boundary degenerations:

(1) $\bigsqcup_{\mathbf{y}'', \chi' + \chi'' - \kappa = \chi} \mathcal{M}^{\mathrm{ind}=0, \chi'}(\mathbf{y}', \mathbf{y}, \mathbf{y}'') \times \mathcal{H}^{\mathrm{ind}=0, \chi''}(\mathbf{q}', \mathbf{y}'', \mathbf{q})$;

(2) $\bigsqcup_{\mathbf{q}'', \chi' + \chi'' - \kappa = \chi} \mathcal{H}^{\mathrm{ind}=0, \chi'}(\mathbf{q}'', \mathbf{y}', \mathbf{q}') \times \mathcal{H}^{\mathrm{ind}=0, \chi''}(\mathbf{q}', \mathbf{y}, \mathbf{q})$;

(3) the set $\partial_n \overline{\mathcal{H}}^{\mathrm{ind}=1, \chi}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ with a nodal degeneration along Σ.

Type (1) is given on the left-hand side of Figure 7 and contributes $\mathcal{F}(\mu^2(\mathbf{y}, \mathbf{y}'))$. Type (2) is given on the right-hand side of Figure 7 and contributes $\mathcal{F}(\mathbf{y})\mathcal{F}(\mathbf{y}')$. A standard gluing argument shows that all contributions to $\mathcal{F}(\mu^2(\mathbf{y}, \mathbf{y}'))$ and $\mathcal{F}(\mathbf{y})\mathcal{F}(\mathbf{y}')$ come from such broken degenerations.
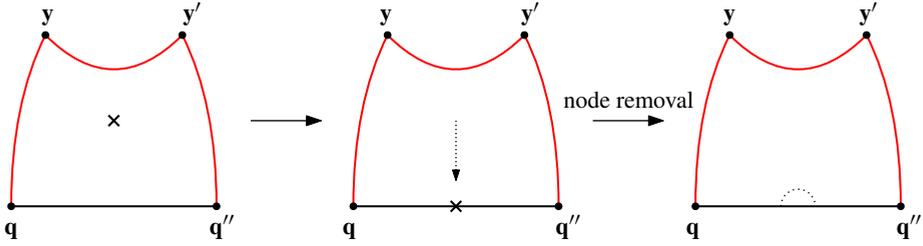
Fig. 8. Nodal degeneration of $\mathcal{H}^{\mathrm{ind}=1,\chi}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ in the T_2 direction: a nodal point on Σ (middle); removal of the nodal point (right).
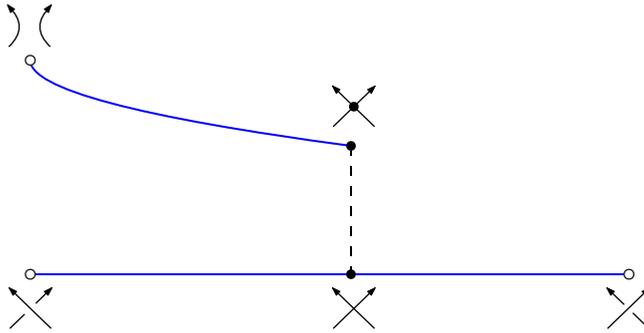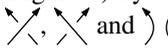

Fig. 9. Representing a curve u by its image $\gamma(u)$ on $[0, 1] \times \Sigma$. The upper arc denotes the moduli space with a nodal degeneration. The lower arc denotes the companion moduli space with χ increased by 1, whose evaluation by $\mathcal{G}$ on $[0, 1] \times \Sigma$ exhibits a crossing of two braid strands.

We now discuss (3), which is given in Figure 8. Let u_s, $s \in [0, 1)$, be a generic 1-parameter family such that $\pi_{T_2} \circ u_s$ has a branch point (generically a double branch point) that limits to ∂T_2 as $s \to 1$. We denote this limiting curve as $u_n \in \partial_n \bar{\mathcal{H}}^{\mathrm{ind}=1,\chi}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$. Moreover, the only component of ∂T_2 that a branch point can approach is $\partial_3 T_2$ (corresponding to the zero section Σ) since all other boundary arcs correspond to disjoint sets of Lagrangians where nodal degenerations cannot occur. By Gromov compactness, $\partial_n \bar{\mathcal{H}}^{\mathrm{ind}=1,\chi}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ is finite. If we continue this family past the nodal curve u_n, then the nodal point is removed and χ increases by 1 as shown on the right-hand side of Figure 8. Interpreted in another way, given $u_n \in \partial_n \bar{\mathcal{H}}^{\mathrm{ind}=1,\chi}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$, there exists a 1-parameter family of curves u'_t, $t \in (-\delta, \delta)$, in $\mathcal{H}^{\mathrm{ind}=1,\chi+1}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ such that u'_0 is obtained from u_n by removing the nodal point and the family $\gamma(u'_t)$ corresponds to a crossing of two paths in $[0, 1] \times \Sigma$. This is illustrated in Figure 9 as a skein relation on $[0, 1] \times \Sigma$.

On the other hand, given a 1-parameter family u'_t, $t \in (-\delta, \delta)$, of curves in the space $\mathcal{H}^{\mathrm{ind}=1,\chi+1}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ such that $\gamma(u'_t)$ exhibits a single crossing of two strands (see the bottom blue line of Figure 9), let u_n be the nodal curve corresponding to u'_0 as in the middle of Figure 8. Suppose the nodal point on the domain of u_n is $p_n \in \dot{F}$. Since

a neighborhood of $u_n(p_n)$ in the ambient symplectic manifold $T_2 \times T^*\Sigma$ is diffeomorphic to $T^*\mathbb{R} \times T^*\mathbb{R}^2 \approx \mathbb{R}^6$, we can construct a preglued curve $\tilde{u}_{1-\varepsilon}$ using a cut-off version of the standard hyperbolic node model of [14, Section 4.1.1], where $\varepsilon \in [0, \varepsilon_0)$ is the pregluing parameter for some small $\varepsilon_0 > 0$ so that $u_n = \tilde{u}_1$. By a standard Newton iteration technique, there exists a unique 1-parameter family of holomorphic curves $u_{1-\varepsilon} \in \bar{\mathcal{H}}^{\text{ind}=1,\chi}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ for each $\varepsilon \in [0, \varepsilon_0)$, where $u_n = u_1$; see [14, Lemma 4.16] for details. Therefore, a small neighborhood of u_n in $\bar{\mathcal{H}}^{\text{ind}=1,\chi}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ is homeomorphic to $[0, \varepsilon_0)$, which corresponds to the upper arc of Figure 9.

Summarizing the above discussion, $\bigsqcup_\chi \bar{\mathcal{H}}^{\text{ind}=1,\chi}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$ is a uni-trivalent graph with trivalent vertices indexed by $u_n \in \partial_n \mathcal{H}^{\text{ind}=1,\chi}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})$. We denote the trivalent neighborhood of u_n (the two blue arcs in Figure 9) by $\mathcal{N}_{u_n}$, where $\partial \bar{\mathcal{N}}_{u_n}$ contains three curves u_+, u_- and u_0 that correspond to $\asymp$, $\times$ and $)($ in Figure 9, respectively. Now

$$\bar{\mathcal{H}}^{\text{ind}=1}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q}) \setminus \bigsqcup_{u_n \in \partial_n \mathcal{H}^{\text{ind}=1}(\mathbf{q}'', \mathbf{y}', \mathbf{y}, \mathbf{q})} \mathcal{N}_{u_n}$$

is a 1-dimensional manifold with boundary. By comparing the count of its boundary curves with different Euler characteristics, we have

$$
\begin{aligned}
&\mathcal{F}(\mathbf{y})\mathcal{F}(\mathbf{y}') - \mathcal{F}(\mu^2(\mathbf{y}, \mathbf{y}')) \\
&= 1 \cdot \Big[\sum_{\chi(u_+)=\kappa} \mathcal{G}(u_+) - \sum_{\chi(u_-)=\kappa} \mathcal{G}(u_-) \Big] \\
&\quad + \hbar \cdot \Big[\sum_{\chi(u_+)=\kappa-1} \mathcal{G}(u_+) - \sum_{\chi(u_-)=\kappa-1} \mathcal{G}(u_-) - \sum_{\chi(u_0)=\kappa-1} \mathcal{G}(u_0) \Big] \\
&\quad + \hbar^2 \cdot \Big[\sum_{\chi(u_+)=\kappa-2} \mathcal{G}(u_+) - \sum_{\chi(u_-)=\kappa-2} \mathcal{G}(u_-) - \sum_{\chi(u_0)=\kappa-2} \mathcal{G}(u_0) \Big] + \cdots . \quad (6.4)
\end{aligned}
$$

The skein relation in Definition 4.4 implies that

$$\mathcal{G}(u_+) - \mathcal{G}(u_-) = \hbar \cdot \mathcal{G}(u_0) \in \mathrm{H}_\kappa(\Sigma, \mathbf{q}) \otimes_{\mathbb{Z}} [\hbar]\mathbb{Z}[\![\hbar]\!].$$

So the right-hand side of (6.4) is zero. This completes the proof. ∎

To show $\mathcal{F}$ is an isomorphism, it suffices to show that $\mathcal{F}$ is a bijection. We use a perturbation argument and start with the case where $\hbar = 0$.

Lemma 6.6. *The restriction of $\mathcal{F}$ to $\hbar = 0$ is an isomorphism*

$$\mathcal{F}|_{\hbar=0}\colon \mathrm{HW}\Big(\bigsqcup_i T^*_{q_i}\Sigma \Big)_c \Big|_{\hbar=0} \to \mathrm{H}_\kappa(\Sigma, \mathbf{q})|_{\hbar=0}.$$

Proof. For simplicity, we write $\mathcal{F}_0 = \mathcal{F}|_{\hbar=0}$ throughout this proof.

We first prove the lemma for $\kappa = 1$. In this case, $\mathrm{H}_1(\Sigma, \mathbf{q})|_{\hbar=0} \cong \mathbb{Z}[\pi_1(\Sigma, q)_c]$. Setting $c = 1$, $\pi_1(\Sigma, q)_c$ is isomorphic to $\pi_1(\Sigma, q)$. In each homotopy class of $\pi_1(\Sigma, q)$, there is a unique generator $y \in \mathrm{HW}(T^*_q\Sigma)$ whose Legendre transform $\mathcal{L}(y)$ represents

this class. By [4, Lemma 5.1], $\mathcal{F}_0|_{c=1}(y) = [\mathcal{L}(y)]$. Hence $\mathcal{F}_0|_{c=1}$ is an isomorphism. Add the parameter c and view $y \in \mathrm{HW}(T_q^* \Sigma)_c$. We have $\mathcal{F}_0(y) = c^d [\mathcal{L}(y)]_c$ for some integer d, where $[\mathcal{L}(y)]_c \in \pi_1(\Sigma, q)_c$ is a lift of $[\mathcal{L}(y)] \in \pi_1(\Sigma, q)$. Hence $\mathcal{F}_0$ is an isomorphism. For later use, we denote the map $\mathcal{F}_0$ for $\kappa = 1$ by

$$\widetilde{\mathcal{F}}_c \colon \mathrm{HW}(T_q^* \Sigma)_c \to \mathbb{Z}[\pi_1(\Sigma, q)_c], \tag{6.5}$$

which is the version of (3.5) with the c-parameter.

We now prove the lemma for $\kappa \geq 1$. The μ^2-operation of $\mathrm{HW}(\bigsqcup_i T_{q_i}^* \Sigma)_c|_{\hbar=0}$ only counts curves with $\chi = \kappa$, i.e., where there are κ trivial pseudoholomorphic disks. Then one can easily compute that $\mathrm{HW}(\bigsqcup_i T_{q_i}^* \Sigma)_c|_{\hbar=0}$ is isomorphic to $(\bigotimes_i \mathrm{HW}(T_{q_i}^* \Sigma)_c) \rtimes S_\kappa$. On the other hand, $\mathrm{H}_\kappa(\Sigma, \mathbf{q})_c|_{\hbar=0}$ degenerates to $(\bigotimes_i \mathbb{Z}[\pi_1(\Sigma, q_i)_c]) \rtimes S_\kappa$ by Lemma 4.6. Here, both tensor products are over $\mathbb{Z}[c^{\pm 1}]$.

Since $\mathrm{HW}(\bigsqcup_i T_{q_i}^* \Sigma)_c|_{\hbar=0}$ is generated by $\bigotimes_i \mathrm{HW}(T_{q_i}^* \Sigma)_c$ and S_κ is an algebra, it suffices to show that

(1) $\mathcal{F}_0|_{\bigotimes_i \mathrm{HW}(T_{q_i}^* \Sigma)_c} = \widetilde{\mathcal{F}}_c^{\otimes \kappa} \colon \bigotimes_i \mathrm{HW}(T_{q_i}^* \Sigma)_c \to \bigotimes_i \mathbb{Z}[\pi_1(\Sigma, q_i)_c]$, where the map $\widetilde{\mathcal{F}}_c$ is in (6.5);

(2) $\mathcal{F}_0|_{S_\kappa} = \mathrm{id} \colon S_\kappa \to S_\kappa$.

By Theorem 3.4, Remark 3.5 and [4, Lemma 5.1], $\widetilde{\mathcal{F}}_c$ maps the time-1 Hamiltonian flow to the homotopy class of its Legendre transform. The first equation follows since $\mathcal{F}_0|_{\bigotimes_i \mathrm{HW}(T_{q_i}^* \Sigma)_c}$ maps κ time-1 Hamiltonian flows to their Legendre transforms.

The symmetric group S_κ is generated by transpositions $\sigma_i = (i, i+1)$. Let $\mathbf{y}_i = \{y_{i1}, \ldots, y_{i\kappa}\} \in \mathrm{HW}(\bigsqcup_i T_{q_i}^* \Sigma)_c|_{\hbar=0}$ be the corresponding generator in Floer homology, where $y_{ij} \in \mathrm{CF}(\phi_{H_V}^1(T_{q_j}^* \Sigma), T_{q_{\sigma_i(j)}}^* \Sigma)$. Since $\hbar = 0$, the map $\mathcal{F}_0(\mathbf{y}_i)$ counts curves with $\chi = \kappa$, that is, κ holomorphic disks, where each disk limits to y_{ij} at the positive puncture and evaluates on Σ as a path from q_j to $q_{\sigma_i(j)}$, for $j = 1, \ldots, \kappa$.

Note that each homotopy class of paths from q_j to $q_{\sigma_i(j)}$ contains a unique V-perturbed geodesic, i.e., $\mathcal{L}(y_{ij})$ (see Definition 3.2). By [4], there is a unique holomorphic disk which limits to y_{ij} at the positive puncture and evaluates on Σ as a path γ_j from q_j to $q_{\sigma_i(j)}$ on the stable manifold of $\mathcal{L}(y_{ij})$. However, since the action of y_{ij} equals that of $\mathcal{L}(y_{ij})$, we see $\gamma_j = \mathcal{L}(y_{ij})$.

Therefore, the evaluation of the boundary of the unique κ-tuple of disks on the zero section $\Sigma \subset T^* \Sigma$ gives a based loop in $\mathrm{UConf}_\kappa(\Sigma, \mathbf{q})$, which consists of two short paths $\mathcal{L}(y_{i,i+1})$ and $\mathcal{L}(y_{i+1,i})$, and trivial paths from q_j to q_j for $j \neq i, i+1$. This loop gives a class in $\mathrm{H}_\kappa(\Sigma, \mathbf{q})_c|_{\hbar=0} \cong (\bigotimes_i \mathbb{Z}[\pi_1(\Sigma, q_i)_c]) \rtimes S_\kappa$ corresponding to $\sigma_i \in S_\kappa$. Hence the second equation follows. ∎

Proof of Theorem 1.5. *Injectivity of* $\mathcal{F}$. Suppose that there exists $\mathbf{a} \neq 0$ such that $\mathcal{F}(\mathbf{a}) = 0$. Let $\mathbf{a} = \sum_{i \geq 0} \hbar^i \mathbf{a}_i$, where $\mathbf{a}_i \in \mathrm{CW}(\bigsqcup_i T_{q_i}^* \Sigma)_c|_{\hbar=0}$. Since $\mathrm{H}_\kappa(\Sigma, \mathbf{q})$ has no $\hbar$-torsion, we may assume that $\mathbf{a}_0 \neq 0$. Setting $\hbar = 0$, we have $\mathcal{F}(\mathbf{a}_0) = \mathcal{F}(\mathbf{a}) = 0$. Therefore, $\mathcal{F}|_{\hbar=0}(\mathbf{a}_0) = 0$, which means $\mathbf{a}_0 = 0$ since $\mathcal{F}|_{\hbar=0}$ is an isomorphism. This leads to contradiction. Thus $\mathcal{F}$ is injective.

Surjectivity of $\mathcal{F}$. It suffices to show that any $\mathbf{b} \in \mathrm{H}_\kappa(\Sigma, \mathbf{q})$ is in $\mathrm{Im}\,\mathcal{F}$. Since $\mathcal{F}|_{\hbar=0}$ is an isomorphism, there exists $\mathbf{a}_0 \in \mathrm{CW}(\bigsqcup_i T^*_{q_i}\Sigma)_c|_{\hbar=0}$ such that $\mathcal{F}(\mathbf{a}_0) \equiv \mathbf{b}$ (mod $\hbar$). Let

$$\mathbf{b}_1 = \left.\frac{(\mathbf{b} - \mathcal{F}(\mathbf{a}_0))}{\hbar}\right|_{\hbar=0}.$$

Then there exists $\mathbf{a}_1 \in \mathrm{CW}(\bigsqcup_i T^*_{q_i}\Sigma)_c|_{\hbar=0}$ such that $\mathcal{F}(\mathbf{a}_1) \equiv \mathbf{b}_1$ (mod $\hbar$). Repeating this procedure, we get $\mathcal{F}(\sum_{i\geq 0}\mathbf{a}_i\hbar^i) = \mathbf{b}$. Hence $\mathcal{F}$ is surjective. ∎

7. Surfaces with punctures

In this section, we show that Theorem 1.5 still holds for $\overset{\circ}{\Sigma}$, which is obtained from a closed oriented surface of genus $g \geq 0$ by removing a finite number (> 0) of punctures. For simplicity we assume that $c = 1$.

In this case, the wrapped Floer homology of the cotangent fibers and the $\mathcal{F}$ map of (6.2) can be defined similarly but need modifications near the punctures. The main issue is the noncompactness of the moduli space of holomorphic curves: if the wrapped Lagrangians on $T^*\overset{\circ}{\Sigma}$ approach the punctures when projected to $\overset{\circ}{\Sigma}$, then a sequence of curves bounded by those wrapped Lagrangians projected to $\overset{\circ}{\Sigma}$ may also approach the punctures.

To remedy this, we confine the wrapped Lagrangians and all involved holomorphic curves to stay over a compact subset of $\overset{\circ}{\Sigma}$. Our approach is a simple application of the partially wrapped Fukaya category by Sylvan [32] and its further development by Ganatra, Pardon and Shende [17, 18]. A similar approach in the context of sutured contact manifolds is due to Colin, Ghiggini, Honda and Hutchings [11].

Let g be the standard flat metric on $\overset{\circ}{\Sigma}$ if $\overset{\circ}{\Sigma}$ is homeomorphic to $\mathbb{R}^2$ or $\mathbb{R} \times S^1$. Otherwise, let g be a complete finite-volume hyperbolic metric on $\overset{\circ}{\Sigma}$, i.e., all the punctures of $\overset{\circ}{\Sigma}$ correspond to cusps.

Suppose Σ is a closed oriented surface and $z \in \Sigma$. We consider the once punctured surface $\overset{\circ}{\Sigma} = \Sigma \setminus \{z\}$ (the case of more than one puncture is similar).

Let $\mathbf{q} = \{q_1, \ldots, q_\kappa\} \subset \overset{\circ}{\Sigma}$ be a κ-tuple of points. Pick an end

$$\mathcal{N} \approx (-\infty, 1)_s \times S^1_\theta$$

near the puncture of $\overset{\circ}{\Sigma}$ so that $\{q_1, \ldots, q_\kappa\} \subset \overset{\circ}{\Sigma} \setminus \mathcal{N}$. We have a trivialization of $T\mathcal{N}$ by $\{\frac{\partial}{\partial s}, \frac{\partial}{\partial \theta}\}$. See Figure 10. Let (p_s, p_θ) be the dual coordinates to (s, θ) so that we have a trivialization of $T^*\mathcal{N}$ by $\{\frac{\partial}{\partial p_s}, \frac{\partial}{\partial p_\theta}\}$. The canonical symplectic form ω on $\overset{\circ}{\Sigma}$ restricts to $\omega = \mathrm{d}s \wedge \mathrm{d}p_s + \mathrm{d}\theta \wedge \mathrm{d}p_\theta$ on $T^*\mathcal{N}$. We fix the trivial almost complex structure $J_\mathcal{N}$ on $T^*\mathcal{N}$ so that $J_\mathcal{N}(\frac{\partial}{\partial s}) = \frac{\partial}{\partial p_s}$ and $J_\mathcal{N}(\frac{\partial}{\partial \theta}) = \frac{\partial}{\partial p_\theta}$. Then there is a $(J_\mathcal{N}, j)$-holomorphic map

$$\pi_s \colon T^*\mathcal{N} \to \mathbb{C}_{\mathrm{Re}<1}, \quad (s, \theta, p_s, p_\theta) \mapsto s + jp_s,$$

where j is the standard complex structure on $\mathbb{C}_{\mathrm{Re}<1}$.
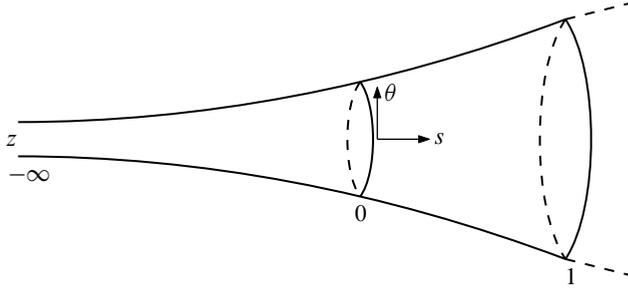
Fig. 10. The end $\mathcal{N}$ near the puncture of $\overset{\circ}{\Sigma}$.

Fix a diffeomorphism $f: (0,1)_s \to (-\infty, 1)_s$ such that $f' > 0$, $f'' \leq 0$, and $f(s) = s$ for $s \geq \frac{1}{2}$. It induces the diffeomorphism $\mathcal{N} \cap \{s > 0\} \overset{\sim}{\to} \mathcal{N}$, $(s, \theta) \mapsto (f(s), \theta)$. Extending by the identity, we obtain a diffeomorphism

$$\widetilde{f}: \overset{\circ}{\Sigma}_0 := \overset{\circ}{\Sigma} \setminus \{s \leq 0\} \to \overset{\circ}{\Sigma}.$$

The pullback metric $\widetilde{f}^* g$ on $\overset{\circ}{\Sigma}_0$ induces a norm $|\cdot|_f$ on $T^* \overset{\circ}{\Sigma}_0$. Choose a time-dependent Hamiltonian $H_{V,f}: [0,1] \times T^* \overset{\circ}{\Sigma}_0 \to \mathbb{R}$ with $|p|_f$ instead of $|p|$ in equation (2.4), where V has compact support in $\overset{\circ}{\Sigma} \setminus \mathcal{N}$ and has small $W^{1,2}$-norm on $[0,1] \times \overset{\circ}{\Sigma}$; $X_{H_{V,f}}$ and $\phi^t_{H_{V,f}}$ are as before with respect to $H_{V,f}$. Note that the wrapped Lagrangians

$$\phi^t_{H_V} \left(\bigsqcup_i T^*_{q_i} \overset{\circ}{\Sigma} \right) \subset \overset{\circ}{\Sigma}_0 \subset \overset{\circ}{\Sigma}$$

for all $t \geq 0$, and hence cannot cross $\{s = 0\}$.

We follow the notation of Section 2.2. As in Definition 2.5, we define the following.

Definition 7.1. The wrapped Heegaard Floer cochain complex of $\mathrm{CW}(\bigsqcup_i T^*_{q_i} \overset{\circ}{\Sigma})$ is $\mathrm{CF}(\phi^1_{H_V} (\bigsqcup_i T^*_{q_i} \overset{\circ}{\Sigma}), \bigsqcup_i T^*_{q_i} \overset{\circ}{\Sigma})$.

Choose sufficiently generic consistent collection of almost complex structures $\{J_{D_m}\}$ satisfying (J1)–(J3) as before and apply the rescaling argument of Section 2.2 to $\{J_{D_m}\}$ to get $\{J''_{D_m}\}$. We also define $\widetilde{L}''_j(D_m)$ as before.

Given $\mathbf{y}_1, \ldots, \mathbf{y}_m \in \mathrm{CW}(\bigsqcup_i T^*_{q_i} \overset{\circ}{\Sigma})$, let $\mathcal{M}(\mathbf{y}_1, \ldots, \mathbf{y}_m, \mathbf{y}_0)$ be the moduli space of maps

$$u: (\dot{F}, j) \to (D_m \times T^* \overset{\circ}{\Sigma}, J_{D_m}),$$

where (F, j) is a compact Riemann surface with boundary and u satisfies the conditions similar to (1)–(5) in Section 2.2.

It is easy to check that all the conclusions in Section 2.2 still hold except for Lemma 2.6 and the A_∞-relation. However, we claim that the standard proof of A_∞-relation (see [12, Proposition 4.0.3]) works by showing the following.

Lemma 7.2. *Fix a sufficiently generic consistent choice of almost complex structures as above. Suppose that d is equal to 0 or 1. Given generators $\mathbf{y}_1, \ldots, \mathbf{y}_m \in \mathrm{CW}(\bigsqcup_i T^*_{q_i} \overset{\circ}{\Sigma})$,*

$\mathcal{M}^{\text{ind}=d,\chi}(\mathbf{y}_1, \ldots, \mathbf{y}_m, \mathbf{y}_0)$ *is empty for all but finitely many* $\mathbf{y}_0$. *When it is nonempty,* $\mathcal{M}^{\text{ind}=d,\chi}(\mathbf{y}_1, \ldots, \mathbf{y}_m, \mathbf{y}_0)$ *(and* $\mathcal{M}^{\text{ind}=d,\chi}(\mathbf{y}_1, \mathbf{y}_0)/\mathbb{R}$ *if* $m = 1$*) admits a compactification for each Euler characteristic* χ.

Proof. We follow [17, Lemma 2.41]. Let $\pi \colon T^*\mathring{\Sigma} \to \mathring{\Sigma}$ be the projection. For each $u \in \mathcal{M}(\mathbf{y}_1, \ldots, \mathbf{y}_m, \mathbf{y}_0)$, we show that $\pi \circ u(\dot{F}) \cap \{s < 0\} = \varnothing$.

Consider the holomorphic map

$$\pi_s \circ u \colon u^{-1}(\pi_s^{-1}(\mathbb{C}_{\text{Re}<0})) \to \mathbb{C}_{\text{Re}<0}.$$

By definition, $\pi \circ u(\partial \dot{F}) \cap \{s \le 0\} = \varnothing$, hence $u^{-1}(\pi_s^{-1}(\mathbb{C}_{\text{Re}<0})) \subset F \setminus \partial F$ is an open subset of the interior of F. Therefore, $K := (\pi_s \circ u)(u^{-1}(\pi_s^{-1}(\mathbb{C}_{\text{Re}<0})))$ is an open subset of $\mathbb{C}_{\text{Re}<0}$ by the open mapping theorem.

On the other hand, $u^{-1}(\pi_s^{-1}(\mathbb{C}_{\text{Re}\le0})) \subset F \setminus \partial F$ is a compact subset of the interior of F. Therefore, the function

$$\text{Re} \circ \pi_s \circ u \colon u^{-1}(\pi_s^{-1}(\mathbb{C}_{\text{Re}\le0})) \to \mathbb{R}$$

attains its minimum on $u^{-1}(\pi_s^{-1}(\mathbb{C}_{\text{Re}\le0}))$. Since K is open, this is only possible if $K = \varnothing$. Hence $u^{-1}(\pi_s^{-1}(\mathbb{C}_{\text{Re}<0})) = \varnothing$ and then $\pi \circ u(\dot{F}) \cap \{s < 0\} = \varnothing$.

We have shown how to prevent curves from crossing the vertical boundary of $T^*\mathring{\Sigma}_0$. The remaining proof is the same as that of Lemma 2.6. ∎

Hence $\text{CW}(\bigsqcup_i T^*_{q_i}\mathring{\Sigma})$ is a well-defined ordinary algebra supported in degree 0.

To modify Section 3, we consider paths in $\mathring{\Sigma}_0$ instead of Σ. We now use the metric $\tilde{f}^*g$. The Legendre transform $L_{f,v}$ is with $|v|_f$ instead of $|v|$ in (3.1), where $|\cdot|_f$ is the norm on $T\mathring{\Sigma}_0$ induced by $\tilde{f}^*g$. Given $q_0, q_1 \in \mathring{\Sigma}_0 \setminus \mathcal{N}$, the Lagrangian action functional $\mathcal{A}_{V,f}$ is defined as in (3.2) with L_V replaced by $L_{V,f}$. By the choice of the metric $\tilde{f}^*g$, it is easy to see that

(1) no V-perturbed geodesics can exit $\mathring{\Sigma}_0$;

(2) the induced negative gradient flow of $\mathcal{A}_V$ always stays inside a compact region of $\mathring{\Sigma}_0$.

Since $\tilde{f} \colon (\mathring{\Sigma}_0, \tilde{f}^*g) \to (\mathring{\Sigma}, g)$ is an isometry, we have the following.

Lemma 7.3. *The Morse homology* $\text{HM}_*(\Omega^{1,2}(\mathring{\Sigma}_0, q_0, q_1))$ *induced by the metric* $\tilde{f}^*g$ *is well defined and is isomorphic to* $\text{HM}_*(\Omega^{1,2}(\mathring{\Sigma}, q_0, q_1))$ *induced by the metric* g.

It is easy to verify that all the definitions and conclusions in Section 4 hold for $\mathring{\Sigma}_0$. Hence the Hecke algebra $H_\kappa(\mathring{\Sigma}_0, \mathbf{q})$ is well defined.

Next we consider the modification of the map $\mathcal{F}$ defined by (6.2). We use the notation from Section 6 with the following modifications:

(1) H_V is replaced by $H_{V,f}$;

(2) choose a sufficiently generic consistent collection $T_{m-1} \mapsto J_{T_{m-1}}$ of compatible almost complex structures on $T_{m-1} \times T^*\mathring{\Sigma}$ for $T_{m-1} \in \mathcal{T}_{m-1}$ and all $m \ge 2$, where $J_{T^*\mathring{\Sigma}}$ coincides with $j_m \times J_\mathcal{N}$ on $T_{m-1} \times T^*\mathcal{N}$.

All of Section 6.1 carries over with the exception of Lemma 6.2. Recall that we denote the set of intersection points between $\bigsqcup_i T_{q_i}^* \overset{\circ}{\Sigma}$ (resp. $\phi_{H_V}^1 (\bigsqcup_i T_{q_i}^* \overset{\circ}{\Sigma})$) and $\overset{\circ}{\Sigma}$ by $\mathbf{q}$ (resp. $\mathbf{q}'$). Suppose $\mathbf{y} \in \mathrm{CF}(\phi_{H_V}^1 (\bigsqcup_i T_{q_i}^* \overset{\circ}{\Sigma}), \bigsqcup_i T_{q_i}^* \overset{\circ}{\Sigma})$. We show the following.

Lemma 7.4. *The moduli space $\mathcal{H}^\chi(\mathbf{q}', \mathbf{y}, \mathbf{q})$ admits a compactification for each Euler characteristic χ.*

Proof. This is similar to the proof of Lemma 7.2. Let $\pi \colon T^* \overset{\circ}{\Sigma} \to \overset{\circ}{\Sigma}$ be the projection. For each $u \in \mathcal{H}^\chi(\mathbf{q}', \mathbf{y}, \mathbf{q})$, consider the holomorphic map

$$\pi_s \circ u \colon u^{-1}(\pi_s^{-1}(\mathbb{C}_{\mathrm{Re}<0})) \to \mathbb{C}_{\mathrm{Re}<0}.$$

By definition, $u(\partial \dot{F}) \cap T^* \mathcal{N}|_{s<0}$ is a subset of the zero section $\mathcal{N}|_{s<0}$. Since

$$u^{-1}(\pi_s^{-1}(\mathbb{C}_{\mathrm{Re}\leq 0})) \subset F \setminus \partial F$$

is an compact closed subset of F, its image $P := (\pi_s \circ u)(u^{-1}(\pi_s^{-1}(\mathbb{C}_{\mathrm{Re}\leq 0}))) \subset \mathbb{C}$ is also compact.

Note that $(\pi_s \circ u)(u^{-1}(\pi_s^{-1}(\mathbb{C}_{\mathrm{Re}<0, \mathrm{Im}\neq 0}))) \subset \mathbb{C}$ is open by the open mapping theorem. As a result, $\partial P \cap \mathbb{C}_{\mathrm{Re}<0, \mathrm{Im}\neq 0} = \varnothing$, hence $P \subset \mathbb{C}_{\mathrm{Re}=0} \cup \mathbb{C}_{\mathrm{Im}=0}$. This implies $u^{-1}(\pi_s^{-1}(\mathbb{C}_{\mathrm{Re}\leq 0})) \subset \partial F$, which is only possible if $P \subset \mathbb{C}_{\mathrm{Re}=0}$. Therefore, we conclude that $\pi \circ u(\dot{F}) \cap \{s < 0\} = \varnothing$.

We have shown that curves in $\mathcal{H}^\chi(\mathbf{q}', \mathbf{y}, \mathbf{q})$ cannot cross the vertical boundary of $T^* \overset{\circ}{\Sigma}_0$. The remaining proof is the same as that of Lemma 6.2. ∎

The parallel modification of Lemma 6.4 and its proof are similar. Also note that Proposition 6.5 still holds. Therefore,

$$\mathcal{F} \colon \mathrm{CW}\Big(\bigsqcup_i T_{q_i}^* \overset{\circ}{\Sigma}\Big) \to \mathrm{H}_\kappa(\overset{\circ}{\Sigma}_0, \mathbf{q})|_{c=1} \otimes_{\mathbb{Z}[\hbar]} \mathbb{Z}[\![\hbar]\!] \cong \mathrm{H}_\kappa(\overset{\circ}{\Sigma}, \mathbf{q})|_{c=1} \otimes_{\mathbb{Z}[\hbar]} \mathbb{Z}[\![\hbar]\!]$$

is well defined.

To modify the proof of Theorem 1.5, it suffices to modify the proof of Lemma 6.6 when $\kappa = 1$. Note that $\hbar = 0$ is automatically satisfied in this case.

Lemma 7.5. *When $\kappa = 1$, the map $\mathcal{F}$ is an isomorphism*

$$\mathcal{F} \colon \mathrm{HW}(T_q^* \overset{\circ}{\Sigma}) \to \mathrm{H}_1(\overset{\circ}{\Sigma}_0, q)|_{c=1} \cong \mathrm{H}_1(\overset{\circ}{\Sigma}, q)|_{c=1}.$$

Proof. This is essentially [4, Lemma 5.1], replacing Σ by $\overset{\circ}{\Sigma}$.

Recall that Abbondandolo and Schwarz [1, Theorem 3.1] constructed a chain isomorphism

$$\Theta \colon \mathrm{CM}_*(\Omega^{1,2}(\Sigma, q)) \to \mathrm{CW}(T_q^* \Sigma)$$

by a specific moduli space $\mathcal{M}_\Omega^+$ of holomorphic curves of index 0 (see [1, p. 35]). Abouzaid then showed that $\mathcal{F}$ is a homotopy inverse of Θ by constructing another moduli space $\mathcal{C}$ of curves of index 1 (see [4, p. 33]).

In the case of $\overset{\circ}{\Sigma}$, we define the moduli spaces $\mathcal{M}_\Omega^+$ and $\mathcal{C}$ in a similar manner. Again, it suffices to show that no curves in $\mathcal{M}_\Omega^+$ or $\mathcal{C}$ can cross the vertical boundary of $T^* \overset{\circ}{\Sigma}_0$. We omit the details which are similar to the proofs of Lemmas 7.2 and 7.4. ∎

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