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A prismatic approach to $(\varphi, \hat{G})$-modules and F-crystals

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Abstract. We give a new construction of $(\varphi, \hat{G})$-modules using the theory of prisms developed by Bhatt and Scholze. We give two applications of our results. Firstly, we provide a new proof for the equivalence between the category of prismatic F-crystals in finite locally free $\mathcal{O}_{\Delta}$-modules over $(\mathcal{O}_K)_{\Delta}$ and the category of lattices in crystalline representations of G_K, where K is a complete discretely valued field of mixed characteristic with perfect residue field. Moreover, we generalize this result to semistable representations using the absolute logarithmic prismatic site defined by Koshikawa.

Keywords: integral p-adic Hodge theory, semistable representation, prismatic F-crystals.

Contents

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1. Introduction

1.1. Overview and main results

Let K be a complete discretely valued field of mixed characteristics with perfect residue field k. Fix a separable closure $\overline{K}$ of K and let G_K be the absolute Galois group of K. The study of stable lattices in crystalline representations of G_K plays an important role in number theory. For example, in many modularity lifting results, one wants to understand liftings of mod p representations of the Galois group of a number field F to Galois representations over $\mathbb{Z}_p$-lattices with nice properties when restricted to the Galois groups of F_v for all places v of F. And a reasonable property at places over p is that the representation of the Galois group of the local field is crystalline. There are various theories about characterizing G_K-stable lattices in crystalline representations, for example, the theory of strongly divisible lattices of Breuil [10], Wach modules [3, 34], Kisin modules [25], Kisin–Ren's theory [26] and the theory of $(\varphi, \hat{G})$-modules [30]. The theories above state that one can describe lattices in crystalline representations using linear algebraic data over certain commutative rings A.

A recent work of Bhatt–Scholze [9] gives a different characterization of the category of lattices in crystalline representations. To explain their result, let $\mathcal{O}_K$ be the ring of integers in K. They consider the absolute prismatic site $(\mathcal{O}_K)_{\Delta}$, which is defined as the opposite category of all bounded prisms over $\mathcal{O}_K$ and equipped with the faithfully flat topology. Let $\mathcal{O}_{\Delta}$ be the structure sheaf on $(\mathcal{O}_K)_{\Delta}$, and $\mathcal{I}_{\Delta} \subset \mathcal{O}_{\Delta}$ be the ideal sheaf of Hodge–Tate divisor, then $\mathcal{O}_{\Delta}$ carries a φ-action coming from the δ-structures. A prismatic F-crystal in finite locally free $\mathcal{O}_{\Delta}$-modules over $(\mathcal{O}_K)_{\Delta}$ is defined as a crystal $\mathfrak{M}_{\Delta}$ over $(\mathcal{O}_K)_{\Delta}$ in finite locally free $\mathcal{O}_{\Delta}$-modules together with an isomorphism $(\varphi^*\mathfrak{M}_{\Delta})[1/\mathcal{I}_{\Delta}] \simeq \mathfrak{M}_{\Delta}[1/\mathcal{I}_{\Delta}]$. The main result in [9] is the following:

Theorem 1.1.1 ([9, Theorem 1.2] and Theorem 4.1.10). *There is an equivalence of the category of prismatic F-crystals in finite locally free $\mathcal{O}_{\Delta}$-modules over $(\mathcal{O}_K)_{\Delta}$ and the category of Galois stable lattices in crystalline representations of G_K.*

It is known that the prismatic theory of Bhatt–Scholze was first developed to give a new cohomology theory in [8] that behaves like the "p-adic motivic cohomology" for varieties X over $\mathbb{Q}_p$. More concretely, prismatic cohomology theory unifies a lot of p-adic cohomology theories in the sense that one can get various cohomology theories using "evaluation maps" [8, Example 1.9]. And such a procedure of evaluation can be seen more clearly in the absolute prismatic cohomology theory developed in [7]. Theorem 1.1.1 should also be regarded as a "unified"-integral p-adic Hodge theory for crystalline representations. That is one should be able to recover classical integral p-adic Hodge theories from the result of Bhatt–Scholze using evaluation maps. If a classical integral p-adic Hodge theory over $\mathcal{O}_K$ is defined using linear algebraic data over a commutative ring A, then one should first realize A as certain prisms (A, I) over $\mathcal{O}_K$, then expect that evaluating the prismatic F-crystals on (A, I) recovers the corresponding theory. For example, Kisin [25] uses the base ring $A = \mathfrak{S} := W(k)[\![u]\!]$ with $\delta(u) = 0$, and he needs to fix a uniformizer ϖ of $\mathcal{O}_K$ which is a zero of an Eisenstein polynomial $E \in W(k)[u]$. Then it is well-known that $(A, (E))$ is the so-called Breuil–Kisin prism, and $(A, (E))$ is inside $(\mathcal{O}_K)_{\Delta}$. Kisin was able to attach to any lattice T in a crystalline representation of G_K a finite free A-module together with an isomorphism $(\varphi^*\mathfrak{M})[\frac{1}{E}] \simeq \mathfrak{M}[\frac{1}{E}]$. Now, if $\mathfrak{M}_{\Delta}$ is the prismatic F-crystal attached to T under Theorem 1.1.1, then Bhatt–Scholze show that the evaluation of $\mathfrak{M}_{\Delta}$ on $(A, (E))$ recovers Kisin's theory [9, Theorem 1.3].

Prismatic $(\varphi, \hat{G})$-modules. Figure 1 shows an expectation of the relation between the prismatic result of Bhatt–Scholze and other works on characterizing lattices in crystalline representations. The first question that we consider here is whether and how one can recover the theory of $(\varphi, \hat{G})$-modules from Theorem 1.1.1. The category of $(\varphi, \hat{G})$-modules, developed by Liu [29], roughly speaking consists of pairs $((\mathfrak{M}, \varphi_{\mathfrak{M}}), \hat{G})$, where $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ is a Kisin module, and $\hat{G}$ is a G_K-action on $\mathfrak{M} \otimes_{\mathfrak{S},\varphi} \widehat{\mathcal{R}}$ that commutes with $\varphi_{\mathfrak{M}}$ and has some additional properties. Here $\widehat{\mathcal{R}}$ is a subring of A_{inf} that is stable under φ and G_K, where $A_{\mathrm{inf}} = W(\mathcal{O}^{\flat}_{\widehat{\overline{K}}})$ is the infinitesimal period ring introduced by Fontaine. However, the period ring $\widehat{\mathcal{R}}$ introduced by Liu is not known to be p-adically complete, and it is even harder to determine whether it can appear as a prism. So in order to relate the category of $(\varphi, \hat{G})$-modules to the category of prismatic F-crystals of Bhatt–Scholze, we develop a theory of prismatic $(\varphi, \hat{G})$-modules, in which the ring $\widehat{\mathcal{R}}$ is replaced by $A^{(2)}_{\mathrm{st}}$, a subring of A_{inf} constructed as a certain prismatic envelope in Section 2.3.

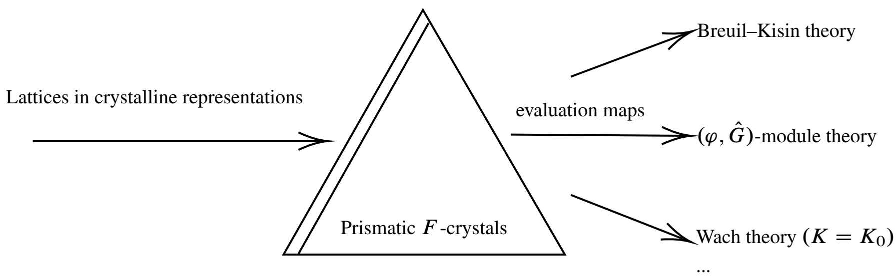

Fig. 1. A cartoon that shows the expectations that prismatic F-crystals will reproduce many classical integral p-adic Hodge theories. And our prismatic $(\varphi, \hat{G})$-module theory can be regarded as reverse engineering of this procedure.

We have the following result.

Theorem 1.1.2 (Theorem 5.0.18). *There is an equivalence between the category of prismatic $(\varphi, \hat{G})$-modules and the category of lattices in semistable representations of G_K. Moreover, we have a necessary and sufficient condition on a prismatic $(\varphi, \hat{G})$-module to determine whether it corresponds to a lattice in crystalline representations of G_K.*

Here a prismatic $(\varphi, \hat{G})$-module, defined similarly to the classical theory, is a pair $((\mathfrak{M}, \varphi_{\mathfrak{M}}), \hat{G})$ consisting of a Kisin module $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ and a G_K-action $\hat{G}$ on $\mathfrak{M} \otimes_{\mathfrak{S}} A^{(2)}_{\mathrm{st}}$ satisfying several additional conditions. The ring $A^{(2)}_{\mathrm{st}}$ here indeed comes from a prism $(A^{(2)}_{\mathrm{st}}, (E))$ inside $(\mathcal{O}_K)_{\Delta}$ that lies over $(A, (E))$, and $(A^{(2)}_{\mathrm{st}}, (E))$ carries an action of

G_K inside $(\mathcal{O}_K)_{\Delta}$. For a G_K-stable lattice T in a crystalline representation, if $\mathfrak{M}_{\Delta}$ is the prismatic F-crystal attached to T, then evaluating $\mathfrak{M}_{\Delta}$ on the diagram $(A,(E)) \to (A^{(2)}_{\mathrm{st}},(E))$ recovers the prismatic $(\varphi,\hat{G})$-module attached to T. We can also show that the map $A^{(2)}_{\mathrm{st}} \to A_{\mathrm{inf}} \xrightarrow{\varphi} A_{\mathrm{inf}}$ factors through $\widehat{\mathcal{R}}$, so the theory of prismatic $(\varphi,\hat{G})$-modules recovers the classical theory. The ring $A^{(2)}_{\mathrm{st}}$ is simpler than $\widehat{\mathcal{R}}$ in many ways. Although it is still very complicated and non-Noetherian, it is p-adic complete and we can give an explicit description of $A^{(2)}_{\mathrm{st}}$ modulo E. In particular, our new theory can be used to fix the gap in [29] indicated by [19, Appendix B].

Another benefit of having the period rings $A^{(2)}$ and $A^{(2)}_{\mathrm{st}}$ in the theory of $(\varphi,\hat{G})$-modules in the absolute prismatic site is the possibility to establish the concept of "$(\varphi,\hat{G})$-module cohomology theory". Specifically, given a smooth formal scheme $\mathfrak{X}$ over $\mathcal{O}_K$, Bhatt–Scholze have demonstrated in [8, Theorem 1.8] that the Breuil–Kisin cohomology $R\Gamma_{\mathfrak{S}}(\mathfrak{X})$ previously studied in [6] can be realized by prismatic cohomology $R\Gamma_{\Delta}(\mathfrak{X}/(\mathfrak{S},(E)))$. This provides a geometric interpretation of the Breuil–Kisin modules associated with the p-adic étale cohomology of the adic generic fiber of $\mathfrak{X}$. Additionally, by making use of the base change property of prismatic cohomology, we can establish that $R\Gamma_{\Delta}(\mathfrak{X}/(A^{(2)}_{\mathrm{st}},(E)))$ yields similarly a geometric interpretation of the $(\varphi,\hat{G})$-modules associated with the p-adic étale cohomology of the adic generic fiber of $\mathfrak{X}$. It is worth noting that $R\Gamma_{\Delta}(\mathfrak{X}/(A^{(2)}_{\mathrm{st}},(E)))$ not only possesses a Frobenius structure, but also admits an action of the full Galois group G_K. We anticipate that this observation will prove useful in studying integral p-adic cohomology theories.

By proving Theorem 1.1.2, we actually provide reverse engineering of the procedure described in Figure 1. That is, using the known equivalence between lattices in semistable representations and prismatic $(\varphi,\hat{G})$-modules, we can establish a functor from the category of prismatic $(\varphi,\hat{G})$-modules that correspond to crystalline representations to prismatic F-crystals. Moreover, we show that this functor is an equivalence, thus giving a different proof of the result of Bhatt–Scholze stated in Theorem 1.1.1.

To be more precise, let T be a G_K-stable lattice in a crystalline representation with positive Hodge–Tate weights, let (A,E) be the Breuil–Kisin prism, and let $(A^{(2)},(E))$ (resp. $(A^{(3)},(E))$) be the self-product (resp. self-triple-product) of $(A,(E))$ in $(\mathcal{O}_K)_{\Delta}$. It is known that evaluating prismatic F-crystals on the diagram $(A,(E)) \xrightarrow{i_1} (A^{(2)},(E)) \xleftarrow{i_2} (A,(E))$ induces an equivalence of the category of prismatic F-crystals and the category of Kisin modules with descent data, that is, pairs $((\mathfrak{M},\varphi_{\mathfrak{M}}),f)$ where $(\mathfrak{M},\varphi_{\mathfrak{M}})$ is a Kisin module and

$$f : \mathfrak{M} \otimes_{\mathfrak{S},i_1} A^{(2)} \simeq \mathfrak{M} \otimes_{\mathfrak{S},i_2} A^{(2)}$$

is an isomorphism of $A^{(2)}$-modules that is compatible with φ and satisfies the cocycle condition over $A^{(3)}$. Using this, to establish an equivalence between prismatic $(\varphi,\hat{G})$-modules that correspond to crystalline representations and prismatic F-crystals, it remains to find a correspondence between the $\hat{G}$-action and the descent isomorphism f.

We will show that the descent isomorphism can be obtained by looking at the G_K-action of the $(\varphi,\hat{G})$-module at a specific element. To be more precise, fix a Kummer tower $K_\infty = \bigcup_{n=1}^{\infty} K(\varpi_n)$ used in the theory of Kisin, where $\{\varpi_n\}_{n\geq 0}$ is a compatible system of p^n-th roots of $\varpi_0 = \varpi$. Choose $\tilde{\tau} \in G_K$ satisfying $\tilde{\tau}(\varpi_n) = \zeta_{p^n}\varpi_n$ where $\{\zeta_{p^n}\}_{n\geq 0}$ is a compatible system of primitive p^n-th roots of 1. Then our slogan is that the descent isomorphism f comes from the $\tilde{\tau}$-action on the Kisin module $\mathfrak{M}$ inside $T^\vee \otimes A_{\mathrm{inf}}$. We will call this the prismatic (φ, τ)-module theory, which can be regarded as the (φ, τ)-module version of the result of Wu [35].

Actually, the maps $u \mapsto [\varpi^\flat]$ and $u \mapsto [\tilde{\tau}(\varpi^\flat)]$ define two morphisms of $(A, (E))$ to $(A_{\mathrm{inf}}, EA_{\mathrm{inf}})$. By the universal property of $(A^{(2)}, (E))$, these two maps induce a morphism $(A^{(2)}, (E)) \to (A_{\mathrm{inf}}, EA_{\mathrm{inf}})$. We can show this map $A^{(2)} \to A_{\mathrm{inf}}$ is injective, and it factors through $A^{(2)}_{\mathrm{st}}$, which is the base ring used in our prismatic $(\varphi, \hat{G})$-module theory. That is, we have a chain of subrings $A \subset A^{(2)} \subset A^{(2)}_{\mathrm{st}}$ of A_{inf} such that $\tilde{\tau}(A)$ is also contained in $A^{(2)}$. We show that a prismatic $(\varphi, \hat{G})$-module corresponds to a crystalline representation if and only if the coefficients of the $\tilde{\tau}$-action on $\mathfrak{M}$ inside $T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}$ lie in $A^{(2)}$. Once this is proved, the $\tilde{\tau}$-action will induce an isomorphism

$$f_\tau : \mathfrak{M} \otimes_{\mathfrak{S}, \tilde{\tau}} A^{(2)} \simeq \mathfrak{M} \otimes_{\mathfrak{S}} A^{(2)}.$$

We will see that f_τ as above gives a descent isomorphism. Consequently, this leads to a new proof for Theorem 1.1.1.

Logarithmic prismatic F-crystals and lattices in semistable representations. Another advantage of our approach is that our new method can be easily generalized to semistable representation cases. It turns out that the prism $(A^{(2)}_{\mathrm{st}}, (E))$ is isomorphic to the self-product of $(A, (E))$ if both are given proper log structures and realized as log prisms inside the absolute logarithmic prismatic site over $\mathcal{O}_K$ defined by Koshikawa [27]; for details see Section 5. Using the equivalence between prismatic $(\varphi, \hat{G})$-modules and lattices in semistable representations of G_K, we will show in Section 5 the following generalization of Theorem 1.1.1 to semistable representations.

Theorem 1.1.3 (Theorem 5.0.18). *There is an equivalence of the category of prismatic F-crystals in finite locally free $\mathcal{O}_{\Delta}$-modules over $(\mathcal{O}_K)_{\Delta_{\log}}$ and the category of Galois stable lattices in semistable representations of G_K.*

Further discussion. As illustrated in Figure 1, Bhatt–Scholze can package classical integral p-adic Hodge theories in prismatic F-crystals, and our work can be thought of as a way of finding minimal data when unpacking prismatic F-crystals while we can still recover the whole package.

Integral p-adic Hodge theory is well-known for its contribution to the study of torsion Galois representations. It is expected in [9] that Theorem 1.1.1 should be upgraded using the stack Σ'' constructed by Drinfeld [15] to study torsion representations. On the other hand, the theory of $(\varphi, \hat{G})$-modules has been shown to be a powerful tool when studying torsion Galois representations [13, 20]. As the theory of $(\varphi, \hat{G})$-modules has been improved, we hope the results of this paper can develop into new ideas and tools to deal with torsion Galois representations.

Another interesting and natural question one can ask is whether Theorems 1.1.1 and 1.1.3 can accommodate more general base schemes. It turns out that our strategy does allow us to treat many relative bases. See [18] for more details. So part of our paper, for example Section 2, does allow specific general base rings.

1.2. Outline

Let us briefly overview the content of each section. We begin with some ring-theoretic foundations in Section 2. We will construct the period rings $A^{(2)}$ and $A^{(2)}_{\mathrm{st}}$ and collect some of their basic properties for later use. Our construction of $A^{(2)}$ and $A^{(2)}_{\mathrm{st}}$ relies heavily on the theory of prisms developed by Bhatt–Scholze; we refer the reader to [8] for the foundations of this theory. Note that we expect some of the results in this paper can allow more general base rings, so most of the discussion in Section 2 is conducted in a quite general setup. In Appendix A, we will compare our setup with those in [11, 24]. In Section 3, we develop the theory of prismatic $(\varphi, \hat{G})$-modules. We state our main result about prismatic $(\varphi, \hat{G})$-modules in Theorem 3.3.3 and Corollary 3.3.4, but both rely on Proposition 3.2.2 which is proved in Section 4. In that section, we review the definition of prismatic F-crystals by Bhatt–Scholze and give a different proof of their prismatic description of crystalline lattices. This is achieved simultaneously with completing the proof of Proposition 3.2.2. And in order to do this, we develop the prismatic (φ, τ)-module theory in Section 4.2. In Section 5, we briefly review the theory of logarithmic prisms developed by Koshikawa [27], and then we generalize the result of Bhatt–Scholze to semistable representations.

2. Ring structures on certain prismatic envelope

Recall that K is a completed discrete valuation field in mix characteristic $(0, p)$ with the ring of integers $\mathcal{O}_K$ and perfect residue field k. Let $W(k)$ be the ring of Witt vectors over k. Let $\varpi \in \mathcal{O}_K$ be a uniformizer and $E = E(u) \in W(k)[u]$ be the Eisenstein polynomial of ϖ. Let $\mathbb{C}_p$ be the p-adic completion of $\overline{K}$, and $\mathcal{O}_{\mathbb{C}_p}$ be the ring of integers. In this paper, for a p-adic complete ring T, we write $T\langle x_1, \ldots, x_d\rangle$ for the p-adic completion of $T[x_1, \ldots, x_d]$. Let R_0 be a $W(k)$-algebra which admits a lifting of the p-th Frobenius $\varphi : R_0 \to R_0$. Set $R := R_0 \otimes_{W(k)} \mathcal{O}_K$. We make the following assumptions on R_0 and R:

(1) Both R_0 and R are p-adically complete integral domains, and $R_0/pR_0 = R/\varpi R$ is an integral domain.
(2) Let $\breve{R}_0 := W(k)\langle t_1, \ldots, t_m\rangle$. Then R_0 is an $\breve{R}_0$-*formally étale* algebra with p-adic topology.
(3) $\breve{R}_0$ admits a Frobenius lift such that $\breve{R}_0 \to R_0$ defined in (2) is φ-equivalent.
(4) The k-algebra R_0/pR_0 has finite p-basis in the sense of [14, Definition 1.1.1].

Our main example is $R_0 = \breve{R}_0 = W(k)$. We will not use the finite p-basis assumption until Section 4. The following are other examples of R_0:

Example 2.0.1. (1) $R_0 = W(k)\langle t_1^{\pm 1}, \ldots, t_m^{\pm 1}\rangle$ with $\varphi(t_j) = t_j^p$.
(2) $R_0 = W(k)[\![t]\!]$ with $\varphi(t) = t^p$ or $(1+t)^p - 1$.

(3) R_0 is an unramified complete DVR with imperfect field κ with finite p-basis. See Appendix A for more discussion.

We reserve $\gamma_i(\cdot)$ for the i-th divided power.

2.1. *Construction of* $A^{(2)}$

Let $A = \mathfrak{S} = R_0[\![u]\!]$ and extend $\varphi : A \to A$ by $\varphi(u) = u^p$. It is well-known that (A, E) is a prism and we can define a surjection $\theta : A \to R$ via $u \mapsto \varpi$. We have $\operatorname{Ker}\theta = (E(u))$. Let $\breve{A} := \breve{R}_0[\![u]\!]$ and define φ and $\breve{\theta} : \breve{A} \to \breve{R} := \mathcal{O}_K \otimes_W \breve{R}_0$ similarly. We set

$$A^{\widehat{\otimes} 2} := A[\![y - x, s_1 - t_1, \dots, s_m - t_m]\!],$$
$$A^{\widehat{\otimes} 3} := A[\![y - x, w - x, \{s_i - t_i, r_i - t_i\}_{j=1}^m]\!].$$

Note that $A^{\widehat{\otimes} 2}$ (resp. $A^{\widehat{\otimes} 3}$) is an $\breve{A} \otimes_{\mathbb{Z}_p} \breve{A}$ (resp. $\breve{A} \otimes_{\mathbb{Z}_p} \breve{A} \otimes_{\mathbb{Z}_p} \breve{A}$)-algebra via $u \otimes 1 \mapsto x$, $1 \otimes u \mapsto y$ and $1 \otimes t_i \mapsto s_i$ (resp. $1 \otimes 1 \otimes u \mapsto w$ and $1 \otimes 1 \otimes t_i \mapsto r_i$). So in this way, we can extend the Frobenius φ of A, which is compatible with that on $\breve{A}$, to $A^{\widehat{\otimes} 2}$ and $A^{\widehat{\otimes} 3}$. Set $J^{(2)} = (E, y - x, \{s_i - t_i\}_{i=1}^m) \subset A^{\widehat{\otimes} 2}$ and $J^{(3)} = (E, y - x, w - x, \{s_i - t_i, r_i - t_i\}_{i=1}^m) \subset A^{\widehat{\otimes} 3}$. Clearly, $A^{\widehat{\otimes} i}/J^{(i)} \simeq R$ for $i = 2, 3$. And $A^{\widehat{\otimes} 2}/(p, E)$ (resp. $A^{\widehat{\otimes} 3}/(p, E)$) is a formal power series ring over the variables $\bar{y} - \bar{x}, \{\bar{s}_i - \bar{t}_i\}_{i=1}^m$ (resp. $\bar{y} - \bar{x}, \bar{w} - \bar{x}, \{\bar{s}_i - \bar{t}_i, \bar{r}_i - \bar{t}_i\}_{i=1}^m$), so $(A, (E)) \to (A^{\widehat{\otimes} i}, J^{(i)})$ satisfies the requirements of [8, Proposition 3.13], and we can construct the prismatic envelope with respect to this map, which will be denoted by $A^{(i)}$. More precisely, $A^{(i)} \simeq A^{\widehat{\otimes} i}\{\frac{J^{(i)}}{E}\}_\delta^\wedge$, where $\{\cdot\}_\delta^\wedge$ means freely adjoining elements in the category of $(p, E(u))$-completed δ-A-algebras. We will see in Section 4.1 that $A^{(i)}$, $i = 2, 3$, are the self-product and the self-triple-product of A in the category R_{Δ}.

2.2. *The ring* $A^{(2)}_{\max}$

Now we set $t_0 = x$, $s_0 = y$ and

$$z_j = \frac{s_i - t_i}{E} \quad \text{and} \quad z_0 = z = \frac{y - x}{E} = \frac{s_0 - t_0}{E}.$$

Note that $A^{(i)}$ are A-algebras via $u \mapsto x$ for $i = 2, 3$.

Definition 2.2.1. Let $O_{\max}$ be the p-adic completion of the A-subalgebra of $A[\frac{1}{p}]$ generated by $p^{-1}E$. And let $A^{(2)}_{\max}$ be the p-adic completion of the A-subalgebra of $A[z_j, \frac{1}{p}, j = 0, \dots, m]$ generated by $p^{-1}E$ and $\{\gamma_i(z_j)\}_{i \geq 1, j = 0, \dots, m}$.

We first note that $A^{(2)}_{\max}$ is an $A^{\widehat{\otimes} 2}$-algebra via $s_j - t_j = Ez_j$, $j = 0, \dots, m$. Write $\iota : A^{\widehat{\otimes} 2} \to A^{(2)}_{\max}$ for the structure map. By construction, it is easy to see that $A^{(2)}_{\max} \subset R_0[\frac{1}{p}][\![E, z_j, j = 0, \dots, m]\!]$. In particular, $A^{(2)}_{\max}$ is a domain and any element $b \in A^{(2)}_{\max}$ can be *uniquely* written as $\sum_{i_0=0}^\infty \cdots \sum_{i_m=0}^\infty b_{i_1, \dots, i_m} \prod_{j=0}^m \gamma_{i_j}(z_j)$ with $b_{i_0, \dots, i_m} \in O_{\max}$ and $b_{i_0, \dots, i_m} \to 0$ p-adically when $i_0 + \cdots + i_m \to \infty$.

Our next goal in this subsection is to show in Lemma 2.2.3 that there is a natural way to extend φ on $A^{\widehat{\otimes}2}$ to $A^{(2)}_{\max}$. More importantly, we will show in Proposition 2.2.8 and Lemma 2.2.10 that $A^{(2)}$ is a closed subring of $A^{(2)}_{\max}$ and is stable under φ.

Lemma 2.2.2. *$c := \varphi(E)/p \in \mathcal{O}_{\max}$ and $c^{-1} \in \mathcal{O}_{\max}$.*

Proof. Since A is a δ-ring, and E is a distinguished element, we have in particular

$$\varphi(E)/p = c_0 + E^p/p,$$

where $c_0 = \delta(E) \in A^\times$. So $c = \varphi(E)/p \in \mathcal{O}_{\max}$, and $c^{-1} = c_0^{-1}\sum_{i=0}^\infty \frac{(-c_0^{-1}E^p)^i}{p^i} \in \mathcal{O}_{\max}$. ■

Lemma 2.2.3. *If we define $\varphi(z) = \varphi(z_0) = \frac{y^p - x^p}{\varphi(E)}$ and $\varphi(z_j) = \frac{\varphi(s_j)-\varphi(t_j)}{\varphi(E)}$, then for $0 \le j \le m$, we have $\gamma_n(\varphi(z_j)) \in A^{(2)}_{\max}$ for $n \ge 0$. Moreover, φ extends to a* ring *map $\varphi: A^{(2)}_{\max} \to A^{(2)}_{\max}$.*

Proof. We have

$$\begin{aligned}\varphi(z) = \frac{y^p - x^p}{\varphi(E)} &= c^{-1}\frac{y^p - x^p}{p} = c^{-1}\frac{(x+Ez)^p - x^p}{p} \\ &= c^{-1}\sum_{i=1}^p x^{p-i}(Ez)^i\binom{p}{i}/p = c^{-1}\sum_{i=1}^p a_i z^i,\end{aligned}$$

where $a_i \in W(k)[\![x]\!][\frac{E^p}{p}] \subset \mathcal{O}_{\max} \subset A^{(2)}_{\max}$ and c is a unit in $\mathcal{O}_{\max}$. In particular, $\varphi(z) \in A^{(2)}_{\max}$. Moreover,

$$\gamma_n(\varphi(z)) = \frac{\varphi(z)^n}{n!} = \frac{z^n}{n!}\Big(c^{-1}\sum_{i=1}^p a_i z^{i-1}\Big)^n$$

is in $A^{(2)}_{\max}$. The argument for $\varphi(z_j)$ for $j > 1$ needs some more details. We claim that in $A^{\widehat{\otimes}2}$, we have $\delta(s_j) - \delta(t_j) = (s_j - t_j)\lambda_j$ for some $\lambda_j \in A^{\widehat{\otimes}2}$. Recall that $\delta(t_j) = \frac{\varphi(t_j) - t_j^p}{p}$ and $p, s_j - t_j$ is a regular sequence in $A^{\widehat{\otimes}2}$ from our definition, so the claim follows from the fact that $\varphi(s_j) - \varphi(t_j)$ and $s_j^p - t_j^p$ are both divisible by $s_j - t_j$. Since $s_j - t_j = Ez_j$, we have

$$\varphi(z_j) = c^{-1}\Big(\frac{s_j^p - t_j^p}{p} + Ez_j\lambda_j\Big). \tag{2.1}$$

The same argument as that for $\varphi(z_0)$ also shows that $\gamma_n(z_j) \in A^{(2)}_{\max}$ for $j = 1, \dots, m$. Since any element $b \in A^{(2)}_{\max}$ can be uniquely written as

$$b = \sum_{i_0=0}^\infty \cdots \sum_{i_m=0}^\infty b_{i_1,\dots,i_m}\prod_{j=0}^m \gamma_{i_j}(z_j)$$

with $b_{i_0,\dots,i_m} \in \mathcal{O}_{\max}$ and $b_{i_0,\dots,i_m} \to 0$ p-adically when $i_0 + \cdots + i_m \to \infty$, this allows us to extend the Frobenius φ on A to a ring map $\varphi : A^{(2)}_{\max} \to A^{(2)}_{\max}$ by letting $\varphi(u) = u^p$, $\varphi(z) = \frac{y^p - x^p}{\varphi(E)}$, $\varphi(z_j) = \frac{\varphi(s_j) - \varphi(t_j)}{\varphi(E)}$, and $\gamma_i(z_j) \mapsto \gamma_i(\varphi(z_j))$ for $i \geq 1$. ∎

Remark 2.2.4. The ring map $\varphi : A^{(2)}_{\max} \to A^{(2)}_{\max}$ is *not* a Frobenius lift of $A^{(2)}_{\max}/p$ because $\varphi(E/p) - (E/p)^p \notin pA^{(2)}_{\max}$. In particular, $A^{(2)}_{\max}$ is not a δ-ring.

Recall that $A^{(2)}_{\max}$ is an $A^{\widehat{\otimes} 2}$-algebra via the map $\iota : A^{\widehat{\otimes} 2} \to A^{(2)}_{\max}$. The above construction of the Frobenius φ on $A^{(2)}_{\max}$ is obviously compatible with ι.

Our next goal is to show that ι induces a map $A^{(2)} \to A^{(2)}_{\max}$ so that $A^{(2)}$ is a subring of $A^{(2)}_{\max}$ which is compatible with the φ-structures and filtration. We need a little preparation. Write $\mathfrak{z}_n = \delta^n(z)$ with $\delta_0(z) = z = \mathfrak{z}_0$, and $A_0 = W(k)[\![u]\!]$.

Lemma 2.2.5.

$$\delta^n(Ez) = b_n \mathfrak{z}_n + \sum_{i=0}^{p} a_i^{(n)} \mathfrak{z}_{n-1}^i,$$

where $a_i^{(n)} \in A_0[\mathfrak{z}_0, \ldots, \mathfrak{z}_{n-2}]$ so that $a_p^{(n)} \in A_0^\times$ and for $0 \leq i \leq p-1$ each monomial of $a_i^{(n)}$ contains a factor $\mathfrak{z}_j^p$ for some $0 \leq j \leq n-2$. Furthermore, $b_{n+1} = p\delta(b_n) + b_n^p$ and $b_1 = p\delta(E) + E^p$.

Proof. Given $f \in A_0[x_1, \ldots, x_m]$, if each monomial of f contains x_j^l for some j and $l \geq p$ then we call f *good*. For example, $f = x_1^p x_2 + 2x_1 x_2^{p+3}$ is good. So we need to show that $a_i^{(n)} \in A_0[\mathfrak{z}_0, \ldots, \mathfrak{z}_{n-2}]$ is good. Before using induction on n, we discuss some properties of good polynomials. It is clear that the set of good polynomials is closed under addition and multiplication. Note that

$$\delta(\mathfrak{z}_l^i) = \frac{\varphi(\mathfrak{z}_l^i) - \mathfrak{z}_l^{pi}}{p} = \frac{(p\mathfrak{z}_{l+1} + \mathfrak{z}_l^p)^i - \mathfrak{z}_l^{pi}}{p} = \sum_{j=1}^{i} \binom{i}{j} (p^{j-1} \mathfrak{z}_l^{p(i-j)}) \mathfrak{z}_{l+1}^j. \tag{2.2}$$

In particular, given an $f \in A_0[\mathfrak{z}_0, \ldots, \mathfrak{z}_m]$, $\delta(\mathfrak{z}_m^p f) = f^p \delta(\mathfrak{z}_m^p) + \mathfrak{z}_m^{p^2} \delta(f) + p\delta(\mathfrak{z}_m^p)\delta(f)$ is a good polynomial in $A[\mathfrak{z}_0, \ldots, \mathfrak{z}_{m+1}]$. Using the fact that $\delta(a+b) = \delta(a) + \delta(b) + F(a, b)$ where $F(X, Y) = \frac{1}{p}(X^p + Y^p - (X+Y)^p) = -\sum_{i=1}^{p-1} \left(\binom{p}{i}/p\right) X^i Y^{p-i}$, together with the above argument for $\delta(\mathfrak{z}_l^p f)$, it is not hard to show that if $g \in A_0[\mathfrak{z}_0, \ldots, \mathfrak{z}_m]$ is good then $\delta(g) \in A_0[\mathfrak{z}_0, \ldots, \mathfrak{z}_m, \mathfrak{z}_{m+1}]$ is also good.

Now we use induction on n. When $n = 1$, we have

$$\delta(Ez) = E^p \mathfrak{z}_1 + z^p \delta(E) + p\delta(E)\mathfrak{z}_1 = (p\delta(E) + E^p)\mathfrak{z}_1 + \delta(E) z^p.$$

Then $b_1 = p\delta(E) + E^p$, $a_p^{(1)} = \delta(E) \in A_0^\times$ and $a_i^{(1)} = 0$ for $1 \leq i \leq p-1$ are required. Now assume the formula is correct for n. Then

$$\begin{aligned} \delta^{n+1}(Ez) &= \delta\Big(b_n \mathfrak{z}_n + \sum_{i=0}^{p} a_i^{(n)} \mathfrak{z}_{n-1}^i\Big) \\ &= \delta(b_n \mathfrak{z}_n) + \delta\Big(\sum_{i=0}^{p} a_i^{(n)} \mathfrak{z}_{n-1}^i\Big) + F\Big(b_n \mathfrak{z}_n, \sum_{i=0}^{p} a_i^{(n)} \mathfrak{z}_{n-1}^i\Big). \end{aligned}$$

Clearly, $F(b_n\mathfrak{z}_n, \sum_{i=0}^p a_i^{(n)}\mathfrak{z}_{n-1}^i) = \sum_{j=1}^{p-1}\tilde{a}_j^{(n)}\mathfrak{z}_n^j$ with $\tilde{a}_j^{(n)}$ being good. An easy induction shows $\delta(\sum_{i=0}^p a_i^{(n)}\mathfrak{z}_{n-1}^i) = \sum_{i=0}^p \delta(a_i^{(n)}\mathfrak{z}_{n-1}^i) + f$ with $f \in A_0[\mathfrak{z}_0, \dots, \mathfrak{z}_{n-1}]$ being good. Plug the formula (2.2) for $\delta(\mathfrak{z}_{n-1}^i)$ into

$$\delta(a_i^{(n)}\mathfrak{z}_{n-1}^i) = (a_i^{(n)})^p\delta(\mathfrak{z}_{n-1}^i) + (\mathfrak{z}_{n-1}^{pi})\delta(a_i^{(n)}) + p\delta(\mathfrak{z}_{n-1}^i)\delta(a_i^{(n)}),$$

and using the fact that $a_i^{(n)}$ good implies $\delta(a_i^{(n)})$ is also good, we conclude that for $0 \le i \le p-1$,

$$\sum_{i=0}^{p-1}\delta(a_i^{(n)}\mathfrak{z}_{n-1}^i) = \sum_{i=0}^{p-1}\alpha_i\mathfrak{z}_n^i$$

with $\alpha_i \in A_0[\mathfrak{z}_0, \dots, \mathfrak{z}_{n-1}]$ being good polynomials. Since $a_p^{(n)} \in A_0^\times$, we find that $\delta(a_p^{(n)}\mathfrak{z}_{n-1}^p) = \sum_{i=0}^p \beta_i\mathfrak{z}_n^i$ with $\beta_p \in pA_0$ and $\beta_j \in A_0[\mathfrak{z}_0, \dots, \mathfrak{z}_{n-1}]$ being good for $1 \le j \le p-1$. Now we only need to analyze $\delta(b_n\mathfrak{z}_n)$, which is $\delta(b_n)\mathfrak{z}_n^p + b_n^p\mathfrak{z}_{n+1} + p\delta(b_n)\mathfrak{z}_{n+1}$. So $b_{n+1} = p\delta(b_n) + b_n^p$ and $a_p^{(n+1)} = \delta(b_n) + \beta_p$. Since $\delta(b_n) \in A_0^\times$, we see that $a_p^{(n+1)} = \delta(b_n) + \beta_p \in A_0^\times$ as required. ■

Let $\widetilde{A^{(2)}} := A^{\hat{\otimes}2}[z_j]_\delta = A^{\hat{\otimes}2}[\delta^n(z_j), n \ge 0, j = 0, \dots, m]$ and consider the natural map $\alpha : \widetilde{A^{(2)}} \to \widetilde{A^{(2)}}[\frac{1}{p}]$ (we do not know whether α is injective at this moment). We need the following result for Lemma 2.2.12 which is crucial for our later applications.[1]

Lemma 2.2.6. *For $i \ge 0$ and $j = 0, 1, \dots, d$, there exists $f_{ij}(X) \in \widetilde{A^{(2)}}[X]$ such that, as elements of $\widetilde{A^{(2)}}[\frac{1}{p}]$ via $\alpha : \widetilde{A^{(2)}} \to \widetilde{A^{(2)}}[\frac{1}{p}]$,*

$$\gamma_i(z_j) = f_{ij}\left(\frac{E}{p}\right).$$

Proof. Write $z = z_j$ for simplicity, and let $\tilde{\gamma}(z) = \frac{z^p}{p}$ and $\tilde{\gamma}^n = \tilde{\gamma} \circ \cdots \circ \tilde{\gamma}$ (n factors). It suffices to show that for each $n \ge 1$, we have $\tilde{\gamma}^n(z) = f_n(\frac{E}{p})$ inside $\widetilde{A^{(2)}}[\frac{1}{p}]$ for some $f_n(X) \in \widetilde{A^{(2)}}[X]$. For an element $x \in A[\delta^i(z)]_{i\ge 0}$, we say that x has δ-*order* $\le n$ if x can be written as a sum of monomials such that *no* term is divisible by $\delta^j(z)$ for $j > n$, so $x \in \sum_{j=0}^n A[\{\delta^i(z)\}_{i=0}^n]\delta^j(z)$.

We claim that the following two equations hold for each $n \ge 1$:

(1) We have

$$\delta^n(z) = \nu_n\tilde{\gamma}^n(z) + P_n\left(\frac{E}{p}\right) + \frac{E^p}{p}d_n\delta^n(z) \tag{2.3}$$

for some $\nu_n \in A^\times$, $d_n \in A$, and $P_n(X) \in (A[\delta^i(z)]_{i\ge 0})[X]$ such that each coefficient of $P_n(X)$ has δ-order $\le n-1$.

[1] We want to note that there was a gap in the proof of Lemma 2.2.12 in our previous preprint. We thank Yong Suk Moon for pointing it out and helping us prove the following lemma.

(2) We have

$$\tilde{\gamma}(\delta^{n-1}(z)) = \mu_{n-1}\tilde{\gamma}^n(z) + Q_{n-1}\left(\frac{E}{p}\right) \tag{2.4}$$

for some $\mu_{n-1} \in A^\times$ and $Q_{n-1}(X) \in (A[\delta^i(z)]_{i\geq 0})[X]$ such that each coefficient of $Q_{n-1}(X)$ has δ-order $\leq n-1$.

We prove claims (1) and (2) by induction. For $n = 1$, since

$$\delta(Ez) = z^p\delta(E) + (p\delta(E) + E^p)\delta(z)$$

and $\delta(E) \in \mathfrak{S}^\times$, we have

$$\delta(z) = -\tilde{\gamma}(z) + \delta(E)^{-1}\frac{\delta(Ez)}{p} - \delta(E)^{-1}\frac{E^p}{p}\delta(z).$$

By easy induction, we also have $\delta^i(Ez) \in (Ez)A$ for each $i \geq 1$. So claim (1) holds. Claim (2) holds for $n = 1$ trivially with $Q_0(X) = 0$.

Suppose that claims (1) and (2) hold for $1 \leq n \leq m$. We will verify them for $n = m+1$. We first consider claim (2). Since each coefficient of $P_m(X)$ has δ-order $\leq m-1$, $\frac{E^p}{p} = p^{p-1}(\frac{E}{p})^p$, and equations (2.3) and (2.4) hold for $1 \leq n \leq m$, applying $\tilde{\gamma}(\cdot)$ to (2.3) for $n = m$ yields

$$\tilde{\gamma}(\delta^m(z)) = \nu_m^p\tilde{\gamma}^{m+1}(z) + Q_m\left(\frac{E}{p}\right)$$

for some $Q_m(X) \in (\mathfrak{S}[\delta^i(z)]_{i\geq 0})[X]$ such that each coefficient of $Q_m(X)$ has δ-order $\leq m$. This proves claim (2) for $n = m+1$.

We now consider claim (1) for $n = m+1$. By the above lemma for $n = m+1$ and since $b_n = p\alpha_n + \beta_n E^p$ for some $\alpha_n \in A^\times$ and $\beta_n \in A$ (via an easy induction on n), we have

$$\begin{aligned}\alpha_{m+1}\delta^{m+1}(z) = \frac{\delta^{m+1}(Ez)}{p} - \beta_{m+1}\frac{E^p}{p}\delta^{m+1}(z) - a_p^{(m+1)}\tilde{\gamma}(\delta^m(z)) \\ - \frac{1}{p}\sum_{j=0}^{p-1} a_j^{(m+1)}(\delta^m(z))^j.\end{aligned}$$

As noted above, we have $\delta^{m+1}(Ez) \in (Ez)A$. Furthermore, by the condition on $a_j^{(m+1)}$, the last term $\frac{1}{p}\sum_{j=0}^{p-1} a_j^{(m+1)}(\delta^m(z))^j$ is a linear combination of terms involving $\tilde{\gamma}(\delta^l(z)) = \frac{1}{p}(\delta^l(z))^p$ for some $0 \leq l \leq m-1$. Thus, by applying (2.3) and (2.4) for $1 \leq n \leq m$, we see that claim (1) also holds for $n = m+1$ with $\nu_{m+1} = -\alpha_{m+1}^{-1}a_p^{(m+1)}\mu_m$ and $d_{m+1} = -\alpha_{m+1}^{-1}\beta_{m+1}$. This completes the induction and proves the lemma. ■

Remark 2.2.7. In the above proof, by (2.4), we even have, for all $i, j \geq 0$, $\gamma_i(\delta^j(z)) = f(\frac{E}{p})$ for some $f \in \widetilde{A^{(2)}}[X]$.

An easy induction using (2.3) implies $\alpha(\delta^n(z)) \in A^{\widehat{\otimes}2}[\{\gamma_i(z_j)\}_{i\geq 0, j=1,\dots,m}, \frac{E}{p}]$ inside $A^{(2)}_{\max}$, which satisfies the equations in Lemma 2.2.5 on replacing $\mathfrak{z}_n$ by $\alpha(\delta^n(z))$ inside $A^{(2)}_{\max}$. It is clear that ι is still Frobenius compatible (because both $A^{\widehat{\otimes}2}$ and $A^{(2)}_{\max}$ are domains). Since $E = p\frac{E}{p}$, ι is continuous for the (p, E)-topology on $\widetilde{A^{(2)}}$ and the p-topology on $A^{(2)}_{\max}$. Finally, we construct a ring map $\iota : A^{(2)} \to A^{(2)}_{\max}$ so that ι is compatible with Frobenius.

Our next goal is to show that ι is injective. We define

$$\mathrm{Fil}^i A^{(2)}_{\max}\left[\tfrac{1}{p}\right] := E^i A^{(2)}_{\max}\left[\tfrac{1}{p}\right].$$

And for any subring $B \subset A^{(2)}_{\max}[\frac{1}{p}]$, set

$$\mathrm{Fil}^i B := B \cap \mathrm{Fil}^i A^{(2)}_{\max}\left[\tfrac{1}{p}\right] = B \cap E^i A^{(2)}_{\max}\left[\tfrac{1}{p}\right].$$

Let D_z be the p-adic completion of $R[\gamma_i(z_j), i \geq 0, j = 0, \dots, m]$.

Proposition 2.2.8. (1) $\widetilde{A^{(2)}}/E = R[\gamma_i(z_j), i \geq 0, j = 0, \dots, m]$.

(2) $A^{(2)}/E \simeq D_z$.

(3) *ι is injective.*

(4) $\mathrm{Fil}^1 A^{(2)} = EA^{(2)}$.

(5) *$A^{(i)}$ are flat over A for $i = 2, 3$.*

Proof. (1) By definition, $\widetilde{A^{(2)}} = A^{\widehat{\otimes}2}[z_j^{(n)}, n \geq 0, j = 0, \dots, m]/J$ where mod J is equivalent to quotienting by the following relations (note that $z_0 = z$): $\delta(z_j^{(n)}) = z_j^{(n+1)}, \delta^n(Ez) = \delta^n(y - x), \delta^n(Ez_j) = \delta^n(s_j - t_j)$ for $n \geq 0$. Since $\delta(x - y) = \frac{(x^p - y^p) - (x-y)^p}{p}$ and $\delta(s_j - t_j) = \frac{\varphi(s_j - t_j) - (s_j - t_j)^p}{p}$, using the fact that $p, s_i - t_j$ and $p, x - y$ are regular sequences, one can prove that $\delta^n(x - y)$ and $\delta^n(s_j - t_j)$ always contains a factor $x - y$, $s_j - t_j$ and hence $\delta^n(x - y), \delta(s_j - t_j) = 0 \bmod E$. Therefore $\delta^n(Ez_j) = 0 \bmod E$. By Lemma 2.2.5, we see that

$$p\mu_n z_j^{(n)} = -\sum_{i=0}^{p} \overline{a_i^{(n)}}(z_j^{(n-1)})^i \bmod E \quad \text{and} \quad pz_j^{(1)} = z_j^p \bmod E$$

where $\overline{a_i^{(n)}} = a_i^{(n)} \bmod E$ and $\mu_n = \frac{\delta(b_n)}{p} \bmod E \in \mathcal{O}_K^\times$. Since $a_p^{(n)} \in A_0^\times$, and $a_i^{(n)}, 1 \leq i \leq p - 1$, are good in the sense that they contain factor of $(z_j^{(l)})^p$ for some $l = 0, \dots, n-2$, we easily see by induction that $\widetilde{A^{(2)}}/E = R[\tilde{\gamma}^n(z_j), n \geq 0, j = 0, \dots, m]$. But it is well-known that $R[\tilde{\gamma}^n(z_j), n \geq 0, j = 0, \dots, m] = R[\gamma_n(z_j), n \geq 0, j = 0, \dots, m]$.

Now we show that the natural map $\iota : \widetilde{A^{(2)}} \to A^{(2)}_{\max}[\frac{1}{p}]$ induced by $\alpha(\delta^n(z_j))$ is injective. Note that $\widetilde{A^{(2)}}$ is the direct limit of $\widetilde{A_n^{(2)}}$ where $\widetilde{A_n^{(2)}} := A^{\widehat{\otimes}2}[\{\delta^i(z_j)\}_{i=1,\dots,n, j=0,\dots,m}]$. A similar argument shows that $\widetilde{A_n^{(2)}}/E$ injects to $A^{(2)}_{\max}[\frac{1}{p}]/E = D_z[\frac{1}{p}]$. Since $\widetilde{A_n^{(2)}}$ is

E-separate and $A_{\max}^{(2)}$ is a domain, this implies that $\widetilde{A_n^{(2)}}$ injects to $A_{\max}^{(2)}[\frac{1}{p}]$. So $\widetilde{A^{(2)}}$ injects to $A_{\max}^{(2)}$ via ι.

(2) Since $A^{(2)}$ is the (p, E)-completion of $\widetilde{A^{(2)}}$,[2] we have a natural map $\bar{\iota}: A^{(2)}/E \to D_z$. The surjectivity of $\bar{\iota}$ is straightforward as $A^{(2)}$ is also p-complete. To see injectivity, given a sequence f_n such that $f_{n+1} - f_n \in (p, E)^n \widetilde{A^{(2)}}$ and $f_n = E g_n$ for all n, we have to show that g_n is a convergent sequence in $A^{(2)}$. Since $E(g_{n+1} - g_n) = \sum_{i=0}^n p^i E^{n-i} h_i$ with $h_i \in \widetilde{A^{(2)}}$, we have $E \mid p^n h_n$. Since $\widetilde{A^{(2)}}/E$ has no p-torsion, we have $E \mid h_n$ and we write $h_n = E h_n'$. Since $\widetilde{A^{(2)}}$ is a domain as it is inside the fraction field of $A^{\widehat{\otimes} 2}$, we see that $g_{n+1} - g_n = p^n h_n' + \sum_{i=0}^{n-1} p^i E^{n-i-1} h_i$. Hence g_n converges in $A^{(2)}$ as required.

(3) It is clear that $A_{\max}^{(2)}[\frac{1}{p}]/E \simeq D_z[\frac{1}{p}]$. So the map ι mod $E(u)$ induces an injection $D_z \hookrightarrow D_z[\frac{1}{p}]$. So for any $x \in \operatorname{Ker} \iota$, we have $x = Ea$ for some $a \in A^{(2)}$. As $A_{\max}^{(2)}$ is E-torsion free and $A^{(2)}$ is E-complete, we see that $x = 0$ as required.

(4) By the definition of $\operatorname{Fil}^1 A^{(2)}$, we see that $EA^{(2)} \subset \operatorname{Fil}^1 A^{(2)}$ and $A^{(2)}/\operatorname{Fil}^1 A^{(2)}$ injects to $A_{\max}^{(2)}[\frac{1}{p}]/E = D_z[\frac{1}{p}]$. But we have seen that $A^{(2)}/E = D_z$ injects to D_z. Then $\operatorname{Fil}^1 A^{(2)} = EA^{(2)}$.

(5) Both $A^{(2)}$ and $A^{(3)}$ are obtained by the construction of [8, Proposition 3.13], which implies that they are (p, E)-complete flat over A. Since A is Noetherian, by [33, Tag 0912], both $A^{(2)}$ and $A^{(3)}$ are A-flat. ∎

Corollary 2.2.9. (1) $\operatorname{Fil}^i A^{(2)} = E^i A^{(2)}$.

(2) *$A^{(i)}$ are bounded prisms for $i = 2, 3$.*

Proof. These follow that $A^{(2)}/EA^{(2)} \simeq D_z$ which is $\mathbb{Z}_p$-flat. For (2), $A^{(2)}$ and $A^{(3)}$ are (p, E)-complete flat over A, so boundedness follows from [8, Lemma 3.7 (2)]. ∎

Lemma 2.2.10. *$A^{(2)}$ is a closed subset inside $A_{\max}^{(2)}$.*

Proof. We need to show the following statement: Given $x \in \widetilde{A^{(2)}}$, if $x = p^n y$ with $y \in A_{\max}^{(2)}$ then $x = \sum_{i=0}^n p^{n-i} E^i x_i$ with $x_i \in \widetilde{A^{(2)}}$. Indeed, since $A^{(2)}/E \simeq A_{\max}^{(2)}/\operatorname{Fil}^1$, there exist $x_0, w_1 \in \widetilde{A^{(2)}}$ such that $x = p^n x_0 + E w_1$. Then $E w_1 \in p^n A_{\max}^{(2)}$. Writing

$$E w_1 = p^n \sum_{i=0}^{\infty} \sum_{j=0}^{m} f_{ij} \gamma_i(z_j),$$

we see that $f_{ij} = \sum_{l \geq 1} a_{ijl} \frac{E^l}{p^l} \in \operatorname{Fil}^1 O_{\max}$. So it is easy to see that $p^n E^{-1} f_{ij} \in p^{n-1} O_{\max}$ and so $w_1 = p^{n-1} x_1$ with $x_1 \in A_{\max}^{(2)}$. Then we may repeat the above argument for w_1, and finally $x = \sum_{i=0}^n p^{n-i} E^i x_i$ with $x_i \in \widetilde{A^{(2)}}$ as required. ∎

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[2] Indeed, $A^{(2)}$ is the *derived* (p, E)-completion. Since $\widetilde{A^{(2)}}/E$ is $\mathbb{Z}_p$-flat, the derived completion coincides with the classical completion, which is used here.

Remark 2.2.11. This remark is prompted by feedback provided by an anonymous referee. We have seen ι induces $\iota'\colon A^{(2)}\langle \frac{E}{p}\rangle \to A^{(2)}_{\max}$, where $A^{(2)}\langle \frac{E}{p}\rangle$ is the p-adic completion of $A^{(2)}[\frac{E}{p}]$. Moreover, since E is divisible by p in both $A^{(2)}\langle \frac{E}{p}\rangle$ and $A^{(2)}_{\max}$, the p-adic topology is the same as the (p, E)-adic topology over these rings. Combining these with Lemma 2.2.6 and equation (2.3), we see that ι' has both left and right inverses and is thus an isomorphism. Notably, $A^{(2)}_{\max} = \mathcal{O}_{\Delta}\langle \frac{\mathcal{I}_{\Delta}}{p}\rangle((A^{(2)}, (E)))$, where $\mathcal{O}_{\Delta}\langle \frac{\mathcal{I}_{\Delta}}{p}\rangle((A, I))$ is equal to the p-adic completion of $A[\frac{I}{p}]$, as a presheaf on R_{Δ}. We remark that $\mathcal{O}_{\Delta}\langle \frac{\mathcal{I}_{\Delta}}{p}\rangle$ bears a close relationship to the sheaf $\Delta_{\bullet}\langle \frac{I}{p}\rangle$ on the (small) quasi-syntomic site over R, which is defined in [9, Section 6]. Additionally, using similar techniques to those in [17], one can show $\mathcal{O}_{\Delta}\langle \frac{\mathcal{I}_{\Delta}}{p}\rangle$ is a sheaf on the absolute *transversal* prismatic site and contains $\mathcal{O}_{\Delta}$ as a subsheaf. This may give a direct verification of the inclusion of $A^{(2)}$ in $A^{(2)}_{\max}$. However, to further our subsequent arguments, it is useful to write down the elements of $A^{(2)}_{\max}$ in a more explicit manner.

Now we realize $A^{(2)}$ as a subring of $A^{(2)}_{\max}$ via ι. We need to introduce some auxiliary rings. By the description of elements in $A^{(2)}_{\max}$, we define $\widetilde{S}$ to be the subring of $A^{(2)}_{\max}$ as follows:

$$\widetilde{S} := A^{(2)}\left[\!\left[\frac{E^p}{p}\right]\!\right] := \left\{\sum_{i\geq 0} a_i \left(\frac{E^p}{p}\right)^i \;\middle|\; a_i \in A^{(2)}\right\}.$$

And when $p = 2$, we define $\widehat{S} := A^{(2)}[\![\frac{E^4}{2}]\!]$ similarly. We have $\widehat{S} \subset \widetilde{S} \subset A^{(2)}_{\max}$. Viewing $\widetilde{S}$ and $\widehat{S}$ as subrings of $A^{(2)}_{\max}$, we give them the filtration induced from $A^{(2)}_{\max}$. We will use the fact that if $\{a_i\}_{i\geq 1}$ is a sequence in $A^{(2)}$ that converges to 0, then for any sequence $\{b_i\}_{i\geq 1}$ inside $\widetilde{S}$, one can show $\sum_i a_i b_i \in \widetilde{S}$ by rewriting it as a formal power series in $\frac{E^p}{p}$.

Lemma 2.2.12. *Fix $h \in \mathbb{N}$.*

(1) *We have $\varphi(A^{(2)}_{\max}) \subset \widetilde{S} \subset A^{(2)}_{\max}$, and when $p = 2$, we have $\varphi(\widetilde{S}) \subset \widehat{S} \subset \widetilde{S}$.*

(2) *$x \in \mathrm{Fil}^h \widetilde{S}$ if and only if x can be written as*

$$x = \sum_{i\geq h} a_i \frac{E^i}{p^{\lfloor i/p\rfloor}} \quad \text{with } a_i \in A^{(2)}.$$

(3) *When $p > 2$, there is an $h_0 > h$ such that $\varphi(\mathrm{Fil}^m \widetilde{S}) \subset A^{(2)} + E^h \mathrm{Fil}^{m+1} \widetilde{S}$ for all $m > h_0$.*

(4) *When $p = 2$, then $x \in \mathrm{Fil}^h \widehat{S}$ if and only if x can be written as*

$$x = \sum_{i\geq h} a_i \frac{E^i}{2^{\lfloor i/4\rfloor}} \quad \text{with } a_i \in A^{(2)}.$$

(5) *When $p = 2$, there is an $h_0 > h$ such that $\varphi(\mathrm{Fil}^m \widehat{S}) \subset A^{(2)} + E^h \mathrm{Fil}^{m+1} \widehat{S}$ for all $m > h_0$.*

Proof. For (1), for any $a \in A_{\max}^{(2)}$, we can write

$$a = \sum_{i_0=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \sum_{l=0}^{\infty} a_{i_0,\dots,i_m,l} \left(\frac{E}{p}\right)^l \prod_{j=0}^{m} \gamma_{i_j}(z_j),$$

where $a_{i_0,\dots,i_m,l} \in A$ and $a_{i_0,\dots,i_m,l} \to 0$ p-adically when $\sum_j i_j + l \to \infty$. Thanks to Lemma 2.2.6, we see that $b_{i_0,\dots,i_m,l} := \varphi((\frac{E}{p})^l \prod_{j=0}^{m} \gamma_{i_j}(z_j)) \in \widetilde{S}$. Consequently, $\varphi(a) = \sum a_{i_0,\dots,i_m,l} b_{i_0,\dots,i_m,l}$ converges in $\widetilde{S}$.

For the claim in (1) for $p = 2$, we have $\varphi(\frac{E^2}{2}) = (E^2 + 2b')^2/2 = \frac{E^4}{2} + 2b$ for some $b, b' \in A$. And for $a = \sum_{i \geq 0} a_i (\frac{E^p}{p})^i \in \widetilde{S}$, we have

$$\varphi(a) = \sum_{i \geq 0} \varphi(a_i) \left(\frac{\varphi(E^2)}{2}\right)^i = \sum_{i \geq 0} \varphi(a_i) \sum_{j=0}^{i} c_{ij} (2b)^{i-j} \left(\frac{E^4}{2}\right)^j$$
$$= \sum_{j \geq 0} \left(\sum_{i=j}^{\infty} \varphi(a_i) c_{ij} (2b)^{i-j}\right) \left(\frac{E^4}{2}\right)^j$$

for some $c_{ij} \in \mathbb{Z}$. We have $\varphi(a) \in \widehat{S}$ since $\sum_{i=j}^{\infty} \varphi(a_i) c_{ij} (2b)^{i-j}$ converges in $A^{(2)}$.

For (2), the "if" part is trivial. For the other direction, for any $x \in \widetilde{S}$, we have

$$x = \sum_{i \geq 0} a_i \frac{E^i}{p^{\lfloor i/p \rfloor}}.$$

And if we also have $x \in \mathrm{Fil}^h A_{\max}^{(2)}[\frac{1}{p}] = E^h A_{\max}^{(2)}[\frac{1}{p}]$, then $\tilde{a}_0 = \sum_{0 \leq i \leq h} a_i \frac{E^i}{p^{\lfloor i/p \rfloor}}$ will lie inside $\mathrm{Fil}^h A^{(2)}[\frac{1}{p}]$. This implies $p^{\lfloor h/p \rfloor} \tilde{a}_0 \in \mathrm{Fil}^h A^{(2)} = E^h A^{(2)}$. That is, $\tilde{a}_0 = p^{-\lfloor h/p \rfloor} E^h b$ for some $b \in A^{(2)}$. So x is of the given form. The proof for (4) is similar.

For (3), we have, by (2), $x \in \mathrm{Fil}^m \widetilde{S}$, so x can be written as

$$x = \sum_{i \geq m} a_i \frac{E^i}{p^{\lfloor i/p \rfloor}}.$$

And using the fact $\varphi(E) = E^p + pb$ for some $b \in A^{(2)}$, we have

$$\varphi(x) = \sum_{i \geq m} \varphi(a_i) \sum_{j=0}^{i} \frac{c_{ij} E^{p(i-j)} p^j}{p^{\lfloor i/p \rfloor}}$$
$$= \sum_{i \geq m} \sum_{j \geq \lfloor i/p \rfloor}^{i} \frac{b_{ij} E^{p(i-j)} p^j}{p^{\lfloor i/p \rfloor}} + \sum_{i \geq m} \sum_{0 \leq j < \lfloor i/p \rfloor} E^h \frac{b_{ij} E^{p(i-j)-h} p^j}{p^{\lfloor i/p \rfloor}}$$

with $b_{ij} \in A^{(2)}$.

In particular, $\sum_{i\geq m}\sum_{j\geq\lfloor i/p\rfloor}^{i}\frac{b_{ij}E^{p(i-j)}p^j}{p^{\lfloor i/p\rfloor}}$ is inside $A^{(2)}$. To prove (3), it is enough to find h_0 such that whenever $m>h_0$, $i\geq m$ and $0\leq j<\lfloor i/p\rfloor$, we have

$$\sum_{i\geq m}\sum_{0\leq j<\lfloor i/p\rfloor}\frac{b_{ij}E^{p(i-j)-h}p^j}{p^{\lfloor i/p\rfloor}}\in \mathrm{Fil}^{m+1}\widetilde{S}.$$

The claim follows if we can find $h_0>h$ such that $\frac{E^{p(i-j)-h}p^j}{p^{\lfloor i/p\rfloor}}\in\widetilde{S}$ and $p(i-j)-h\geq m+1$ for all $m>h_0$, $i\geq m$ and $0\leq j<\lfloor i/p\rfloor$. That is, $\lfloor\frac{p(i-j)-h}{p}\rfloor+j\geq\lfloor i/p\rfloor$ and $p(i-j)-h\geq m+1$ for all i,j,m in this range. And solving this we find it is enough to choose $h_0>\max\{h,\frac{p(h+1)+1}{p(p-2)}\}$, which is valid for $p>2$.

Statement (5) is similar to (3). Any $x\in\mathrm{Fil}^m\widehat{S}$ can be written as

$$x=\sum_{i\geq m}a_i\frac{E^i}{2^{\lfloor i/4\rfloor}}.$$

We have $\varphi(E)=E^2+2b$ for some $b\in A^{(2)}$, so

$$\begin{aligned}\varphi(x)&=\sum_{i\geq m}\varphi(a_i)\sum_{j=0}^{i}\frac{c_{ij}E^{2(i-j)}2^j}{2^{\lfloor i/4\rfloor}}\\&=\sum_{i\geq m}\sum_{j\geq\lfloor i/4\rfloor}^{i}\frac{b_{ij}E^{2(i-j)}2^j}{2^{\lfloor i/4\rfloor}}+\sum_{i\geq m}\sum_{0\leq j<\lfloor i/4\rfloor}E^h\frac{b_{ij}E^{2(i-j)-h}2^j}{2^{\lfloor i/4\rfloor}}.\end{aligned}$$

Similar to the argument for (3), it is enough to find h_0 such that whenever $m>h_0$, $i\geq m$ and $0\leq j<\lfloor i/4\rfloor$, we have $\lfloor(i-j)-h/2\rfloor+j\geq\lfloor i/4\rfloor$ and $2(i-j)-h\geq m+1$. It is enough to choose $h_0>2(h+2)$. ∎

If A is a ring then we denote by $\mathrm{M}_d(A)$ the set of $d\times d$-matrices with entries in A.

Proposition 2.2.13. *Let $Y\in\mathrm{M}_d(A^{(2)}_{\max})$ so that $E^hY=B\varphi(Y)C$ with B and C in $\mathrm{M}_d(A^{(2)})$. Then Y is in $\mathrm{M}_d(A^{(2)}[\frac{1}{p}])$.*

Proof. First, we claim that there is a constant s only depending on h such that the entries of p^sY are in $\widetilde{S}$. By Lemma 2.2.12 (1), the entries of E^hY are in $\widetilde{S}$. So for each entry a of Y, we can write $E^ha=\sum_{i=0}^{\infty}a_i\frac{E^{pi}}{p^i}$ with $a_i\in A^{(2)}$. It is clear that $E^hp^ha=a'+E^h\sum_{i\geq h}a_j\frac{E^{pi-h}}{p^i}$ so that $a'\in A^{(2)}$. Therefore, $a'\in\mathrm{Fil}^hA^{(2)}=E^hA^{(2)}$ by Corollary 2.2.9. So writing $a'=E^hb$, we have $p^ha=b+\sum_{i\geq h}a_j\frac{E^{pi-h}}{p^i}$. In particular, $p^{2h}a\in\widetilde{S}$, which proves our claim. When $p=2$, we may repeat the above argument and we can assume p^sY is in $\mathrm{M}_d(\widehat{S})$.

Let $\mathfrak{R}=\widetilde{S}$ when $p>2$ and $\mathfrak{R}=\widehat{S}$ when $p=2$. Then we may assume Y is inside $\mathrm{M}_d(\mathfrak{R})$. We claim there is another constant r, only depending on h, such that for each entry a of p^rY, there is a sequence $\{b_i\}_{i\geq 1}$ in $A^{(2)}$ such that $a-\sum_{i=0}^{m}b_iE^i\in\mathrm{Fil}^{m+1}\mathfrak{R}$. Note that once this is known, we will see that $\sum_{i=0}^{m}b_iE^i$ converges to an element b in $A^{(2)}$, and $a-b=0$ since it is in $\mathrm{Fil}^m\mathfrak{R}$ for all $m\in\mathbb{N}$.

It remains to show our claim. When $p > 2$, let h_0 be the integer in Lemma 2.2.12 (3). Then it is easy to show there is a constant r only depending on h_0 (so only on h) and a sequence $\{b_i\}_{i=1}^{h_0}$ such that for each entry a of $Y' := p^r Y$, we have

$$a - \sum_{i=0}^{h_0} b_i E^i \in \mathrm{Fil}^{h_0+1}\mathfrak{R}.$$

Now we show our claim by induction, assuming that for each entry a in Y', there is a sequence $\{b_i\}_{i=1}^{m}$ such that

$$a - \sum_{i=0}^{m} b_i E^i \in \mathrm{Fil}^{m+1}\mathfrak{R}$$

for some $m \geq h_0$. So we can write Y' as

$$\sum_{i=0}^{m} Y_i E^i + Z_{m+1}$$

with $Y_i \in \mathrm{M}_d(A^{(2)})$ and $Z_{m+1} \in \mathrm{M}_d(\mathrm{Fil}^{m+1}\mathfrak{R})$. If we write $X_m = \sum_{i=0}^{m} Y_i E^i$, then $E^h Y' = B\varphi(Y')C$ implies

$$E^h Z_{m+1} = B\varphi(X_m)C - E^h X_m + B\varphi(Z_{m+1})C.$$

By Lemma 2.2.12 (3), we have $B\varphi(Z_{m+1})C = A_{m+1} + E^h B_{m+1}$ with $A_{m+1} \in \mathrm{M}_d(A^{(2)})$ and $B_{m+1} \in \mathrm{M}_d(\mathrm{Fil}^{m+2}\mathfrak{R})$. One has $B\varphi(X_m)C - E^h X_m + A_{m+1} \in \mathrm{M}_d(\mathrm{Fil}^{h+m+1}A^{(2)})$, so $B\varphi(X_m)C - E^h X_m + A_{m+1} = E^{h+m+1}Y_{m+1}$ with $Y_{m+1} \in \mathrm{M}_d(A^{(2)})$. Moreover, we have $Y - \sum_{i=0}^{m+1} Y_i E^i = B_{m+1} \in \mathrm{M}_d(\mathrm{Fil}^{m+2}\mathfrak{R})$ as required.

At last, let $p = 2$. We know we can assume Y is inside $\mathrm{M}_d(\widehat{S})$. Then by repeating the above arguments with (3) of Lemma 2.2.12 replaced by (5), we can also prove our claim. ■

Remark 2.2.14. The above proposition will be used to prove the "boundedness of descent data at the boundary", similar to the results exhibited in [9, Section 6.3]. Specifically, with $R = \mathcal{O}_K$, in the proof of Lemma 3.2.3, we will establish that any φ-equivariant descent data of a Kisin module defined over $A^{(2)}_{\max}$ will automatically descend to $A^{(2)}[\frac{1}{p}]$. Alternatively, if $(A^{(2)}_{\inf}, (d))$ is the initial prism over the quasi-regular semiperfectoid ring $\mathcal{O}_{\mathbb{C}_p} \hat{\otimes} \mathcal{O}_{\mathbb{C}_p}$, whose existence is ensured by [8, Proposition 7.2] and which admits two natural maps from $(A_{\inf}, \mathrm{Ker}\,\theta)$, and since we mentioned that $A^{(2)}$ is a certain product of A in the absolute prismatic site over R (see Proposition 4.1.3 for the explicit statement), we can construct a universal map $A^{(2)} \to A^{(2)}_{\inf}$ once we fix a map $A \to A_{\inf}$. Moreover, in [9, Proposition 6.10], it is demonstrated that any φ-equivariant descent data of a Breuil–Kisin–Fargues module defined over $A^{(2)}_{\inf}\langle \frac{d}{p}\rangle[\frac{1}{p}]$ will automatically descend to $A^{(2)}_{\inf}[\frac{1}{p}]$. Additionally, if the Breuil–Kisin–Fargues module comes from the base change of Kisin module, then these two results on descent morphisms are actually equivalent using quasi-syntomic descent, providing that $A \to A_{\inf}$ is faithfully flat (so $A_{\inf}$ also applies to Proposition 4.1.6), and the description of $A^{(2)}_{\max}$ from Remark 2.2.11. However, the proof provided in [9] crucially uses the Beilinson fiber square developed in [2]. On the other hand, our proof is entirely explicitly algebraic and mainly uses the finite E-height condition of Kisin modules.

2.3. *The ring* $A^{(2)}_{\mathrm{st}}$

In the following two subsections we assume that $R = \mathcal{O}_K$. For our later use for semistable representations, we construct $A^{(2)}_{\mathrm{st}}$ as follows: Define φ on $W(k)[\![x, \mathfrak{y}]\!]$ by $\varphi(x) = x^p$ and $\varphi(\mathfrak{y}) = (1+\mathfrak{y})^p - 1$ and set $w = \frac{\mathfrak{y}}{E}$. Set $A^{(2)}_{\mathrm{st}} := W(k)[\![x, \mathfrak{y}]\!]\{w\}^{\wedge}_{\delta}$ where $\wedge$ means (p, E)-completion. Similarly, we define $A^{(3)}_{\mathrm{st}} = W(k)[\![x, \mathfrak{y}, \mathfrak{z}]\!]\{\frac{\mathfrak{y}}{E}, \frac{\mathfrak{z}}{E}\}^{\wedge}_{\delta}$, with the δ-structure on $W(k)[\![x, \mathfrak{y}, \mathfrak{z}]\!]$ given by $\delta(x) = 0$, $\varphi(\mathfrak{y}) = (\mathfrak{y}+1)^p - 1$ and $\varphi(\mathfrak{z}) = (\mathfrak{z}+1)^p - 1$. Define $A^{(2)}_{\mathrm{st,max}}$ to be the p-adic completion of $W(k)[\![x, \mathfrak{y}]\!][w, \frac{E}{p}, \gamma_i(w), i \geq 0]$. It is clear that any $f \in A^{(2)}_{\mathrm{st,max}}$ can be written uniquely as $f = \sum_{i=0}^{\infty} f_i \gamma_i(w)$ with $f_i \in O_{\max}$ and $f_i \to 0$ p-adically. For any subring $B \subset A^{(2)}_{\mathrm{st,max}}[\frac{1}{p}]$, we set $\mathrm{Fil}^i B := B \cap E^i A^{(2)}_{\mathrm{st,max}}[\frac{1}{p}]$ and let D_w be the p-adic completion of $\mathcal{O}_K[\gamma_i(w), i \geq 0]$.

It turns out that $A^{(2)}$ and $A^{(2)}_{\mathrm{st}}$ share almost the same properties by replacing z with w. So we summarize all these properties in the following:

Proposition 2.3.1. (1) *One can extend the Frobenius from A to $A^{(2)}_{\mathrm{st,max}}$.*

(2) *There exists an embedding $\iota : A^{(2)}_{\mathrm{st}} \hookrightarrow A^{(2)}_{\mathrm{st,max}}$ that commutes with the Frobenius.*

(3) $A^{(2)}_{\mathrm{st}} \cap E^i A^{(2)}_{\mathrm{st,max}}[\frac{1}{p}] = E A^{(2)}_{\mathrm{st}}$.

(4) $A^{(2)}_{\mathrm{st}}/E \simeq D_w = A^{(2)}_{\mathrm{st,max}}/\mathrm{Fil}^1 A^{(2)}_{\mathrm{st,max}}$.

(5) *$A^{(2)}_{\mathrm{st}}$ is closed in $A^{(2)}_{\mathrm{st,max}}$.*

(6) *$A^{(2)}_{\mathrm{st}}$ and $A^{(3)}_{\mathrm{st}}$ are flat over A, and in particular they are bounded.*

(7) *Proposition 2.2.13 holds on replacing $A^{(2)}_{\max}$ and $A^{(2)}$ by $A^{(2)}_{\mathrm{st}}$ and $A^{(2)}_{\mathrm{st,max}}$ respectively.*

Proof. All the previous proof applies by noting the following difference:

$$\varphi(w) = \varphi\left(\frac{\mathfrak{y}}{E}\right) = c^{-1}\frac{1}{p}\sum_{i=1}^{p}\binom{p}{i}\mathfrak{y}^i = c^{-1}\sum_{i=1}^{p-1}\mathfrak{y}^i\binom{p}{i}/p + c^{-1}\frac{E^p w^p}{p}.$$

Also $\delta(\mathfrak{y}) = \sum_{i=1}^{p-1}\mathfrak{y}^i\binom{p}{i}/p$ always contains a $\mathfrak{y}$-factor and this is a key input for the analogy of Lemma 2.2.6.

For the boundedness of $A^{(3)}_{\mathrm{st}}$, we have

$$W(k)[\![x, \mathfrak{y}, \mathfrak{z}]\!]/(p, E) \simeq (\mathcal{O}_K/p)[\![\bar{\mathfrak{y}}, \bar{\mathfrak{z}}]\!],$$

so $\{\mathfrak{y}, \mathfrak{z}\}$ form a (p, E)-complete regular sequence, and by [8, Proposition 3.13], $A^{(3)}_{\mathrm{st}}$ is also A-flat, and this implies $A^{(3)}_{\mathrm{st}}$ is bounded by [8, Lemma 3.7 (2)]. ∎

Note that $A^{\widehat{\otimes}2} = W(k)[\![x, y]\!] \subset W(k)[\![x, \mathfrak{y}]\!]$ via $y = x(\mathfrak{y}+1)$ or equivalently $\mathfrak{y} = \frac{y}{x} - 1$. It is clear that this inclusion is a map of δ-rings. By the universal property of prismatic envelope to construct $A^{(2)}$, the inclusion induces a map of prisms $\alpha : A^{(2)} \to A^{(2)}_{\mathrm{st}}$. Since $z = xw$, we easily see that $A^{(2)}_{\max} \subset A^{(2)}_{\mathrm{st},\max}$. So $A^{(2)} \subset A^{(2)}_{\mathrm{st}}$ via α. We will see in Sections 4.1 and 5 that $A^{(2)}$ (resp. $A^{(2)}_{\mathrm{st}}$) is the self-product of A in the category X_{Δ} (resp. $(X, M_X)_{\Delta_{\log}}$). Then the existence of $\alpha : A^{(2)} \to A^{(2)}_{\mathrm{st}}$ can be explained by the universal property of self-product. See Section 5 for details.

To simplify our notation, let $B^{(2)}_{\mathrm{st}}$ (resp. $B^{(3)}_{\mathrm{st}}$, $B^{(2)}$, $B^{(3)}$) be the p-adic completion of $A^{(2)}_{\mathrm{st}}[\frac{1}{E}]$ (resp. $A^{(3)}_{\mathrm{st}}[\frac{1}{E}]$, $A^{(2)}[\frac{1}{E}]$, $A^{(3)}[\frac{1}{E}]$).

Lemma 2.3.2. (1) *$A^{(i)}_{\mathrm{st}} \subset B^{(i)}_{\mathrm{st}} \subset B^{(i)}_{\mathrm{st}}[\frac{1}{p}]$ and $A^{(i)} \subset B^{(i)} \subset B^{(i)}[\frac{1}{p}]$ for $i = 2, 3$.*

(2) *$B^{(2)}_{\mathrm{st}} \cap A^{(2)}_{\mathrm{st}}[\frac{1}{p}] = A^{(2)}_{\mathrm{st}}$ and $B^{(2)} \cap A^{(2)}[\frac{1}{p}] = A^{(2)}$.*

Proof. Here we only prove the case $A^{(2)}$; the proofs for $A^{(2)}_{\mathrm{st}}$, $A^{(3)}$ and $A^{(3)}_{\mathrm{st}}$ are almost the same.

By Proposition 2.2.8, $A^{(2)}$ is a subring of $A^{(2)}_{\max} \subset K_0[\![x, z]\!]$. So $A^{(2)}$, and hence $A^{(2)}[\frac{1}{E}]$, is an integral domain. Then $B^{(2)}$ has no p-torsion: Assume that $x \in B^{(2)}$ with $px = 0$. Suppose that $x_n \in A^{(2)}[\frac{1}{E}]$ is such that $x = x_n \bmod p^n$. Then $px_n = 0 \bmod p^n A^{(2)}[\frac{1}{E}]$. Since $A^{(2)}[\frac{1}{E}]$ is a domain, $x_n = 0 \bmod p^{n-1}$. Hence $x = 0$. As $B^{(2)}$ has no p-torsion, we see that $B^{(2)} \subset B^{(2)}[\frac{1}{p}]$. To see that the natural map $A^{(2)} \to B^{(2)}$ is injective, it suffices to show that $A^{(2)}/pA^{(2)}$ injects to $A^{(2)}/pA^{(2)}[\frac{1}{u}] = B^{(2)}/pB^{(2)}$. Clearly, this is equivalent to $A^{(2)}/pA^{(2)}$ having no u-torsion. Note that $A^{(2)}$ is obtained by taking the prismatic envelope of $A^{\widehat{\otimes}2} = W(k)[\![x, z]\!]$ for the ideal $I = (z)$. As mentioned before, we can apply [8, Proposition 3.13] to our situation. So $A^{(2)}$ is flat over A and hence $A^{(2)}/pA^{(2)}$ has no u-torsion as desired.

Now we can regard $B^{(2)}$ and $A^{(2)}[\frac{1}{p}]$ as subrings of $B^{(2)}[\frac{1}{p}]$. In particular, $B^{(2)} \cap A^{(2)}[\frac{1}{p}]$ makes sense and contains $A^{(2)}$. For any $x \in B^{(2)} \cap A^{(2)}[\frac{1}{p}]$, if $x \notin A^{(2)}$ but $px \in A^{(2)}$, then the image of $y = px$ inside $A^{(2)}/pA^{(2)}$ is nonzero but the image of y in $B^{(2)}/pB^{(2)}$ is zero. This contradicts $A^{(2)}/pA^{(2)}$ injecting to $B^{(2)}/pB^{(2)}$. So such an x cannot exist and we have $B^{(2)} \cap A^{(2)}[\frac{1}{p}] = A^{(2)}$ as required. ∎

By [8, Lemma 3.9], any prism (B, J) admits a perfection $(B, J)_{\mathrm{perf}} = (B_{\mathrm{perf}}, JB_{\mathrm{perf}})$.

Remark 2.3.3. In [8], the underlying δ-ring of $(B, J)_{\mathrm{perf}}$ is denoted by (B_∞, JB_∞), and B_{perf} is defined as the direct perfection of B in the category of δ-rings. In this paper, we write B_{perf} as the (p, J)-adic completion of $\mathrm{colim}_\varphi B$, which also coincides with the derived (p, I)-completion of $\mathrm{colim}_\varphi B$ (see [8, Lemma 3.9]).

Lemma 2.3.4. *$(A^{(2)})_{\mathrm{perf}}$ and $(A^{(2)}_{\mathrm{st}})_{\mathrm{perf}}$ are A-flat.*

Proof. We have seen that $A^{(2)}$ is A-flat via i_1. And it is easy to see that φ on A is flat. Since i_1 is a δ-map, we have $\varphi^n \circ i_1 = i_1 \circ \varphi^n$, which is flat. So $\mathrm{colim}_\varphi A^{(2)}$ is flat over A. In particular, A_{perf} is (p, E)-complete flat over A. Now since A is Noetherian, by [33, Tag 0912] we find that $(A^{(2)})_{\mathrm{perf}}$ is A-flat. The proof for $(A^{(2)}_{\mathrm{st}})_{\mathrm{perf}}$ is the same. ∎

2.4. Embedding $A^{(2)}$ and $A^{(2)}_{\mathrm{st}}$ to A_{inf}

Let $A_{\mathrm{inf}} = W(\mathcal{O}^{\flat}_{\mathbb{C}_p})$. Then there is a surjection $\theta : A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_p}$ and $\operatorname{Ker}\theta = (E)$. Let B^+_{dR} be the $\operatorname{Ker}\theta$-adic completion of $A_{\mathrm{inf}}[\frac{1}{p}]$.

Definition 2.4.1. Let $\mathbb{A}_{\max}$ be the p-adic completion of the A_{inf}-subalgebra of B^+_{dR} generated by E/p.

It can be easily seen that $\varphi(E/p) := \varphi(E)/p \in A_{\mathrm{cris}} \subset \mathbb{A}_{\max}$ is well-defined and it extends the Frobenius structure on A_{inf} to an endomorphism on $\mathbb{A}_{\max}$.

Let $\{\varpi_n\}_{n\geq 0}$ be a compatible system of p^n-th roots of $\varpi_0 = \varpi$ and $\{\zeta_n\}_{n\geq 0}$ be a compatible system of p^n-th roots of 1. Write $\varpi^{\flat} := (\varpi_n)_{n\geq 0}$ and $\zeta^{\flat} := (\zeta_n)_{n\geq 0}$ as elements in $\mathcal{O}^{\flat}_{\mathbb{C}_p}$, and let $u = [\varpi^{\flat}]$, $\epsilon = [\zeta^{\flat}]$, $v = \epsilon u$ and $\mu = \epsilon - 1$ be elements inside A_{inf}. We can regard $W(k)[\![x, y]\!]$ as a subring of A_{inf} via $x \mapsto u$ and $y \mapsto v$. Consider $z' = \frac{u-v}{E} \in A_{\mathrm{inf}}[\frac{1}{E}]$. Since $u - v = u(\epsilon - 1)$ is clearly inside $\operatorname{Ker}\theta$ and $\operatorname{Ker}\theta = EA_{\mathrm{inf}}$, we conclude that $z' \in A_{\mathrm{inf}}$. Hence we have a natural map (of δ-rings) $\iota_A : \widetilde{A^{(2)}} \to A_{\mathrm{inf}}$ via $z \mapsto z'$, which naturally extends to $\iota_A : A^{(2)} \to A_{\mathrm{inf}}$ because the (p, E)-topology of $A^{(2)}$ matches the weak topology of A_{inf}. Similarly, we have a map of δ-rings $\iota_{\mathrm{st}} : A^{(2)}_{\mathrm{st}} \to A_{\mathrm{inf}}$ via $x \mapsto u$ and $\mathfrak{y} \mapsto \epsilon - 1$ and $w \mapsto \frac{\epsilon-1}{E}$.

Remark 2.4.2. Once we know that $A^{(2)}$ is the self-product of A inside X_{Δ} with $X = \operatorname{Spf}(\mathcal{O}_K)$ as explained in Section 4.1, the map ι_A can be constructed as follows: First we fix an embedding $A \to A_{\mathrm{inf}}$ by sending $x \mapsto u = [\varpi^{\flat}]$. Then $A \to A_{\mathrm{inf}}$ with $x \mapsto v = \epsilon u$ is another map of prisms. By the universal property of $A^{(2)}$, these two maps define a map $\iota_A : A^{(2)} \to A_{\mathrm{inf}}$. Clearly, ι_A depends on the choice of $\varpi^{\flat} = (\varpi_n)_{n\geq 0}$ and $\zeta^{\flat} = (\zeta_n)_{n\geq 0}$. Also ι_A is a special case of $\iota^{(2)}_{\gamma}$ defined by (4.5) in Section 4.3. Indeed, if $\gamma([w^{\flat}]) = [\zeta^{\flat}][w^{\flat}]$ then $\iota_A = \iota^{(2)}_{\gamma}$. Similar comments also apply to ι_{st}.

Proposition 2.4.3. *There is a unique embedding $A^{(2)}_{\max} \hookrightarrow \mathbb{A}_{\max}$ such that*

$$\begin{array}{ccccc} W(k)[\![x,y]\!] & \hookrightarrow & A_{\mathrm{inf}} & & \\ \downarrow & & \downarrow & & \\ A^{(2)}_{\max} & \hookrightarrow & \mathbb{A}_{\max} & \hookrightarrow & B^+_{\mathrm{dR}} \end{array}$$

commutes. Furthermore, $\operatorname{Fil}^i B^+_{\mathrm{dR}} \cap A^{(2)}_{\max} = \operatorname{Fil}^i A^{(2)}_{\max}$. The same holds when $A^{(2)}_{\max}$ is replaced by $A^{(2)}_{\mathrm{st},\max}$.

Proof. In the following, we only treat the case of $A^{(2)}_{\mathrm{st},\max}$; the proof for $A^{(2)}_{\max}$ is the same by noting that $z = uw$ in A_{inf}.

The uniqueness is apparent. To show the existence of the embedding, it is enough to show $\gamma_i(w) \in \mathbb{A}_{\max}$ for all $i \geq 1$.

It is well-known that $\mathbb{A}_{\max}$ is isomorphic to the p-adic completion of $A_{\mathrm{inf}}[\frac{u^e}{p}]$, and $\mathbb{A}_{\max}[\frac{1}{p}]$ is a Banach $\mathbb{Q}_p$-algebra, which is the completion of $A_{\mathrm{inf}}[\frac{1}{p}]$ under the norm $|\cdot|_{p^{-1}}$

such that

$$|x|_{p^{-1}} = \sup_n p^{-n} |x_n|_{\mathcal{O}_C^\flat},$$

where $x = \sum_{n \gg 0} [x_n] p^n \in A_{\inf}[\frac{1}{p}]$. And for $x \in \mathbb{A}_{\max}[\frac{1}{p}]$, we have $x \in \mathbb{A}_{\max}$ if and only if $|x|_{p^{-1}} \leq 1$. Moreover, $|\cdot|_{p^{-1}}$ is multiplicative. So now it is enough to show that for $x = \gamma_i(w)$ considered as an element inside $\mathbb{A}_{\max}[\frac{1}{p}]$, we have $|x^{p-1}|_{p^{-1}} \leq 1$. Indeed, by [5, Proposition 3.17], $\xi := \mu/\varphi^{-1}(\mu)$ is a generator of $\operatorname{Ker}\theta$ with $\mu = \epsilon - 1$. In particular, $w = \mu/E = a\varphi^{-1}(\mu) \in A_{\inf}$ with $a \in A_{\inf}^\times$. And we can check $\overline{w}^{p-1} = c\overline{u}^e$ inside $\mathcal{O}_C^\flat = A_{\inf}/pA_{\inf}$, with c a unit. So $w^{p-1} = au^e + bp$ with $a, b \in A_{\inf}$, and

$$x^{p-1} = \frac{(au^e + bp)^i}{(i!)^{p-1}}.$$

Using the fact that $v_p(i!) < \frac{i}{p-1}$, one can show each term in the binomial expansion on the right-hand side of the equation has $|\cdot|_{p^{-1}}$-norm less than or equal to 1, so in particular $|x^{p-1}|_{p^{-1}} \leq 1$.

To prove that $\operatorname{Fil}^i B_{\mathrm{dR}}^+ \cap A_{\mathrm{st,max}}^{(2)} = \operatorname{Fil}^i A_{\mathrm{st,max}}^{(2)}$, it suffices to show $EB_{\mathrm{dR}}^+ \cap A_{\mathrm{st,max}}^{(2)}[\frac{1}{p}] = EA_{\mathrm{st,max}}^{(2)}[\frac{1}{p}]$. By Proposition 2.2.8, it remains to prove that the map

$$\theta : D_w = A_{\mathrm{st,max}}^{(2)}[\tfrac{1}{p}]/E \to B_{\mathrm{dR}}^+/E = \mathbb{C}_p$$

is injective. Let $f(w) = \sum_{i \geq 0} a_i \gamma_i(w) \in \operatorname{Ker}\theta$ where $a_i \in \mathcal{O}_K$ limits to 0 p-adically. Then $f(w_0) = 0$ with $w_0 := \theta(w) = \theta(\frac{\epsilon-1}{E}) \in \mathbb{C}_p$. Note $v_p(w_0) \geq \frac{1}{p-1}$ since $\frac{\epsilon-1}{\varphi^{-1}(\epsilon)-1}$ is another generator of the kernel of $\theta : A_{\inf} \to \mathcal{O}_{\mathbb{C}_p}$, and we have $v_p(w_0) = v_p(\theta(\varphi^{-1}(\epsilon) - 1)) = \frac{1}{p-1}$. Since we aim to show that $f = 0$, without loss of generality we can assume that K contains $p_1 = p^{\frac{1}{p-1}}$. Noting that $v_p(i!) \leq \frac{1}{p-1}$, we conclude that $\frac{w_0}{p_1}$ is a root of $f(p_1 w)$ which is in $\mathcal{O}_K\langle w\rangle$. By the Weierstrass preparation theorem, w_0 is algebraic over K unless $f = 0$. By the lemma below, $w_0 := \theta(w) \in \mathbb{C}_p$ is transcendental over K and hence $f = 0$. ∎

Lemma 2.4.4. *$w_0 = \theta(\frac{\epsilon-1}{E})$ is transcendental over K.*

Proof. If w_0 is contained in an algebraic extension L over K, we define $L_{0,\infty} = \bigcup_n L(\varpi_n)$. For $g \in G_{L_{0,\infty}}$, we have

$$\theta\left(g\left(\frac{\epsilon - 1}{E}\right)\right) = g(w_0) = w_0 = \theta\left(\frac{\epsilon - 1}{E}\right).$$

Since $G_{L_{0,\infty}}$ fixes E, $\theta(\frac{g(\epsilon-1)-(\epsilon-1)}{E}) = 0$. This implies $g(\epsilon - 1) - (\epsilon - 1) \in \operatorname{Fil}^2 B_{\mathrm{dR}}^+$. Recall that for $t = \log \epsilon$, $t - (\epsilon - 1) \in \operatorname{Fil}^2 B_{\mathrm{dR}}^+$, so we have $g(t) - t \in \operatorname{Fil}^2 B_{\mathrm{dR}}^+$. But this cannot be true. Indeed, since $L_{0,\infty}$ can only contain finitely many p^n-th roots of 1, for $g \in G_{L_{0,\infty}}$, we have $g(t) = c(g)t$ with $c(g) \in \mathbb{Q}_p$ and $c(g) \neq 1$. This implies $g(t) - t = (c(g) - 1)t \in \operatorname{Fil}^1 B_{\mathrm{dR}}^+ \setminus \operatorname{Fil}^2 B_{\mathrm{dR}}^+$. ∎

Corollary 2.4.5. *The natural maps $\iota_A : A^{(2)} \to A_{\inf}$ and $\iota_{\mathrm{st}} : A_{\mathrm{st}}^{(2)} \to A_{\inf}$ are injective.*

To summarize, we have the following commutative diagram of rings inside B_{dR}^{+}:

$$\begin{array}{ccccc} A^{(2)} & \hookrightarrow & A_{\mathrm{st}}^{(2)} & \hookrightarrow & A_{\mathrm{inf}} \\ \downarrow & & \downarrow & & \downarrow \\ A_{\max}^{(2)} & \hookrightarrow & A_{\mathrm{st},\max}^{(2)} & \hookrightarrow & \mathbb{A}_{\max} \end{array}$$

3. Application to semistable Galois representations

In this section, we assume that $R = \mathcal{O}_K$. We explain how to use the period ring $A^{(2)}$ and $A_{\mathrm{st}}^{(2)}$ to understand lattices in crystalline and semistable representations. Roughly speaking, we are going to use $A^{(2)}$ and $A_{\mathrm{st}}^{(2)}$ to replace $\widehat{\mathcal{R}}$ in the theory of $(\varphi, \hat{G})$-modules developed in [30].

Let $K_\infty = \bigcup_{n=1}^{\infty} K(\varpi_n)$, $G_\infty := \mathrm{Gal}(\overline{K}/K_\infty)$ and $G_K := \mathrm{Gal}(\overline{K}/K)$. Recall that $A = \mathfrak{S} = W(k)[\![u]\!]$. Let S be the p-adic completion of $W(k)[\![u, \frac{E^i}{i!}, i \geq 1]\!]$, which is the PD envelope of $W(k)[u]$ for the ideal (E). It is clear that $S \subset O_{\max}$. We define φ and Fil^i on S induced from those on $O_{\max}$, in particular, $\mathrm{Fil}^i S = S \cap E^i O_{\max}[\frac{1}{p}]$. Note that A embeds to A_{inf} via $u \mapsto [\varpi^\flat]$, which is not stable under the G_K-action but only under the G_∞-action. For any $g \in G_K$, define $\underline{\varepsilon}(g) = \frac{g(u)}{u}$. It is clear that $\underline{\varepsilon}(g) = \epsilon^{a(g)}$ with $a(g) \in \mathbb{Z}_p$. We define *two* differential operators N_S and ∇_S on S by $N_S(f) = \frac{df}{du} u$ and $\nabla_S(f) = \frac{df}{du}$. We need ∇_S to treat crystalline representations.

3.1. *Kisin module attached to a semistable representation*

Fix $h \geq 0$, a *Kisin module of height h* is a finite free A-module $\mathfrak{M}$ with a semilinear endomorphism $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ such that $\mathrm{Coker}(1 \otimes \varphi_{\mathfrak{M}})$ is killed by E^h, where $1 \otimes \varphi_{\mathfrak{M}} : \mathfrak{M}^* := A \otimes_{\varphi, A} \mathfrak{M} \to \mathfrak{M}$ is the linearization of $\varphi_{\mathfrak{M}}$. Note that here we are using the classical setting of Kisin modules used in [30] but it is good enough for this paper. The following summarizes the results on Kisin modules attached to G_K-stable $\mathbb{Z}_p$-lattices in semistable representations. The details and proofs of these facts can be found in [30].

Let T be a G_K-stable $\mathbb{Z}_p$-lattice inside a semistable representation V of G_K with Hodge–Tate weights in $[0, h]$. We set

$$D := D_{\mathrm{st}}^*(V) = \mathrm{Hom}_{\mathbb{Q}_p, G_K}(V, B_{\mathrm{st}}),$$

which is the filtered (φ, N)-module attached to V, and $D_K := K \otimes_{K_0} D$. Then there exists a unique Kisin module $\mathfrak{M} := \mathfrak{M}(T)$ of height h attached to T such that:

(1) $\mathrm{Hom}_{\varphi, A}(\mathfrak{M}, A_{\mathrm{inf}}) \simeq T|_{G_\infty}$.

(2) There exists an S-linear isomorphism

$$\iota_S : S\left[\tfrac{1}{p}\right] \otimes_{\varphi, A} \mathfrak{M} \simeq D \otimes_{W(k)} S$$

compatible with φ on both sides.

(3) ι_S also induces an isomorphism $\mathrm{Fil}^h(S[\frac{1}{p}] \otimes_{\varphi,A} \mathfrak{M}) \simeq \mathrm{Fil}^h(D \otimes_{W(k)} S)$. The filtration on both sides are defined as follows:

$$\mathrm{Fil}^h\big(S\big[\tfrac{1}{p}\big] \otimes_{\varphi,A} \mathfrak{M}\big) := \big\{x \in S\big[\tfrac{1}{p}\big] \otimes_{\varphi,A} \mathfrak{M} \;\big|\; (1 \otimes \varphi_{\mathfrak{M}})(x) \in \mathrm{Fil}^h S\big[\tfrac{1}{p}\big] \otimes_A \mathfrak{M}\big\}.$$

To define a filtration on $\mathcal{D} := S \otimes_{W(k)} D$, we first extend the monodromy operator $N_D = N$ on D to $N_{\mathcal{D}}$ (resp. $\nabla_{\mathcal{D}}$) on $\mathcal{D}$ by letting $N_{\mathcal{D}} = 1 \otimes N_D + N_S \otimes 1$ (resp. $\nabla_{\mathcal{D}} = 1 \otimes N_D + \nabla_S \otimes 1$). Then we define $\mathrm{Fil}^i \mathcal{D}$ by induction: set $\mathrm{Fil}^0 \mathcal{D} = \mathcal{D}$ and

$$\mathrm{Fil}^i \mathcal{D} := \{x \in \mathcal{D} \mid N_{\mathcal{D}}(x) \in \mathrm{Fil}^{i-1} \mathcal{D},\ f_{\varpi}(x) \in \mathrm{Fil}^i D_K\},$$

where $f_{\varpi} : \mathcal{D} \to D_K$ is induced by $S \to \mathcal{O}_K$ via $u \mapsto \varpi$.

Remark 3.1.1 (Griffith transversality). We have $N_{\mathcal{D}}(\mathrm{Fil}^i \mathcal{D}) \subset \mathrm{Fil}^{i-1} \mathcal{D}$ from the construction of $\mathrm{Fil}^i \mathcal{D}$. This property is called *Griffith transversality*.

We only use $\nabla_{\mathcal{D}}$ when $N_D = 0$, that is, when V is crystalline. In this case, it is clear that $N_{\mathcal{D}} = u\nabla_{\mathcal{D}}$. So it is clear that $\nabla_{\mathcal{D}}(\mathrm{Fil}^i \mathcal{D}) \subset \mathrm{Fil}^{i-1} \mathcal{D}$.

For ease of notation, we will write $N = N_{\mathcal{D}}$ and $\nabla = \nabla_{\mathcal{D}}$ in the following. Let $T^\vee := \mathrm{Hom}_{\mathbb{Z}_p}(T, \mathbb{Z}_p)$ and $V^\vee := T^\vee \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ denote the dual representations. Then there exists an A_{inf}-linear injection

$$\iota_{\mathfrak{M}} : A_{\mathrm{inf}} \otimes_A \mathfrak{M} \to T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}, \tag{3.1}$$

which is compatible with the G_∞-actions (G_∞ acts on $\mathfrak{M}$ trivially) and φ on both sides. Applying $S \otimes_{\varphi,A} -$ and using $\iota_S := S \otimes_{\varphi,A} \iota_{\mathfrak{M}}$, we obtain the commutative diagram

$$\begin{array}{ccc} A_{\mathrm{cris}}\big[\tfrac{1}{p}\big] \otimes_{\varphi,A} \mathfrak{M} & \xrightarrow{S \otimes_{\varphi,A} \iota_{\mathfrak{M}}} & V^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \\ \wr\big\downarrow A_{\mathrm{cris}} \otimes_S \iota_S & & \big\| \\ A_{\mathrm{cris}} \otimes_{W(k)} D & \xrightarrow{\alpha} & V^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}} \end{array}$$

where the map α is built by the classical comparison

$$B_{\mathrm{st}} \otimes_{K_0} D^*_{\mathrm{st}}(V) \simeq V^\vee \otimes_{\mathbb{Q}_p} B_{\mathrm{st}},$$

and α is G_K-stable on both sides. The left side of α is defined by

$$\forall g \in G_K,\ \forall x \in D, \quad g(x) = \sum_{i=0}^{\infty} N^i(x)\gamma_i(\log(\underline{\varepsilon}(g)))$$

Therefore, if we regard $\mathfrak{M}^* := A \otimes_{\varphi,A} \mathfrak{M}$ as an A-submodule of $V^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{cris}}$ via the injection $\iota^* := S \otimes_{\varphi,A} \iota_A$, one can show that

$$\forall g \in G_K,\ \forall x \in \mathfrak{M}^*, \quad g(x) = \sum_{i=0}^{\infty} N^i_{\mathcal{D}}(x)\gamma_i(\log(\underline{\varepsilon}(g))). \tag{3.2}$$

When V is crystalline, or equivalently $N_D = 0$, we have [28, Section 8.1]

$$\forall g \in G_K,\ \forall x \in \mathfrak{M}^*, \quad g(x) = \sum_{i=0}^{\infty} \nabla^i_{\mathcal{D}}(x)\gamma_i(u\underline{\varepsilon}(g)). \tag{3.3}$$

3.2. *Descent of the G_K-action*

Let us first discuss the G_K-action on $\mathfrak{M} \subset T^\vee \otimes_{\mathbb{Z}_p} A_{\inf}$ via $\iota_{\mathfrak{M}}$ in (3.1) in more detail. We select an A-basis $e_1, \dots, e_d$ of $\mathfrak{M}$ so that $\varphi(e_1, \dots, e_d) = (e_1, \dots, e_d)\mathfrak{A}$ with $\mathfrak{A} \in \mathrm{M}_d(A)$. Then there exists a matrix $B \in \mathrm{M}_d(A)$ such that $\mathfrak{A}B = B\mathfrak{A} = E^h I_d$. For any $g \in G_K$, let X_g be the matrix such that

$$g(e_1, \dots, e_d) = (e_1, \dots, e_d)X_g.$$

In this section, we are interested in where the entries of X_g lie.

Theorem 3.2.1. *The entries of X_g are in $A^{(2)}_{\mathrm{st}}$. If V is crystalline and $g(u) - u = Ez$ then $X_g \in \mathrm{M}_d(A^{(2)})$.*

First, it is well-known that $W(\mathbb{C}_p^\flat) \otimes_{A_{\inf}} \iota_{\mathfrak{M}}$ is an isomorphism. So $X_g \in \mathrm{M}_d(W(\mathbb{C}_p^\flat))$. Since the G_K-actions and φ commute, we have

$$\mathfrak{A}\varphi(X_g) = X_g g(\mathfrak{A}).$$

Define

$$\mathrm{Fil}^h \mathfrak{M}^* := \{x \in \mathfrak{M}^* \mid (1 \otimes \varphi_{\mathfrak{M}})(x) \in E^h \mathfrak{M}\}.$$

Since $\mathfrak{M}$ has E-height h, it is easy to show that $\mathrm{Fil}^h \mathfrak{M}^*$ is a finite free A-module and $\mathrm{Fil}^h \mathcal{D}$ is generated by $\mathrm{Fil}^h \mathfrak{M}^*$.

To be more precise, let $\{e_i^* := 1 \otimes e_i \mid i = 1, \dots, d\}$ be an A-basis of $\mathfrak{M}^*$. It is easy to check that $(\alpha_1, \dots, \alpha_d) = (e_1^*, \dots, e_d^*)B$ is an A-basis of $\mathrm{Fil}^h \mathfrak{M}^*$, and it is also an $S[\frac{1}{p}]$-basis of $\mathrm{Fil}^h \mathcal{D}$. So for any $g \in G_K$, we have $g(\alpha_j) = \sum_{i=0}^{\infty} N^i(\alpha_j)\gamma_i(\log(\underline{\varepsilon}(g)))$. By Griffith transversality in Remark 3.1.1, $N(\mathrm{Fil}^i \mathcal{D}) \subset \mathrm{Fil}^{i-1} \mathcal{D}$, we have

$$g(\alpha_j) = \sum_{i=0}^{h} N^i(\alpha_j)E^i \gamma_i\left(\frac{\log(\underline{\varepsilon}(g))}{E}\right) + \sum_{i>h}^{\infty} N^i(\alpha_j)\gamma_i(E)\left(\frac{\log(\underline{\varepsilon}(g))}{E}\right)^i. \tag{3.4}$$

Since $N^i(\alpha_j)E^i \in \mathrm{Fil}^h \mathcal{D}$ and $\gamma_i(E)$ in $O_{\max}$, and $\{w^n\}$ converges to 0 inside $A^{(2)}_{\mathrm{st},\max}$, we see that $g(\alpha_1, \dots, \alpha_d) = (\alpha_1, \dots, \alpha_d)Y_g$ with $Y_g \in A^{(2)}_{\mathrm{st},\max}[\frac{1}{p}]$.

If V is crystalline, using (3.3), we have

$$g(\alpha_j) = \sum_{i=0}^{h} \nabla^i(\alpha_j)E^i \gamma_i\left(\frac{u\underline{\varepsilon}(g)}{E}\right) + \sum_{i>h}^{\infty} \nabla^i(\alpha_j)\gamma_i(E)\left(\frac{u\underline{\varepsilon}(g)}{E}\right)^i.$$

If g is chosen so that $g(u) - u = Ez$, then a similar argument shows that $g(\alpha_1, \dots, \alpha_d) = (\alpha_1, \dots, \alpha_d)Y_g^\nabla$ with $Y_g^\nabla \in A^{(2)}_{\max}[\frac{1}{p}]$.

Now $g(e_1^*, \dots, e_d^*) = (e_1^*, \dots, e_d^*)\varphi(X_g)$. Using similar arguments, we see that the $\varphi(X_g)$'s entries are in $A^{(2)}_{\mathrm{st},\max}[\frac{1}{p}]$ and $A^{(2)}_{\max}[\frac{1}{p}]$ respectively. Since $(\alpha_1, \dots, \alpha_d) = (e_1^*, \dots, e_d^*)B$ by definition of B, we conclude that

$$\varphi(X_g)g(B) = BY_g.$$

Using the formulas $\mathfrak{A}\varphi(X_g) = X_g g(\mathfrak{A})$ and $\mathfrak{A}B = B\mathfrak{A} = E^h I_d$, we conclude that $Y_g = (\frac{g(E)}{E})^h X_g$. Write $r = \frac{g(E)}{E}$. We claim that r is a unit in $A^{(2)}_{\mathrm{st}}$. Indeed, $\frac{g(E)}{E} = \frac{E(u\epsilon^{a(g)})}{E(u)} = \sum_{i=0}^{e} E^{(i)}(u)\frac{u^i(\epsilon^{a(g)}-1)^i}{Ei!}$ is again inside $A^{(2)}_{\mathrm{st}}$, where $E^{(i)}$ means the i-th derivative of E. And it is easy to show that $g(E)$ is also a distinguished element of $A^{(2)}_{\mathrm{st}}$, so by [8, Lemma 2.24], r is a unit. Similarly, when $g(u) - u = Ez$, we will have $r = \frac{g(E)}{E} \in (A^{(2)})^\times$. Hence

$$E^h X_g = r^{-h}\mathfrak{A}\varphi(X_g)g(B). \tag{3.5}$$

Now we can apply Propositions 2.2.13 and 2.3.1 (5) to the above formula; we conclude that for $g \in G_K$ (resp. $g \in G_K$ such that $g(u) - u = Ez$ and V is crystalline), X_g has entries in $A^{(2)}_{\mathrm{st}}[\frac{1}{p}]$ (resp. $A^{(2)}[\frac{1}{p}]$).

To complete the proof of Theorem 3.2.1, it suffices to show that the entries of X_g are in $A^{(2)}_{\mathrm{st}}$ (resp. $A^{(2)}$). Unfortunately, removing "$\frac{1}{p}$" is much harder; it needs Sections 4.2 and 4.3. For the remainder of this subsection, we only show that the proof of Theorem 3.2.1 can be reduced to the case where $g = \tilde{\tau}$ for a special $\tilde{\tau} \in G_K$.

Let $L = \bigcup_{n=1}^{\infty} K_\infty(\zeta_{p^n})$, $K_{1^\infty} := \bigcup_{n=1}^{\infty} K(\zeta_{p^n})$, $\hat{G} := \mathrm{Gal}(L/K)$ and $H_K := \mathrm{Gal}(L/K_\infty)$. If $p > 2$ then it is known that $\hat{G} \simeq \mathrm{Gal}(L/K_{1^\infty}) \rtimes H_K$ with $\mathrm{Gal}(L/K_{1^\infty}) \simeq \mathbb{Z}_p$. Let τ be a topological generator of $\mathrm{Gal}(L/K_{1^\infty})$. We have $\tau(u) = \epsilon^a u$ with $a \in \mathbb{Z}_p^\times$. Without loss of generality, we may assume that $\tau(u) = \epsilon u$. If $p = 2$ then we can still select $\tau \in \hat{G}$ so that $\tau(u) = \epsilon u$ and τ, H_K topologically generate $\hat{G}$. Pick a lift $\tilde{\tau} \in G_K$ of τ. Clearly, we have $\tilde{\tau}(u) - u = Ez$.

Proposition 3.2.2. *For $g = \tilde{\tau}$, the entries of X_g are in $A^{(2)}_{\mathrm{st}}$, and if further V is crystalline, then $X_g \in \mathrm{M}_d(A^{(2)})$.*

Lemma 3.2.3. *Proposition 3.2.2 is equivalent to Theorem 3.2.1.*

Proof. Since $\hat{G}$ is topologically generated by τ and H_K, it follows that G_K is topologically generated by G_∞ and $\tilde{\tau}$. And we have $\tau(u) - u = (\epsilon - 1)u = Ez$. Now if $X_{\tilde{\tau}}$ has coefficients in $A^{(2)}_{\mathrm{st}}$ and $X_g = I_d$ for all $g \in G_\infty$ then to show that $X_g \in A^{(2)}_{\mathrm{st}}$ for all $g \in G_K$, it suffices to show that $X_{\tilde{\tau}^{p^n}}$ converges to I_d inside $\mathrm{M}_d(A^{(2)}_{\mathrm{st}})$. Since $A^{(2)}_{\mathrm{st}}$ is closed in $A^{(2)}_{\mathrm{st,max}}$ by Proposition 2.3.1 (5), it suffices to show that $X_{\tilde{\tau}^{p^n}}$ converges inside $A^{(2)}_{\mathrm{st,max}}$. Since $X_g = (\frac{E}{g(E)})^r Y_g$ and Y_g is defined by (3.4), we easily check that $X_{\tilde{\tau}^{p^n}}$ converges to I_d in $A^{(2)}_{\mathrm{st,max}}$ by using that $\underline{\varepsilon}(\tilde{\tau}^{p^n})$ converges to 0 in the $(p, \epsilon - 1)$-topology. The proof for the crystalline case is similar by replacing $A^{(2)}_{\mathrm{st}}$ with $A^{(2)}$. ■

So to complete the proof of Theorem 3.2.1, it remains to prove Proposition 3.2.2. We will do this in Section 4.3. Briefly speaking, for $g = \tilde{\tau}$, we have shown that the linearization of the g-action defines a φ-equivariant isomorphism

$$f_g : \mathfrak{M} \otimes_{A,\iota_g} A^{(2)}_{\mathrm{st}}[\tfrac{1}{p}] \simeq \mathfrak{M} \otimes_A A^{(2)}_{\mathrm{st}}[\tfrac{1}{p}]$$

of $A^{(2)}_{\mathrm{st}}[\frac{1}{p}]$-modules. Since $g(u) - u = Ez$ and V is crystalline, f_g defines a φ-equivariant isomorphism

$$f_g : \mathfrak{M} \otimes_{A,\iota_g} A^{(2)}[\tfrac{1}{p}] \simeq \mathfrak{M} \otimes_A A^{(2)}[\tfrac{1}{p}]$$

of $A^{(2)}[\frac{1}{p}]$-modules. Here $\iota_g : A \to A^{(2)}_{\mathrm{st}}$ (resp. $\iota_g : A \to A^{(2)}$) is defined by $u \mapsto g(u)$. On the other hand, by [35, Theorem 5.6], we will see that the g-action on $T^\vee \otimes W(\mathbb{C}_p^\flat)$ also descends to a φ-equivariant morphism c of $B^{(2)}$-modules, where $B^{(2)}$ is the p-adic completion of $A^{(2)}[\frac{1}{E}]$. Then by comparing c and f_g using the technique developed in Section 4.2, we will deduce Proposition 3.2.2 from Lemma 2.3.2.

Remark 3.2.4. Our original strategy to prove Theorem 3.2.1 was to demonstrate that $A^{(2)}_{\mathrm{st}}[\frac{1}{p}] \cap W(\mathbb{C}_p^\flat) = A^{(2)}_{\mathrm{st}}$ (resp. $A^{(2)}[\frac{1}{p}] \cap W(\mathcal{O}^\flat_{\mathbb{C}_p}) = A^{(2)}$). This is equivalent to $A^{(2)}/p$ and $A^{(2)}_{\mathrm{st}}/p$ injecting in $\mathbb{C}_p^\flat$. Unfortunately, it has not worked out, though we can show that $\widetilde{A^{(2)}}/p$ and $\widetilde{A^{(2)}_{\mathrm{st}}}/p$ inject in $\mathbb{C}_p^\flat$.

3.3. *Prismatic $(\varphi, \hat{G})$-modules*

In this subsection, we show that the base ring $\widehat{\mathcal{R}}$ for the theory of $(\varphi, \hat{G})$-modules can be replaced by $A^{(2)}_{\mathrm{st}}$. To this end, we build a new theory of $(\varphi, \hat{G})$-modules with a new base ring $A^{(2)}_{\mathrm{st}}$. Since the idea of this new theory is almost the same as that of the old one, we will use *classical* to indicate we are using the theory over $\widehat{\mathcal{R}}$. For example, when we say classical $(\varphi, \hat{G})$-module, we mean a $(\varphi, \hat{G})$-module over $\widehat{\mathcal{R}}$. Recall $L = \bigcup_{n=1}^\infty K_\infty(\zeta_{p^n})$, $\hat{G} := \mathrm{Gal}(L/K)$ and $H_K := \mathrm{Gal}(L/K_\infty)$. Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}^\flat_{\mathbb{C}_p}$ and set $I_+ = W(\mathfrak{m})$ so that $A_{\mathrm{inf}}/I_+ = W(\bar{k})$. For any subring $B \subset A_{\mathrm{inf}}$ set $I_+B = B \cap I_+$. Let $t = \log \epsilon$, $t^{(i)} = t^{r(i)}\gamma_{\tilde{q}(i)}(\frac{t^{p-1}}{p})$ where $i = (p-1)\tilde{q}(i) + r(i)$ with $0 \le r(i) < p-1$. Recall that $\widehat{\mathcal{R}} := A_{\mathrm{inf}} \cap \mathcal{R}_{K_0}$ where

$$\mathcal{R}_{K_0} := \left\{ \sum_{i=0}^{\infty} f_i t^{(i)} \;\middle|\; f_i \in S[\tfrac{1}{p}], f_i \to 0 \text{ } p\text{-adically} \right\}.$$

Lemma 3.3.1. (1) *As a subring of A_{inf}, $A^{(2)}_{\mathrm{st}}$ is stable under the G_K-action, and the G_K-action factors through $\hat{G}$.*

(2) $A^{(2)}_{\mathrm{st}}/I_+A^{(2)}_{\mathrm{st}} = W(k)$.

(3) $I_+A^{(2)} \subset uA^{(2)}_{\mathrm{st}}$.

(4) $\varphi(A^{(2)}_{\mathrm{st}}) \subset \widehat{\mathcal{R}}$.

Proof. (1) It is clear that the G_K-action is stable on $W(k)[\![u, \epsilon - 1]\!]$. Since $A^{(2)}_{\mathrm{st}}$ is the (p, E)-completion of $W(k)[\![u, \epsilon - 1]\!][\delta^i(w), i \ge 0]$, to show that $A^{(2)}_{\mathrm{st}}$ is G_K-stable, it suffices to show that $g(w) \in A^{(2)}_{\mathrm{st}}$ (since g and δ commute, if $g(x)$ is in $A^{(2)}_{\mathrm{st}}$ then so is $g(\delta(x))$). Now $Ew = \epsilon - 1$, so $g(E)g(w) = g(\epsilon - 1) = \epsilon^{a(g)} - 1$. Thus $g(w) = \frac{E}{g(E)}\frac{\epsilon^{a(g)}-1}{E}$. By [8, Lemma 2.24], $E/g(E)$ is a unit in $A^{(2)}_{\mathrm{st}}$, so $g(w) \in A^{(2)}_{\mathrm{st}}$.

(2) It is clear that both $u, \epsilon - 1$ are in I_+. Hence $w \in I_+$ because $Ew = \epsilon - 1 \in I_+$ and $E = p \bmod I_+$. For any $x = \sum_{i=0}^\infty p^i[x_i] \in A_{\mathrm{inf}}$, $x \in I_+$ if and only if $x_i \in \mathfrak{m}$. Then it is easy to check that $\delta(I_+) \subset I_+$, and consequently all $\delta^i(w)$ are in I_+. So $I_+A^{(2)}_{\mathrm{st}}$ is topologically generated by $u, y = \epsilon - 1, \delta^i(w), i \ge 0$, and hence $A^{(2)}_{\mathrm{st}}/I_+A^{(2)}_{\mathrm{st}} = W(k)$ as required.

(3) $I_+A^{(2)}$ is topologically generated by $u, v = \epsilon u, \delta^i(z), i \geq 0$. Hence (3) follows from $z = uw$ and $\delta^n(z) = u^{p^n}\delta^n(w)$.

(4) Since $A^{(2)}_{\mathrm{st}} \subset A^{(2)}_{\mathrm{st,max}}$, it suffices to show that $\varphi(A^{(2)}_{\mathrm{st,max}}) \subset \mathcal{R}_{K_0}$. Since $\varphi(O_{\max}) \subset A[\![\frac{E^p}{p}]\!] \subset S$, it suffices to show that $\varphi(\gamma_n(w)) \in \mathcal{R}_{K_0}$. Note that $\varphi(E) = p\nu$ with $\nu \in A[\![\frac{E^p}{p}]\!]^\times$ and $\gamma_i(\epsilon - 1) \in \mathcal{R}_{K_0}$. And we have

$$\varphi(w) = \varphi\left(\frac{\epsilon - 1}{E}\right) = \nu^{-1}(\epsilon - 1)\sum_{i=1}^{p}\left(\binom{p}{i}/p\right)(\epsilon - 1)^{i-1},$$

which is a polynomial with coefficients in $\mathbb{Z}$ and in variables ν^{-1} and $\gamma_i(\epsilon - 1)$. In particular, $\varphi(\gamma_n(w)) \in \mathcal{R}_{K_0}$ by the basic properties of divided powers. ∎

Definition 3.3.2. A (*finite free*) $(\varphi, \hat{G})$-*module of height* h is a (finite free) Kisin module $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ of height h together with an $A^{(2)}_{\mathrm{st}}$-semilinear $\hat{G}$-action on $\widehat{\mathfrak{M}} := A^{(2)}_{\mathrm{st}} \otimes_A \mathfrak{M}$ such that

(1) the actions of φ and $\hat{G}$ on $\widehat{\mathfrak{M}}$ commute;

(2) $\mathfrak{M} \subset \widehat{\mathfrak{M}}^{H_K}$;

(3) $\hat{G}$ acts on $\widehat{\mathfrak{M}}/I_+A^{(2)}_{\mathrm{st}}$ trivially.

The category of $(\varphi, \hat{G})$-modules consists of the above objects, and a morphism between two $(\varphi, \hat{G})$-modules is a morphism of Kisin modules that commutes with the actions of $\hat{G}$. Given a $(\varphi, \hat{G})$-module $\widehat{\mathfrak{M}} := (\mathfrak{M}, \varphi, \hat{G})$, we define a $\mathbb{Z}_p$-representation of G_K,

$$\widehat{T}(\widehat{\mathfrak{M}}) := \mathrm{Hom}_{A^{(2)}_{\mathrm{st}},\varphi}(A^{(2)}_{\mathrm{st}} \otimes_A \mathfrak{M}, A_{\mathrm{inf}}).$$

Since $\varphi(A^{(2)}_{\mathrm{st}}) \subset \widehat{\mathcal{R}}$, given a $(\varphi, \hat{G})$-module $\widehat{\mathfrak{M}} := (\mathfrak{M}, \varphi, \hat{G})$ defined as above, $(\mathfrak{M}, \varphi)$ together with the $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,A} \mathfrak{M}$ is a *classical* $(\varphi, \hat{G})$-module $\widehat{\mathfrak{M}}_c$. It is easy to check that $\widehat{T}(\widehat{\mathfrak{M}}) = \widehat{T}(\widehat{\mathfrak{M}}_c) := \mathrm{Hom}_{\widehat{\mathcal{R}},\varphi}(\widehat{\mathcal{R}} \otimes_{\varphi,A} \mathfrak{M}, A_{\mathrm{inf}})$.

Theorem 3.3.3. *The functor* $\widehat{T}$ *from the category of* $(\varphi, \hat{G})$*-modules of height* h *to the category of* G_K*-stable* $\mathbb{Z}_p$*-lattices in semistable representations with Hodge–Tate weights in* $[0, h]$ *is an anti-equivalence.*

Proof. Given a $(\varphi, \hat{G})$-module $\widehat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G})$, $\widehat{\mathfrak{M}}_c$ is a classical $(\varphi, \hat{G})$-module. So $\widehat{T}(\widehat{\mathfrak{M}}) = \widehat{T}(\widehat{\mathfrak{M}}_c)$ is a lattice inside semistable representations with Hodge–Tate weights in $[0, h]$. Conversely, given a lattice in a semistable representation T with Hodge–Tate weights in $[0, h]$, following the proof of the existence of a classical $(\varphi, \hat{G})$-module $\widehat{\mathfrak{M}}$ such that $\widehat{T}(\widehat{\mathfrak{M}}) = T$, it suffices to show that for any $g \in G_K$, $g(\mathfrak{M}) \subset A^{(2)}_{\mathrm{st}} \otimes_A \mathfrak{M}$; here $\mathfrak{M}$ and $A^{(2)}_{\mathrm{st}} \otimes_A \mathfrak{M}$ are regarded as submodules of $T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}$ via $\iota_{\mathfrak{M}}$ in (3.1) and using the G_K-action on $T^\vee \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}$. This follows from Theorem 3.2.1. ∎

Now let us discuss when $\widehat{T}(\widehat{\mathfrak{M}})$ becomes a crystalline representation. Recall that τ is a selected topological generator of $\mathrm{Gal}(L/K_{1^\infty})$; moreover, $\tau(u) = \epsilon u$ and τ, H_K topologically generate $\hat{G}$.

Corollary 3.3.4. *Select $\tau \in \hat{G}$ as above. Then $\widehat{T}(\widehat{\mathfrak{M}})$ is crystalline if and only if $\tau(\mathfrak{M}) \subset A^{(2)} \otimes_A \mathfrak{M}$.*

Proof. Clearly for the selected τ, we have $\tau(u) - u = Ez$. If $T := \widehat{T}(\widehat{\mathfrak{M}})$ is crystalline then Theorem 3.2.1 proves that $\tau(\mathfrak{M}) \subset A^{(2)} \otimes_A \mathfrak{M}$. Conversely, suppose that $\tau(\mathfrak{M}) \subset A^{(2)} \otimes_A \mathfrak{M}$. Then we see that $(\tau - 1)\mathfrak{M} \subset uA_{\mathrm{inf}} \otimes_A \mathfrak{M}$ by Lemma 3.3.1 (3). And this is enough to show that $\widehat{T}(\widehat{\mathfrak{M}})$ is crystalline. For example, $\mathfrak{M} \otimes_A (A_{\mathrm{inf}}[\frac{1}{p}]/\mathfrak{p})$ has a G_K-fixed basis given by a basis of $\mathfrak{M}$, where the ideal $\mathfrak{p}$ is defined as $\mathfrak{p} := \bigcup_{n\in\mathbb{N}} \varphi^{-n}(u)A_{\mathrm{inf}}[\frac{1}{p}] \subset A_{\mathrm{inf}}[\frac{1}{p}]$. Then one can use the same method as in [32, Theorem 3.8] or directly use [16, Theorem 4.2.1] to show that T is crystalline. ∎

Remark 3.3.5. Though $A^{(2)}_{\mathrm{st}}$ is still complicated, for example, it is not Noetherian, it is better than the old $\widehat{\mathcal{R}}$: at least it has explicit topological generators. Furthermore, $A^{(2)}_{\mathrm{st}}$ is p-adic complete. This can help to close the gap in [29] mentioned in [19, Appendix B]. Indeed, as indicated in [19, Remark B.0.5], if $\widehat{\mathcal{R}}$ can be shown to be p-adic complete then the gap in [29] can be closed. So by replacing $\widehat{\mathcal{R}}$ by $A^{(2)}_{\mathrm{st}}$, we close the gap of [29] ([19] provides another similar way to close that gap).

4. Crystalline representations and prismatic F-crystals

In this section, we re-prove the theorem of Bhatt and Scholze on the equivalence of prismatic F-crystals and lattices in crystalline representations of G_K and complete the proof of Theorem 3.2.1. We start with some general facts on the absolute prismatic site (which allows general base rings).

4.1. Prismatic F-crystals in finite projective modules

Let $R = R_0 \otimes_W \mathcal{O}_K = R_0[u]/E$ be as at the beginning of Section 2, and $X = \mathrm{Spf}(R)$ with the p-adic topology.

Definition 4.1.1. The (*absolute*) *prismatic site* X_{Δ} of X is the opposite of the category of bounded prisms (A, I) that are (p, I)-complete together with a map $R \to A/I$, and a morphism of prisms $(A, I) \to (B, J)$ is a *covering* if $A \to B$ is (p, I)-completely faithfully flat.

Define the following functors:

$$\mathcal{O}_{\Delta} : (A, I) \mapsto A,$$

and for all $h \in \mathbb{N}$, let

$$\mathcal{I}^h_{\Delta} : (A, I) \mapsto I^h.$$

It is known from [8] that these are sheaves on X_{Δ}, and $\mathcal{O}_{\Delta}$ admits an endomorphism φ_{Δ} coming from the δ-structure. We will also use $\mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p$ to denote the functor that assigns to (A, I) the p-adic completion of A with I inverted.

Now we verify $A^{(2)}$ (resp. $A^{(3)}$) constructed in Section 2.1 is indeed the self-product (resp. self-triple-product) of A in X_{Δ}. We mainly discuss the situation of $A^{(2)}$; the proof for $A^{(3)}$ is almost the same. Recall that $\breve{A} = \breve{R}_0[\![u]\!] = W\langle t_1, \ldots, t_m\rangle[\![u]\!]$.

First, we make a remark on the existence of nonempty self-coproduct in the category of prisms over R. We thank Peter Scholze for answering our question on Mathoverflow, and we will repeat his answer here. Let (A_i, I_i) for $i = 1, 2$ be prisms over R, and let $A_0 = A_1 \mathbin{\widehat{\otimes}}_{\mathbb{Z}_p} A_2$ where the completion is taken in the (p, I_1, I_2)-adic topology. Let J be the kernel of the map

$$A_0 \to A_1/I_1 \otimes_R A_2/I_2.$$

Let (A, I) be the prismatic envelope of $(A_1, I_1) \to (A_0, J)$. One can check this is the initial object in the category of prisms over R that admit maps from (A_i, I_i) such that the two $R \to A_i/I_i \to A/I$ agree. Also note that in general we do not know if the boundedness of (A_1, I_1) and (A_2, I_2) implies the boundedness of their coproduct. But we have seen that $A^{(2)}$ and $A^{(3)}$ are indeed bounded by Corollary 2.2.9.

To start, note that there exists a W-linear map $\breve{i}_2 : \breve{A} \to A^{\widehat{\otimes} 2}$ induced by $u \mapsto y$ and $t_i \mapsto s_i$. We claim that $\breve{i}_2$ uniquely extends to $i_2 : A \to A^{\widehat{\otimes} 2}$ which is compatible with the δ-structures. Indeed, consider the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\bar{i}_2} & A^{\widehat{\otimes} 2}/(p, J^{(2)}) \\ \uparrow & \overset{i_{2,n}}{\searrow} & \uparrow \\ \breve{A} & \xrightarrow{\breve{i}_{2,n}} & A^{\widehat{\otimes} 2}/(p, J^{(2)})^n \end{array}$$

Here $\breve{i}_{2,n} = \breve{i}_2 \bmod (p, J^{(2)})^n$ and $\bar{i}_2 : A \to A/(p, E) \simeq A^{\widehat{\otimes} 2}/(p, J^{(2)})$. Since $\breve{i}_2(u) = y = x + (y - x)$ and $\breve{i}_2(t_i) = s_i = t_i + (s_i - t_i)$, the above (outer) diagram commutes. Since A is formally étale over $\breve{A}$ in the (p, u)-adic topology, we conclude that there exists a unique map $i_{2,n} : A \to A^{\widehat{\otimes} 2}/(p, J^{(2)})^n$ such that the above diagram commutes. Since $A^{\widehat{\otimes} 2}$ is $(p, J^{(2)})$-complete, there exists a unique $i_2 : A \to A^{\widehat{\otimes} 2}$ which extends $\breve{i}_2$. To see that i_2 is compatible with the δ-structures, it suffices to show that $\varphi \circ i_2 = i_2 \circ \varphi$. But both $\varphi \circ i_2$ and $i_2 \circ \varphi$ extend $\breve{A} \overset{\varphi}{\to} \breve{A} \to A^{\widehat{\otimes} 2}$. Again by formal étaleness of A over $\breve{A}$, we see that $\varphi \circ i_2 = i_2 \circ \varphi$. Hence we obtain a map $1 \otimes i_2 : A \otimes_{\mathbb{Z}_p} A \to A^{\widehat{\otimes} 2}$. Define $\theta^{\otimes 2} : A \otimes_{\mathbb{Z}_p} A \to R$ via $\theta^{\otimes 2}(a \otimes b) = \theta(a)\theta(b)$. By the construction of i_2, we have the commutative diagram

$$\begin{array}{ccc} A \otimes_{\mathbb{Z}_p} A & \xrightarrow{1 \otimes i_2} & A^{\widehat{\otimes} 2} \\ \Big\downarrow{\scriptstyle \theta^{\otimes 2}} & & \Big\downarrow \\ R & \xrightarrow{\sim} & A^{\widehat{\otimes} 2}/J^{(2)} \end{array}$$

Let $\widehat{A^{\otimes 2}}$ be the $(p, \mathrm{Ker}(\theta^{\otimes 2}))$-completion of $A^{\otimes 2} := A \otimes_{\mathbb{Z}_p} A$. Hence $1 \otimes i_2$ induces a map $\hat{i}_2$ from $\widehat{A^{\otimes 2}}$ to $A^{\widehat{\otimes} 2}$ because $A^{\widehat{\otimes} 2}$ is clearly $(p, J^{(2)})$-complete. To treat $A^{\widehat{\otimes} 3}$, we construct $i_3 : A \to A^{\widehat{\otimes} 3}$ by extending $\breve{i}_3 : A \to A^{\widehat{\otimes} 3}$ via $u \mapsto w$ and $t_j \mapsto r_j$. The same method shows that i_3 is compatible with the δ-structures and we obtain a map $1 \otimes$

$i_2 \otimes i_3 : A^{\otimes 3} \to A^{\widehat{\otimes} 3}$ with $A^{\otimes 3} : A \otimes_{\mathbb{Z}_p} A \otimes_{\mathbb{Z}_p} A$. Similarly, we obtain a natural map $\hat{i}_3 : \widehat{A^{\otimes 3}} \to A^{\widehat{\otimes} 3}$.

Lemma 4.1.2. *For $s = 2, 3$, $\hat{i}_s : \widehat{A^{\otimes s}} \to A^{\widehat{\otimes} s}$ are isomorphisms.*

Proof. We need to construct an inverse of $\hat{i}_s$. We only handle $\hat{i}_2$; the proof for $\hat{i}_3$ is the same. Let $g : A^{\widehat{\otimes} 2} \to \widehat{A^{\otimes 2}}$ be the A-linear map defined by $y - x \mapsto 1 \otimes u - u \otimes 1$ and $s_j - t_j \mapsto 1 \otimes t_j - t_j \otimes 1$. Clearly g is well-defined because $1 \otimes u - u \otimes 1$ and $1 \otimes t_j - t_j \otimes 1$ are in $\mathrm{Ker}(\theta^{\otimes 2})$. Since $i_2(u) = y$ and $i_2(t_j) = s_j$, $\hat{i}_2 \circ g$ is the identity on $A^{\widehat{\otimes} 2}$. Now it suffices to show that $h := g \circ \hat{i}_2$ is the identity. Write $K = (p, \mathrm{Ker}(\theta^{\otimes 2}))$. Note that we have a map $A \otimes_{\mathbb{Z}_p} \breve{A} \to \widehat{A^{\otimes 2}} \xrightarrow{h} \widehat{A^{\otimes 2}}$ induced by h which we still call $\breve{h}$ for simplicity. Now we have the commutative diagram

$$\begin{array}{ccc} A \otimes_{\mathbb{Z}_p} A & \xrightarrow{\bmod K} & (A \otimes_{\mathbb{Z}_p} A)/K \\ \uparrow & \overset{\pi_n}{\underset{h_n}{\dashrightarrow}} & \uparrow \\ A \otimes_{\mathbb{Z}_p} \breve{A} & \xrightarrow[\breve{h} \bmod K^n]{} & (A \otimes_{\mathbb{Z}_p} A)/K^n \end{array}$$

where h_n is induced by h mod K^n, and π_n is the modulo K^n map. We see that both h_n and π_n on the dashed arrows can make the diagram commute. Then by the formal étaleness of A over $\breve{A}$, we conclude that $h_n = \pi_n$ and h is the identity map. ■

Proposition 4.1.3. *$A^{(2)}$ and $A^{(3)}$ are the self-product and self-triple-product of A in $X_{\mathbb{A}}$.*

Proof. We only treat the case of $A^{(2)}$; the proof for $A^{(3)}$ is the same. We need to prove that for any $B = (B, J)$ in $X_{\mathbb{A}}$,

$$\mathrm{Hom}_{X_{\mathbb{A}}^{\mathrm{opp}}}(A^{(2)}, B) = \mathrm{Hom}_{X_{\mathbb{A}}^{\mathrm{opp}}}(A, B) \times \mathrm{Hom}_{X_{\mathbb{A}}^{\mathrm{opp}}}(A, B).$$

By the above lemma, we have the natural maps $A \otimes_{\mathbb{Z}_p} A \to \widehat{A^{\otimes 2}} \simeq A^{\widehat{\otimes} 2}$. Combining this with the natural map $A^{\widehat{\otimes} 2} \to A^{(2)}$ as $A^{(2)}$ is the prismatic envelope of $A^{\widehat{\otimes} 2}$ for the ideal $J^{(2)}$, we have a map $\alpha : A \otimes_{\mathbb{Z}_p} A \to A^{(2)}$ which is compatible with the δ-structures. Then α induces a map

$$\beta : \mathrm{Hom}_{X_{\mathbb{A}}^{\mathrm{opp}}}(A^{(2)}, B) \to \mathrm{Hom}_{X_{\mathbb{A}}^{\mathrm{opp}}}(A, B) \times \mathrm{Hom}_{X_{\mathbb{A}}^{\mathrm{opp}}}(A, B).$$

To prove the surjectivity of β, given $f_i \in \mathrm{Hom}_{X_{\mathbb{A}}}(A, B)$ for $i = 1, 2$, we obtain a map $f_1 \otimes f_2 : A \otimes_{\mathbb{Z}_p} A \to B$. It is clear that $(f_1 \otimes f_2)(\mathrm{Ker}(\theta^{\otimes 2})) \subset J$. Since B is (p, J)-derived complete, $f \otimes f_2$ extends to a map $f_1 \widehat{\otimes} f_2 : \widehat{A^{\otimes 2}} \simeq A^{\widehat{\otimes} 2} \to B$ which is compatible with the δ-structures. Hence $f_1 \widehat{\otimes} f_2$ is a morphism of δ-algebras. Finally, by the universal properties of prismatic envelope, $f_1 \widehat{\otimes} f_2$ extends to a map of prisms $f_1 \widehat{\otimes}_{\mathbb{A}} f_2 : A^{(2)} \to B$ as required.

Finally, we need to show that β is injective. It suffices to show that the A-algebra structure maps $i_1 : A \to A^{(2)}$ and $i_2' : A \xrightarrow{i_2} A^{\widehat{\otimes} 2} \to A^{(2)}$ are both injective. Since all rings here are (p, E)-complete integral domains, it suffices to check that i_1, i_2' mod (p, E) are injective. By Proposition 2.2.8, we see that i_1 mod (p, E) is $R/pR \to R/pR[\{\gamma_i(z_j)\}]$, so it is injective. By the construction of i_2' and i_2, we see that i_2' mod (p, E) is the same as $A/(p, E) \to A^{\widehat{\otimes} 2}/(p, J^{(2)}) \to A^{(2)}/(p, E)$, which is the same as $R/pR \to R/pR[\{\gamma_i(z_j)\}]$. So it is injective. ∎

Remark 4.1.4. When $R = \mathcal{O}_K$ is a complete DVR with perfect residue field k, we know a priori that the self-product $A^{(2)}$ of $(A, (E))$ in X_{Δ} can be constructed as the prismatic envelope of $(A, (E)) \to (B, I)$, where B is the $(p, E(u), E(v))$-adic completion of $W(k)[\![u]\!] \otimes_{\mathbb{Z}_p} W(k)[\![v]\!]$ and I is the kernel of the map

$$B \to A/(E) \otimes_R A/(E) = R.$$

On the other hand, $W(k)$ is formally étale over $\mathbb{Z}_p$ for the p-adic topology, so for all $(C, J) \in X_{\Delta}$, the map $W(k) \to R \to C/J$ lifts uniquely to a map $W(k) \to C$. In particular, for all $(C, J) \in X_{\Delta}$, C has a natural $W(k)$-algebra structure. So when we construct the self-product, we can also consider $A^{(2)}$ as the prismatic envelope of $(A, (E)) \to (C, J)$, where C is the $(p, E(u), E(v))$-adic completion of $A \otimes_{W(k)} A$ and J is the kernel of the map

$$C \to A/(E) \otimes_R A/(E) = R.$$

We have $C \simeq W(k)[\![u, v]\!]$, $J = (E(u), u - v)$ and $A^{(2)} = W(k)[\![u, v]\!]\{\frac{u-v}{E}\}_{\delta}^{\wedge}$.

Definition 4.1.5. (1) A *prismatic crystal over X_{Δ} in finite locally free $\mathcal{O}_{\Delta}$-modules* (resp. *$\mathcal{O}_{\Delta}[1/I]_p^{\wedge}$-modules*) is a finite locally free $\mathcal{O}_{\Delta}$-module (resp. $\mathcal{O}_{\Delta}[1/I]_p^{\wedge}$-module) $\mathfrak{M}_{\Delta}$ such that every morphism $f : (A, I) \to (B, J)$ of prisms induces an isomorphism

$$f^*\mathfrak{M}_{\Delta,A} := \mathfrak{M}_{\Delta}((A, I)) \otimes_A B \simeq \mathfrak{M}_{\Delta,B} := \mathfrak{M}_{\Delta}((B, J))$$
$$(\text{resp. } f^*\mathfrak{M}_{\Delta,A} := \mathfrak{M}_{\Delta}((A, I)) \otimes_{A[1/I]_p^{\wedge}} B[1/I]_p^{\wedge} \simeq \mathfrak{M}_{\Delta,B} := \mathfrak{M}_{\Delta}((B, J))).$$

(2) A *prismatic F-crystal over X_{Δ} of height h in finite locally free $\mathcal{O}_{\Delta}$-modules* is a prismatic crystal $\mathfrak{M}_{\Delta}$ in finite locally free $\mathcal{O}_{\Delta}$-modules together with a φ_{Δ}-semilinear endomorphism $\varphi_{\mathfrak{M}_{\Delta}}$ of the $\mathcal{O}_{\Delta}$-module $\mathfrak{M}_{\Delta}$ such that the cokernel of the linearization $\varphi_{\Delta}^*\mathfrak{M}_{\Delta} \to \mathfrak{M}_{\Delta}$ is killed by $\mathcal{I}^h$.

Proposition 4.1.6. *Suppose the sheaf represented by (B, I) in* $\mathrm{Shv}(X_{\Delta})$ *covers the final object $*$ in* $\mathrm{Shv}(X_{\Delta})$*, i.e., for any (C, J) in X_{Δ}, there is a (P, J) that lies over (B, I) and covers (C, J). Also assume that the self-coproduct $B^{(2)}$ and the self-triple-coproduct $B^{(3)}$ of (B, I) are inside X_{Δ}, i.e., they are bounded. Then a prismatic crystal $\mathfrak{M}_{\Delta}$ over X in finite locally free $\mathcal{O}_{\Delta}$-modules is the same as a finite projective module $\mathfrak{M}$ over B together with a descent datum $\psi : \mathfrak{M} \otimes_{i_1,B} B^{(2)} \simeq \mathfrak{M} \otimes_{i_2,B} B^{(2)}$ that satisfies the cocycle condition. Here $i_j : B \to B^{(2)}$ $(j = 1, 2)$ are the two natural maps.*

Proof. Let $\mathfrak{M}$ be a prismatic crystal in finite projective modules. Define $\mathfrak{M} = \mathfrak{M}_{\Delta}((B, I))$, and the descent datum comes from the crystal property:

$$\psi : \mathfrak{M} \otimes_{i_1,B} B^{(2)} \simeq \mathfrak{M}_{\Delta}((B^{(2)}, I)) \simeq \mathfrak{M} \otimes_{i_2,B} B^{(2)}.$$

Now, given $(\mathfrak{M}, \psi)$, for any (C, J) in X_{Δ} we need to construct a finite projective module over C. We choose (P, J) as in the assumption, let $\mathfrak{M}_P = \mathfrak{M} \otimes_B P$, and consider the following diagram:

$$\begin{array}{ccccccc}
C & \longrightarrow & P & & \xrightarrow{\quad f_1 \quad} & & P_C^{(2)} \\
 & & \uparrow & & \nearrow^{f} & & \uparrow \\
 & & B & \xrightarrow{i_1} & B^{(2)} & & \Big| f_2 \\
 & & & & i_2 \uparrow & & \\
 & & & & B & \longrightarrow & P \\
 & & & & & & \uparrow \\
 & & & & & & C
\end{array}$$

Here $(P_C^{(2)}, J)$ is the self-coproduct of (P, J) in the category of prisms over (C, J); its existence comes from [8, Corollary 3.12], where it is also shown that $P_C^{(2)}$ is the derived (p, J)-completion of $P \otimes^{\mathbb{L}}_C P$ and $(P_C^{(2)}, J)$ is bounded. As a bounded prism over (C, J), $(P_C^{(2)}, J)$ is naturally inside X_{Δ}, so f exists by the universal property of $B^{(2)}$. So if we take the base change of ψ along f, we get

$$f^*\psi : (\mathfrak{M} \otimes_{i_1, B} B^{(2)}) \otimes_{B^{(2)}, f} P_C^{(2)} \simeq (\mathfrak{M} \otimes_{i_2, B} B^{(2)}) \otimes_{B^{(2)}, f} P_C^{(2)},$$

which is the same as an isomorphism

$$\psi_C : \mathfrak{M}_P \otimes_{P, f_1} P_C^{(2)} \simeq \mathfrak{M}_P \otimes_{P, f_2} P_C^{(2)}.$$

Similar arguments show that ψ_C satisfies the cocycle condition. And $\mathfrak{M}_P$ descends to a finite projective module over C by [1, Proposition A.3]. ■

Remark 4.1.7. Note that the structure of finite nonempty coproducts in the category of bounded prisms over a prism (A, I) is much simpler than the structure of finite nonempty products in the category $(R/A)_{\Delta}$ (cf. [4, Lecture V, Corollary 5.2]).

Lemma 4.1.8. *The prism $(A, (E))$ defined in Section 2.1 covers the final object $*$ in* $\mathrm{Shv}(X_{\Delta})$ *in the sense of Proposition 4.1.6. And $A^{(2)}$ and $A^{(3)}$ are bounded.*

Proof. The proof is similar to that of [1, Lemma 5.14]; we need to show that for R defined as in Section 2.1, there exists a quasi-syntomic perfectoid cover of R. We will construct such a cover similar to [24, Section 7.1].

First recall that $R = \mathcal{O}_K \otimes_W R_0$, and fix a compatible system $\{\varpi_n\}_{n \geq 0}$ of p^n-th roots of a uniformizer ϖ_0 of $\mathcal{O}_K$ inside E. Let $\widehat{K}_\infty$ be the p-adic completion of $\bigcup_n K(\varpi_n)$; we know $\widehat{K}_\infty$ is perfectoid. We use $\overline{R}_0[\![u]\!]$ to denote $A/(p) = R/(\varpi) = R_0/(p)[\![u]\!]$, and let $\overline{R}_0[\![u]\!]^{\wedge}_{\mathrm{perf}}$ be the u-adic completion of the direct perfection of $\overline{R}_0[\![u]\!]$. It can be checked directly that $(\overline{R}_0[\![u]\!]^{\wedge}_{\mathrm{perf}}[1/u], \overline{R}_0[\![u]\!]^{\wedge}_{\mathrm{perf}})$ is a perfectoid affinoid $\widehat{K}^{\flat}_\infty$-algebra, and by tilt equivalence, there is a corresponding perfectoid affinoid $\widehat{K}_\infty$-algebra. More explicitly, let $\tilde{R}_\infty = W(\bar{R}_0[\![u]\!]^\wedge_{\mathrm{perf}}) \otimes_{W(\mathcal{O}^\flat_{\hat{K}_\infty}),\theta} \mathcal{O}_{\hat{K}_\infty}$. Then $\tilde{R}_\infty$ is naturally an R-algebra, and we claim it is a quasi-syntomic cover of R.

To show this, by [24, Section 7.1.2], we have

$$\tilde{R}_\infty = (R_0 \hat{\otimes}_W \mathcal{O}_{\hat{K}_\infty}) \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p\langle T_i^{-p^\infty}\rangle,$$

where $T_i \in R_0$ is any lift of a p-basis of $R_0/(p)$. As $\mathcal{O}_K \to \mathcal{O}_{\hat{K}_\infty}$ is a quasi-syntomic cover, by [6, Lemma 4.16 (2)], $R \to R_0 \hat{\otimes}_W \mathcal{O}_{\hat{K}_\infty}$ is also a quasi-syntomic cover. And $S = \mathbb{Z}_p\langle T_i^{-p^\infty}\rangle$ is a quasi-syntomic ring, which can be seen by constructing a perfectoid quasi-syntomic cover of it, so by [6, Lemma 4.34] the complex $\mathbb{L}_{S/\mathbb{Z}_p} \in D(S)$ has p-complete Tor amplitude in $[-1, 0]$. In particular, $\mathbb{Z}_p \to \mathbb{Z}_p\langle T_i^{-p^\infty}\rangle$ is also a quasi-syntomic cover, so by [6, Lemma 4.16], $R \to \tilde{R}_\infty$ is a quasi-syntomic perfectoid cover.

The boundedness of $A^{(2)}$ and $A^{(3)}$ is stated in Corollary 2.2.9 (2). ∎

Corollary 4.1.9. *Assume the the base $X = \mathrm{Spf}(R)$ satisfies the condition in Section 2, and let A, $A^{(2)}$ and $A^{(3)}$ be as defined in Section 2.1. Then a prismatic F-crystal $(\mathfrak{M}_\Delta, \varphi_{\mathfrak{M}_\Delta})$ in finite locally free $\mathcal{O}_\Delta$-modules of height h over X is the same as a Kisin module $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ of height h over A with a descent datum*

$$f : \mathfrak{M} \otimes_{A,i_1} A^{(2)} \simeq \mathfrak{M} \otimes_{A,i_2} A^{(2)}$$

that is compatible with the φ-structure and satisfies the cocycle condition over $A^{(3)}$.

Theorem 4.1.10 ([9, Theorem 1.2]). *Let T be a crystalline representation of G_K over a $\mathbb{Z}_p$-lattice of Hodge–Tate weights in $[0, h]$. Then there is a prismatic F-crystal $\mathfrak{M}_\Delta(T)$ over X_Δ of height h over X such that $\mathfrak{M}_\Delta((A, E))$ is the Kisin module associated to T. Moreover, the association $T \mapsto \mathfrak{M}_\Delta(T)$ induces an equivalence of the above two categories.*

We will prove this theorem in Section 4.3.

Remark 4.1.11. Theorem 4.1.10 was first established by Bhatt–Scholze in [9, Theorem 1.2]. The harder direction of [9, Theorem 1.2] is to show that to all $\mathbb{Z}_p$-lattices inside crystalline representations of G_K, one can attach a prismatic F-crystal. Using the theory of $(\varphi, \hat{G})$-modules, we have shown in Section 3.2 that to a crystalline representation of G_K over a $\mathbb{Z}_p$-lattice T, we can attach a Kisin module $\mathfrak{M}$ and a descent datum[3]

$$f_{\tilde{\tau}} : \mathfrak{M} \otimes_{A,i_1} A^{(2)}\big[\tfrac{1}{p}\big] \simeq \mathfrak{M} \otimes_{A,i_2} A^{(2)}\big[\tfrac{1}{p}\big]$$

coming from the τ-action. We just show this is a φ-equivariant isomorphism, and we need to show it gives rise to a descent datum over $A^{(2)}$. As mentioned in Remark 3.2.4, we have not been able find a direct ring-theoretic proof of this. Our idea is to use the result of [35] or [9, Corollary 3.7]: the underlying Galois representation T gives a descent datum over $A^{(2)}[\frac{1}{E}]^\wedge_p$. To finish our proof, we need to compare this descent datum with $f_{\tilde{\tau}}$ over

[3]Strictly speaking, Section 3.2 only constructs an isomorphism but we have not checked that it satisfies the cocycle condition, which will be done in Section 4.3.

$A^{(2)}[\frac{1}{E}]^\wedge_p[\frac{1}{p}]$. This leads us to develop a "prismatic" (φ, τ)-module theory in the next subsection, where we will have Lemmas 4.2.12 and 4.2.16 to help us compare descent data over $A^{(2)}[\frac{1}{E}]^\wedge_p$ and $A^{(2)}[\frac{1}{E}]^\wedge_p[\frac{1}{p}]$ via an evaluation map to $W(\mathcal{O}^\flat_{\hat{L}})$.

4.2. *Prismatic (φ, τ)-module theory*

In this subsection, we make some preparations to prove Proposition 3.2.2 and Theorem 4.1.10. So we restrict to the case that $R = \mathcal{O}_K$ is a complete DVR with perfect residue field.

Definition 4.2.1. An *étale φ-module over $A[\frac{1}{E}]^\wedge_p$* is a pair $(\mathcal{M}, \varphi_{\mathcal{M}})$ such that $\mathcal{M}$ is a finite free module over $A[\frac{1}{E}]^\wedge_p$, and $\varphi_{\mathcal{M}}$ is an isomorphism

$$\varphi_{\mathcal{M}} : \varphi^*\mathcal{M} := A[\tfrac{1}{E}]^\wedge_p \otimes_{\varphi, A[\frac{1}{E}]^\wedge_p} \mathcal{M} \simeq \mathcal{M}$$

of $A[\frac{1}{E}]^\wedge_p$-modules. And we define an *étale φ-module over $A[\frac{1}{E}]^\wedge_p[\frac{1}{p}]$* to be a φ-module over $A[\frac{1}{E}]^\wedge_p[\frac{1}{p}]$ obtained from an étale φ-module over $A[\frac{1}{E}]^\wedge_p$ by base change.

An *étale φ-module over $A[\frac{1}{E}]^\wedge_p$* (resp. $A[\frac{1}{E}]^\wedge_p[\frac{1}{p}]$) *with descent datum* is a triple $(\mathcal{M}, \varphi_{\mathcal{M}}, c)$ with $(\mathcal{M}, \varphi_{\mathcal{M}})$ an étale φ-module over $A[\frac{1}{E}]^\wedge_p$ (resp. $A[\frac{1}{E}]^\wedge_p[\frac{1}{p}]$), and c an isomorphism

$$c : \mathcal{M} \otimes_{A[\frac{1}{E}]^\wedge_p, i_1} B^{(2)} \simeq \mathcal{M} \otimes_{A[\frac{1}{E}]^\wedge_p, i_2} B^{(2)}$$

$$(\text{resp. } c : \mathcal{M} \otimes_{A[\frac{1}{E}]^\wedge_p[\frac{1}{p}], i_1} B^{(2)}[\tfrac{1}{p}] \simeq \mathcal{M} \otimes_{A[\frac{1}{E}]^\wedge_p[\frac{1}{p}], i_2} B^{(2)}[\tfrac{1}{p}])$$

which is compatible with the φ-structure and satisfies the cocycle condition over $B^{(3)}$ (resp. $B^{(3)}[\frac{1}{p}]$). Here for $j = 1, 2$, $i_j : A[\frac{1}{E}]^\wedge_p \to B^{(2)}$ is the map induced by $i_j : (A, (E)) \to (A^{(2)}, (E))$.

Remark 4.2.2. It is the main result in [35] and [9, Section 2] that there is an equivalence of the category of lattices in representations of G_K and the category of prismatic F-crystals in finite locally free $\mathcal{O}_{\Delta}[1/I]^\wedge_p$-modules over $\mathcal{O}_K$. Also by [9, Proposition 2.7], one can show that prismatic F-crystals in finite locally free $\mathcal{O}_{\Delta}[1/I]^\wedge_p$-modules is the same as étale φ-modules over $A[\frac{1}{E}]^\wedge_p$ with descent data.

The aim of this subsection is to use the ideas in [35] and [23, Section 5.5] to show that étale φ-modules over $A[\frac{1}{E}]^\wedge_p$ (resp. $A[\frac{1}{E}]^\wedge_p[\frac{1}{p}]$) with descent data are equivalent to $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ (resp. $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$). More importantly, for all $\gamma \in \hat{G}$, we will construct an evaluation-at-γ map

$$e_\gamma : B^{(2)} \to W(\hat{L}^\flat)$$

and use it to study φ-equivariant morphisms between finite free $B^{(2)}$ and $B^{(2)}[\frac{1}{p}]$-modules. We will see that the evaluation-at-τ map will play a crucial role in our proof of Proposition 3.2.2 and Theorem 4.1.10 below.

Recall that in Section 3.3, we defined $L = \bigcup_{n=1}^\infty K_\infty(\zeta_{p^n})$, $\hat{G} := \mathrm{Gal}(L/K)$ and $H_K := \mathrm{Gal}(L/K_\infty)$. Moreover, we define $\widehat{K}_{1^\infty}$ to be the p-adic completion of

$\bigcup_{n\geq 0} K(\zeta_{p^n})$, and we let $\hat{L}$ be the p-adic completion of L. It is clear that $A[\frac{1}{E}]^\wedge_p \subset W(\hat{L}^\flat)^{H_K}$. Recall the following definition and theorem from [12]:

Theorem 4.2.3. *An étale (φ, τ)-module is a triple $(\mathcal{M}, \varphi_{\mathcal{M}}, \hat{G})$ where*

- *$(\mathcal{M}, \varphi_{\mathcal{M}})$ is an étale φ-module over $A[\frac{1}{E}]^\wedge_p$;*
- *$\hat{G}$ is a continuous $W(\hat{L}^\flat)$-semilinear $\hat{G}$-action on*

$$\hat{\mathcal{M}} := W(\hat{L}^\flat) \otimes_{A[\frac{1}{E}]^\wedge_p} \mathcal{M}$$

  such that $\hat{G}$ commutes with $\varphi_{\mathcal{M}}$;
- *regarding $\mathcal{M}$ as an $A[\frac{1}{E}]^\wedge_p$-submodule of $\hat{\mathcal{M}}$, we have $\mathcal{M} \subset \hat{\mathcal{M}}^{H_K}$.*

Then there is an anti-equivalence of the category of étale (φ, τ)-modules and $\mathrm{Rep}_{\mathbb{Z}_p}(G_K)$ *such that if T corresponds to $(\mathcal{M}, \varphi_{\mathcal{M}}, \hat{G})$, then*

$$T^\vee = (\hat{\mathcal{M}} \otimes_{W(\hat{L}^\flat)} W(\mathbb{C}_p^\flat))^{\varphi=1}.$$

One of the basic facts used in the theory of étale (φ, τ)-modules developed in [12] is that $\mathrm{Gal}(\hat{L}/\widehat{K}_{1^\infty}) \simeq \mathbb{Z}_p$, and we write τ to be a topological generator of $\mathrm{Gal}(\hat{L}/K_{1^\infty})$ determined by $\tau(\varpi_n) = \zeta_{p^n}\varpi_n$ as in the discussion before Corollary 3.3.4. Also $\hat{G}$ is topologically generated by τ and H_K, so in particular the $\hat{G}$-action on $\hat{\mathcal{M}}$ is determined by the action of τ on $\mathcal{M}$ inside $\hat{\mathcal{M}}$. As discussed before, we will provide a direct correspondence of the category of étale (φ, τ)-modules and the category of étale φ-modules over $A[\frac{1}{E}]^\wedge_p$ with descent data. Moreover, we will construct an evaluation-at-τ map

$$e_\tau : B^{(2)} \to W(\hat{L}^\flat),$$

and show that the τ-action on $\mathcal{M}$ inside $\hat{\mathcal{M}}$ is given by the base change of the descent datum along e_τ.

Remark 4.2.4. In [35, Theorem 5.2], a similar equivalence is proved, but for étale (φ, Γ)-modules. The theory of étale (φ, Γ)-modules is defined for the cyclotomic tower K_{1^∞} over K, while the theory of étale (φ, τ)-modules is defined using the Kummer tower K_∞. We will use a lot of ideas and results developed in [35] when proving our claims in this subsection. The main difficulty in our situation is that the Kummer tower K_∞ is not a Galois tower over K. To deal with this, we have to use the idea in [23, Section 5.5]. Roughly speaking, we will take the Galois closure L of K_∞, prove results over L, and descend back to K_∞ using $K_\infty = L^{H_K}$.

One should be able to construct the evaluation map in the context of [35] the same way as we do in this subsection. This map will give a more direct correspondence of the descent data and the Γ-actions on étale (φ, Γ)-modules.

By [8, Lemma 3.9], any prism (B, J) admits a map into its perfection $(B_{\mathrm{perf}}, JB_{\mathrm{perf}})$. The following theorem is the key to understanding perfect prisms.

Theorem 4.2.5 ([8, Theorem 3.10]). *$(A, I) \to A/I$ induces an equivalence of the category of perfect prisms over $\mathcal{O}_K$ with the category of integral perfectoid rings over $\mathcal{O}_K$.*

Let $(A,(E))$ be the Breuil–Kisin prism defined in Section 2.1. We have the following result.

Lemma 4.2.6. $A_{\mathrm{perf}} \simeq W(\mathcal{O}^{\flat}_{\widehat{K}_\infty})$.

Proof. The same as the proof of [35, Lemma 2.17]. ■

Lemma 4.2.7. *Let* Perfd_K *be the category of perfectoid* K*-algebras. Then* Perfd_K *admits finite nonempty coproducts.*

Proof. Let R and S be two perfectoid K-algebras. It follows from [22, Corollary 3.6.18] that the uniform completion $(R \otimes_K S)^u$ of the tensor product $R \otimes_K S$ is again a perfectoid K-algebra, and it is easy to show that this is the coproduct of R and S in the category of perfectoid K-algebras. ■

For $i \in \mathbb{N}_{>0}$, let $(A^{(i)},(E))$ (resp. $(A_{\inf}(\mathcal{O}_{\hat{L}})^{(i)},(E))$) denote the i-th self-coproduct of $(A,(E))$ (resp. $(A_{\inf}(\mathcal{O}_{\hat{L}}),(E))$) in the category of prisms over $\mathcal{O}_K$, where $A_{\inf}(\mathcal{O}_{\hat{L}}) := W(\mathcal{O}^{\flat}_{\hat{L}})$. The following is a description of $(A^{(i)})_{\mathrm{perf}}[\frac{1}{E}]^{\wedge}_p$ and $(A_{\inf}(\mathcal{O}_{\hat{L}})^{(i)})_{\mathrm{perf}}[\frac{1}{E}]^{\wedge}_p$.

Lemma 4.2.8. *Let* $\widehat{K}^{(i)}_\infty$ *(resp.* $\hat{L}^{(i)}$*) be the* i*-th self-coproduct of* $\widehat{K}_\infty$ *(resp.* $\hat{L}$*) in* Perfd_K*. Then* $(A^{(i)})_{\mathrm{perf}}[\frac{1}{E}]^{\wedge}_p \simeq W((\widehat{K}^{(i)}_\infty)^{\flat})$ *and* $(A_{\inf}(\mathcal{O}_{\hat{L}})^{(i)})_{\mathrm{perf}}[\frac{1}{E}]^{\wedge}_p \simeq W((\hat{L}^{(i)})^{\flat})$.

Proof. We will only prove the lemma for $(A^{(i)})_{\mathrm{perf}}[\frac{1}{E}]^{\wedge}_p$; the case of $\hat{L}^{(i)}$ is similar.

We use similar arguments to those for [35, Lemma 5.3]. Fix i. First we observe that $(A^{(i)})_{\mathrm{perf}}$ is the i-th self-coproduct of $(A_{\mathrm{perf}},(E))$ in the category of perfect prisms over $\mathcal{O}_K$, i.e. $(A^{(i)})_{\mathrm{perf}} = (A_{\mathrm{perf}})^{(i)}_{\mathrm{perf}}$. By Theorem 4.2.5, Lemma 4.2.6 and [35, Proposition 2.15], if we let $S = (A^{(i)})_{\mathrm{perf}}/E$, then $S[\frac{1}{p}]$ is the i-th self-coproduct of $\widehat{K}_\infty$ in the category of perfectoid K-algebras. Now we have

$$(A^{(i)})_{\mathrm{perf}}[\tfrac{1}{E}]^{\wedge}_p \simeq W(S^{\flat})[1/[\varpi^{\flat}]]^{\wedge}_p = W(S^{\flat}[1/\varpi^{\flat}]) \simeq W((\widehat{K}^{(i)}_\infty)^{\flat}). \qquad ■$$

Remark 4.2.9. There is another way to view $\widehat{K}^{(i)}_\infty$ in terms of diamonds over $\mathrm{Spd}(K,\mathcal{O}_K)$, which is used in [35, proof of Lemma 5.3]: there exists a ring $\widehat{K}^{(i),+}_\infty$ of integral elements in $\widehat{K}^{(i)}_\infty$ such that

$$\mathrm{Spa}(\widehat{K}^{(i)}_\infty, \widehat{K}^{(i),+}_\infty)^{\diamond} \simeq \underbrace{\mathrm{Spa}(\widehat{K}_\infty, \widehat{K}^{+}_\infty)^{\diamond} \times_{\mathrm{Spd}(K,\mathcal{O}_K)} \cdots \times_{\mathrm{Spd}(K,\mathcal{O}_K)} \mathrm{Spa}(\widehat{K}_\infty, \widehat{K}^{+}_\infty)^{\diamond}}_{i \text{ copies of } \mathrm{Spa}(K_\infty, K^{+}_\infty)^{\diamond}}. \tag{4.1}$$

And similar results hold for $\hat{L}$. Using this description and the fact that the functor from perfectoid spaces over $\mathrm{Spa}(K,\mathcal{O}_K)$ to diamonds over $\mathrm{Spd}(K,\mathcal{O}_K)$ is an equivalence, we find that $\hat{L}^{(i)}$ has a natural action of $\hat{G}^i$ coming from the action on the diamond spectrum. Since $\hat{L}^{H_K} = \widehat{K}_\infty$, we have

$$\begin{aligned}\mathrm{Spa}(\widehat{K}^{(i)}_\infty, \widehat{K}^{(i),+}_\infty)^{\diamond} &\simeq \big(\mathrm{Spa}(\hat{L},\mathcal{O}_{\hat{L}})^{\diamond} \times_{\mathrm{Spd}(K,\mathcal{O}_K)} \cdots \times_{\mathrm{Spd}(K,\mathcal{O}_K)} \mathrm{Spa}(\hat{L},\mathcal{O}_{\hat{L}})^{\diamond}\big)^{H_K^i} \\ &\simeq (\mathrm{Spa}(\hat{L}^{(i)}, \hat{L}^{(i),+})^{\diamond})^{H_K^i}.\end{aligned}$$

That is, $(\hat{L}^{(i)})^{H_K^i} = \widehat{K}^{(i)}_\infty$.

Now we use ideas in [35] and [23, Section 5.5] to study étale φ-modules over $A[\frac{1}{E}]^\wedge_p$ with descent data. We will show this category is the same as generalized (φ, Γ)-modules in the work of Kedlaya–Liu. The following is a quick review of [23, Examples 5.5.6, 5.5.7].

Firstly, one has $\hat{L}^{(i)} \simeq \mathrm{Cont}(\hat{G}^{i-1}, \hat{L})$, where Cont means the set of continuous functions. One can see this from [35, proof of Theorem 5.6]. When $i = 2$, we choose two canonical maps $i_1, i_2 : \hat{L} \to \hat{L}^{(2)}$ corresponding to $j_1, j_2 : \hat{L} \to \mathrm{Cont}(\hat{G}, \hat{L})$ given by

$$j_1(x) : \gamma \mapsto \gamma(x) \quad \text{and} \quad j_2(x) : \gamma \mapsto x. \tag{4.2}$$

From Remark 4.2.9, there is a natural action of $\hat{G}^2$ on $\hat{L}^{(2)}$. One can check this corresponds to the $\hat{G}^2$-action on $\mathrm{Cont}(\hat{G}, \hat{L})$ given by

$$(\sigma_1, \sigma_2)(f)(\gamma) = \sigma_2 f(\sigma_2^{-1} \gamma \sigma_1).$$

Remark 4.2.10. We have interchanged the roles of j_1 and j_2 compared with the isomorphism defined in [23, Example 5.5.6], so the $\hat{G}^2$-action is different from that in [23, Example 5.5.7]; we will see that our definition is more convenient when relating descent data to semilinear group actions.

One can show that $\mathrm{Cont}(\hat{G}, -)$ commutes with tilting and the Witt vector functor, as discussed in [35, Lemma 5.3], so in particular we have

$$W((\hat{L}^{(i)})^\flat) \simeq \mathrm{Cont}(\hat{G}^{i-1}, W(\hat{L}^\flat)).$$

For $i = 2$, we still use j_1 and j_2 to represent the two canonical maps from $W(\hat{L}^\flat)$ to $\mathrm{Cont}(\hat{G}, W(\hat{L}^\flat))$ that come from (4.2). The above isomorphism is also compatible with the action of $\hat{G}^2$, so we have

$$W((\widehat{K}_\infty^{(2)})^\flat) \simeq \mathrm{Cont}(\hat{G}, W(\hat{L}^\flat))^{H_K^2}. \tag{4.3}$$

Now let $\mathcal{M}$ be an étale φ-module over $W(\widehat{K}_\infty^\flat)$ with a descent datum

$$\psi : \mathcal{M} \otimes_{W(\widehat{K}_\infty^\flat), j_1} W((\widehat{K}_\infty^{(2)})^\flat) \simeq \mathcal{M} \otimes_{W(\widehat{K}_\infty^\flat), j_2} W((\widehat{K}_\infty^{(2)})^\flat)$$

as étale φ-modules over $W((\widehat{K}_\infty^{(2)})^\flat)$ and suppose ψ satisfies the cocycle condition over $W((\widehat{K}_\infty^{(3)})^\flat)$. Using (4.3), we find that ψ is the same as a descent datum

$$\hat{\psi} : \mathcal{M} \otimes_{W(\widehat{K}_\infty^\flat), j_1} \mathrm{Cont}(\hat{G}, W(\hat{L}^\flat))^{H_K^2} \simeq \mathcal{M} \otimes_{W(\widehat{K}_\infty^\flat), j_2} \mathrm{Cont}(\hat{G}, W(\hat{L}^\flat))^{H_K^2}. \tag{4.4}$$

For each $\gamma \in \hat{G}$, we have an evaluation map $\tilde{e}_\gamma : \mathrm{Cont}(\hat{G}, W(\hat{L}^\flat)) \to W(\hat{L}^\flat)$ at γ. Using (4.2), one can check $\tilde{e}_\gamma \circ j_2 : W(\widehat{K}_\infty^\flat) \to W(\hat{L}^\flat)$ is given by the natural embedding and $\tilde{e}_\gamma \circ j_1 : W(\widehat{K}_\infty^\flat) \to W(\hat{L}^\flat)$ is given by $x \mapsto \gamma(x)$. So for each $\gamma \in \hat{G}$, if we tensor (4.4) with the evaluation map $\tilde{e}_\gamma$, we get an isomorphism

$$\psi_\gamma : \mathcal{M} \otimes_{W(\widehat{K}_\infty^\flat), \gamma} W(\hat{L}^\flat) \simeq \mathcal{M} \otimes_{W(\widehat{K}_\infty^\flat)} W(\hat{L}^\flat).$$

And similar to the classical Galois descent theory, the cocycle condition for ψ implies that $\{\psi_\gamma\}_\gamma$ satisfies
$$\psi_{\sigma\gamma} = \psi_\sigma \circ \sigma^* \psi_\gamma.$$
Hence $\{\psi_\gamma\}_\gamma$ defines a continuous semilinear action of $\hat{G}$ on $\hat{\mathcal{M}} := \mathcal{M} \otimes_{W(\widehat{K}^\flat_\infty)} W(\hat{L}^\flat)$. One can check that for $\gamma \in H_K$, the composition
$$W(\widehat{K}^\flat_\infty) \xrightarrow{j_k} W((\widehat{K}^{(2)}_\infty)^\flat) \to \mathrm{Cont}(\hat{G}, W(\hat{L}^\flat)) \xrightarrow{\tilde{e}_\gamma} W(\hat{L}^\flat)$$
is the natural embedding $W(\widehat{K}^\flat_\infty) \hookrightarrow W(\hat{L}^\flat)$ for $k = 1, 2$. And using the cocycle condition, one can show $\psi_\gamma = \mathrm{id}$ for $\gamma \in H_K$, so in particular $\mathcal{M} \subset \hat{\mathcal{M}}^{H_K}$. Conversely, given a semilinear action of $\hat{G}$ on $\hat{\mathcal{M}}$ such that $\mathcal{M} \subset \hat{\mathcal{M}}^{H_K}$, $\{\psi_\gamma\}_\gamma$ defines a descent datum ψ over $\mathrm{Cont}(\hat{G}, W(\hat{L}^\flat))^{H_K^2}$ if and only if the semilinear action is continuous. In summary, we have the following result.

Theorem 4.2.11. (1) *The category of étale φ-modules over $A[\frac{1}{E}]^\wedge_p$ with descent data over $A^{(2)}[\frac{1}{E}]^\wedge_p$ is equivalent to the category of étale (φ, τ)-modules over $A[\frac{1}{E}]^\wedge_p$.*

(2) *Given a descent datum f of an étale φ-module $\mathcal{M}$ over $A[\frac{1}{E}]^\wedge_p$, and $\gamma \in \hat{G}$, we can define the evaluation f_γ of f at γ, defined by the base change of f along*
$$e_\gamma : A^{(2)}\big[\tfrac{1}{E}\big]^\wedge_p \to (A^{(2)})_{\mathrm{perf}}\big[\tfrac{1}{E}\big]^\wedge_p \xrightarrow{\tilde{e}_\gamma} W(\hat{L}^\flat),$$
which defines an isomorphism
$$f_\gamma : \mathcal{M} \otimes_{A[\frac{1}{E}]^\wedge_p, \tilde{\iota}_\gamma} W(\hat{L}^\flat) \simeq \mathcal{M} \otimes_{A[\frac{1}{E}]^\wedge_p} W(\hat{L}^\flat),$$
where $\tilde{\iota}_\gamma : A[\frac{1}{E}]^\wedge_p \to W(\hat{L}^\flat) \xrightarrow{\gamma} W(\hat{L}^\flat)$. Suppose that $(\mathcal{M}, f)$ corresponds to a $\mathbb{Z}_p$-representation T of G_K. Then f_γ corresponds to the semilinear action of γ on $\mathcal{M}$ inside $\mathcal{M} \otimes_{A[\frac{1}{E}]^\wedge_p} W(\mathbb{C}_p^\flat) \simeq T^\vee \otimes W(\mathbb{C}_p^\flat)$. Moreover, two descent data f, g are equal if and only if $f_\tau = g_\tau$.

Proof. The discussion preceding the theorem establishes the equivalence between the category of étale φ-modules over $A_{\mathrm{perf}}[\frac{1}{E}]^\wedge_p$ with descent data over $(A^{(2)})_{\mathrm{perf}}[\frac{1}{E}]^\wedge_p$ and the category of étale (φ, τ)-modules over $A[\frac{1}{E}]^\wedge_p$. Now (1) follows from [35, Theorem 4.6] which shows that the category of étale φ-modules over $B[\frac{1}{I}]^\wedge_p$ is equivalent to the category of étale φ-modules over $B_{\mathrm{perf}}[\frac{1}{I}]^\wedge_p$ for a bounded prism (B, I) such that $\varphi(I)$ mod p is generated by a non-zero-divisor in B/p. Thus it just remains to prove the last statement in (2). Actually one can check (2) by chasing all the functors used in (1), and use the fact that for any étale (φ, τ)-module, the $\hat{G}$-action on $\hat{\mathcal{M}}$ is determined by the τ-action on $\mathcal{M}$. ∎

Lemma 4.2.12. *Given two finite free étale φ-modules $\mathcal{M}, \mathcal{N}$ over $A^{(2)}[\frac{1}{E}]^\wedge_p$ and two morphisms $f, g : \mathcal{M} \to \mathcal{N}$ of étale φ-modules over $A^{(2)}[\frac{1}{E}]^\wedge_p$. For any $\gamma \in \hat{G}$, let f_γ, g_γ be the base changes of f, g along the map*
$$e_\gamma : A^{(2)}\big[\tfrac{1}{E}\big]^\wedge_p \to (A^{(2)})_{\mathrm{perf}}\big[\tfrac{1}{E}\big]^\wedge_p \simeq \mathrm{Cont}\big(\hat{G}, W((\hat{L}^{(2)})^\flat)\big)^{H_K^2} \xrightarrow{\tilde{e}_\gamma} W(\hat{L}^\flat).$$
Then $f = g$ if and only if $f_{\tau^m} = g_{\tau^m}$ for all positive integers m.

Proof. We take the natural base change of f and g along $A^{(2)}[\frac{1}{E}]^{\wedge}_p \to (A^{(2)})_{\mathrm{perf}}[\frac{1}{E}]^{\wedge}_p$, and get two morphisms ψ and ψ' between étale φ-modules over $(A^{(2)})_{\mathrm{perf}}[\frac{1}{E}]^{\wedge}_p$. Since the base change functor between étale φ-modules over $A^{(2)}[\frac{1}{E}]^{\wedge}_p$ and $(A^{(2)})_{\mathrm{perf}}[\frac{1}{E}]^{\wedge}_p$ is an equivalence of categories, it remains to show that $\psi = \psi'$ if and only if their base changes along

$$\tilde{e}_\tau : (A^{(2)})_{\mathrm{perf}}\big[\tfrac{1}{E}\big]^{\wedge}_p \simeq \mathrm{Cont}\big(\hat{G}, W((\hat{L}^{(2)})^\flat)\big)^{H^2_K} \to W(\hat{L}^\flat)$$

are equal. Since $\mathcal{M}$ and $\mathcal{N}$ are finite free, it is enough to show that the evaluation map

$$\tilde{e}_\tau : \mathrm{Cont}\big(\hat{G}, W((\hat{L}^{(2)})^\flat)\big)^{H^2_K} \to W((\hat{L}^{(2)})^\flat)$$

is injective. Suppose $h \in \mathrm{Cont}(\hat{G}, W((\hat{L}^{(2)})^\flat))^{H^2_K}$ satisfies $h(\tau^m) = 0$. Then

$$(\sigma_1, \sigma_2)(h)(\tau^m) = \sigma_2 h(\sigma_2^{-1}\tau^m\sigma_1) = 0$$

for $(\sigma_1, \sigma_2) \in H^2_K$. Since $H_K\tau^m H_K = H_K\tau^{m\chi(H_K)}$ with χ the p-adic cyclotomic character, $\hat{G}$ is topologically generated by H_K and τ, and the set of positive integers is dense in $\mathbb{Z}_p$, we get $h \equiv 0$. ∎

Now we give the $\mathbb{Q}$-isogeny versions of Theorem 4.2.11 and Lemma 4.2.12. Recall that the category of étale (φ, τ)-modules over $A[\frac{1}{E}]^{\wedge}_p[\frac{1}{p}]$ is equivalent to the category $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, and recall the following definition of étale (φ, τ)-modules over $B[\frac{1}{J}]^{\wedge}_p[\frac{1}{p}]$ for a prism $(B, J) \in X_{\Delta}$.

Definition 4.2.13. A *(globally) étale φ-module* $\mathcal{M}$ over $B[\frac{1}{J}]^{\wedge}_p[\frac{1}{p}]$ is a (finite projective) φ-module over $B[\frac{1}{J}]^{\wedge}_p[\frac{1}{p}]$ that arises by base extension from an étale φ-module $B[\frac{1}{J}]^{\wedge}_p$.

From this definition, we immediately deduce the following result by [35, Theorem 4.6].

Proposition 4.2.14. *For any prism $(B, J) \in X_{\Delta}$ such that $\varphi(J)$ mod p is generated by a non-zero-divisor in B/p, base change defined by $B[\frac{1}{J}]^{\wedge}_p[\frac{1}{p}] \to B_{\mathrm{perf}}[\frac{1}{J}]^{\wedge}_p[\frac{1}{p}]$ induces an equivalence between the category of étale φ-modules over $B[\frac{1}{J}]^{\wedge}_p[\frac{1}{p}]$ and the category of étale φ-modules over $B_{\mathrm{perf}}[\frac{1}{J}]^{\wedge}_p[\frac{1}{p}]$.*

And similar to Theorem 4.2.11 and Lemma 4.2.12, we have the following result.

Theorem 4.2.15. *The category of étale φ-modules over $A[\frac{1}{E}]^{\wedge}_p[\frac{1}{p}]$ with descent data over $A^{(2)}[\frac{1}{E}]^{\wedge}_p[\frac{1}{p}]$ is equivalent to the category of étale (φ, τ)-modules over $A[\frac{1}{E}]^{\wedge}_p[\frac{1}{p}]$. Moreover,*

$$\mathrm{Cont}\big(\hat{G}, W(\hat{L}^\flat)\big[\tfrac{1}{p}\big]\big)^{H^2_K} \simeq W(\widehat{K}^{(2)}_\infty)^\flat\big[\tfrac{1}{p}\big].$$

For $\gamma \in \hat{G}$, we can define the evaluation map

$$\tilde{e}_\gamma : \mathrm{Cont}\big(\hat{G}, W(\hat{L}^\flat)\big[\tfrac{1}{p}\big]\big) \to W(\hat{L}^\flat)\big[\tfrac{1}{p}\big].$$

And given a descent datum f of an étale φ-module $\mathcal{M}$ over $A[\frac{1}{E}]^{\wedge}_p[\frac{1}{p}]$, and $\gamma \in \hat{G}$, we can define the evaluation f_γ of f at γ by the base change of f along

$$e_\gamma : A^{(2)}\big[\tfrac{1}{E}\big]^\wedge_p\big[\tfrac{1}{p}\big] \to (A^{(2)})_{\mathrm{perf}}\big[\tfrac{1}{E}\big]^\wedge_p\big[\tfrac{1}{p}\big] \xrightarrow{\tilde{e}_\gamma} W(\hat{L}^\flat)\big[\tfrac{1}{p}\big],$$

which defines an isomorphism

$$\mathcal{M} \otimes_{A[\frac{1}{E}]^\wedge_p[\frac{1}{p}],\tilde{\iota}_\gamma} W(\hat{L}^\flat)\big[\tfrac{1}{p}\big] \simeq \mathcal{M} \otimes_{A[\frac{1}{E}]^\wedge_p[\frac{1}{p}]} W(\hat{L}^\flat)\big[\tfrac{1}{p}\big]$$

where $\tilde{\iota}_\gamma : A[\frac{1}{E}]^\wedge_p[\frac{1}{p}] \to W(\hat{L}^\flat)[\frac{1}{p}] \xrightarrow{\gamma} W(\hat{L}^\flat)[\frac{1}{p}]$. *If* V *in* $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ *corresponds to* $(\mathcal{M}, f)$, *then* f_γ *is the semilinear action of* γ *on* $\mathcal{M}$ *inside* $V^\vee \otimes W(\mathbb{C}_p^\flat)[\frac{1}{p}]$. *Moreover, two descent data* f, g *are equal if and only if* $f_\tau = g_\tau$.

Lemma 4.2.16. *Given two finite free étale* φ*-modules* $\mathcal{M}, \mathcal{N}$ *over* $A^{(2)}[\frac{1}{E}]^\wedge_p[\frac{1}{p}]$ *and two morphisms* $f, g : \mathcal{M} \to \mathcal{N}$ *of étale* φ*-modules over* $A^{(2)}[\frac{1}{E}]^\wedge_p[\frac{1}{p}]$. *For any* $\gamma \in \hat{G}$, *let* f_γ, g_γ *be the base changes of* f, g *along the map*

$$e_\gamma : A^{(2)}\big[\tfrac{1}{E}\big]^\wedge_p\big[\tfrac{1}{p}\big] \to (A^{(2)})_{\mathrm{perf}}\big[\tfrac{1}{E}\big]^\wedge_p\big[\tfrac{1}{p}\big] \xrightarrow{\tilde{e}_\gamma} W(\hat{L}^\flat)\big[\tfrac{1}{p}\big].$$

Then $f = g$ *if and only if* $f_{\tau^m} = g_{\tau^m}$ *for all positive integers* m.

Proof. The proofs are the same as the proofs of Theorem 4.2.11 and Lemma 4.2.12, plus the fact that

$$\mathrm{Cont}\big(\hat{G}, W(\hat{L}^\flat)\big[\tfrac{1}{p}\big]\big) = \mathrm{Cont}(\hat{G}, W(\hat{L}^\flat))\big[\tfrac{1}{p}\big],$$

which can be shown by using the compactness of $\hat{G}$. ■

4.3. *Proofs of Proposition 3.2.2 and Theorem 4.1.10*

In this subsection, we keep the assumption that $R = \mathcal{O}_K$ is a mixed characteristic complete DVR with perfect residue field, and keep the notations of Section 2.1.

Let us first prove Proposition 3.2.2 using Lemma 2.3.2 and the results of Section 4.2. First, we give a different interpretation of the "evaluation map"

$$e_\gamma : A^{(2)}\big[\tfrac{1}{E}\big]^\wedge_p \to (A^{(2)})_{\mathrm{perf}}\big[\tfrac{1}{E}\big]^\wedge_p \simeq \mathrm{Cont}\big(\hat{G}, W((\hat{L}^{(2)})^\flat)\big)^{H_K^2} \xrightarrow{\tilde{e}_\gamma} W(\hat{L}^\flat)$$

in Theorem 4.2.11 when restricted to $A^{(2)}$. Recall that we fixed a compatible system $\{\varpi_n\}_n$ of p^n-th roots of a uniformizer $\varpi \in \mathcal{O}_K$; this defines a map of prisms $\iota : (A, (E)) \to (A_{\mathrm{inf}}, (E))$ sending u to $[\varpi^\flat]$, and given a $\gamma \in G_K$, we define ι_γ to be the composition of ι with $\gamma : (A_{\mathrm{inf}}, (E)) \to (A_{\mathrm{inf}}, (E))$ where the second map is defined as $a \mapsto \gamma(a)$. Since $(E) \subset A_{\mathrm{inf}}$ is equal to $\mathrm{Ker}\,\theta$ and θ is G_K-equivariant, γ is a well-defined map of δ-pairs. By the universal property of $A^{(2)}$, we can define a map of prisms $\iota_\gamma^{(2)} : (A^{(2)}, (E)) \to (A_{\mathrm{inf}}, (E))$ so that the following diagram commutes:

$$\begin{array}{ccccc} (A,(E)) & \xrightarrow{i_1} & (A^{(2)},(E)) & \xleftarrow{i_2} & (A,(E)) \\ & {\scriptstyle \iota_\gamma}\searrow & \downarrow{\scriptstyle \iota_\gamma^{(2)}} & \swarrow{\scriptstyle \iota} & \\ & & (A_{\mathrm{inf}},(E)) & & \end{array} \tag{4.5}$$

The map $\iota_\gamma^{(2)}$ induces a morphism $\tilde{\iota}_\gamma^{(2)} : A^{(2)}[\frac{1}{E}]_p^\wedge \to W(\mathbb{C}_p^\flat)$. We claim that for all $\gamma \in G_K$, $\tilde{\iota}_\gamma^{(2)}$ is the same as

$$A^{(2)}\big[\tfrac{1}{E}\big]_p^\wedge \to (A^{(2)})_{\mathrm{perf}}\big[\tfrac{1}{E}\big]_p^\wedge \simeq \mathrm{Cont}\big(\hat{G}, W((\hat{L}^{(2)})^\flat)\big)^{H_K^2} \xrightarrow{\tilde{e}_\gamma} W(\hat{L}^\flat) \hookrightarrow W(\mathbb{C}_p^\flat).$$

To see this, by the universal property of direct perfection, we see that (4.5) factorizes as

$$\begin{array}{ccccc}
(A,(E)) & \xrightarrow{i_1} & (A^{(2)},(E)) & \xleftarrow{i_2} & (A,(E)) \\
\downarrow & & \downarrow & & \downarrow \\
(A_{\mathrm{perf}},(E)) & \xrightarrow{i_1'} & ((A^{(2)})_{\mathrm{perf}},(E)) & \xleftarrow{i_2'} & (A_{\mathrm{perf}},(E)) \\
& \searrow^{\iota_\gamma'} & \downarrow{\scriptstyle \iota_\gamma'^{(2)}} & \swarrow^{\iota'} & \\
& & (A_{\mathrm{inf}},(E)) & &
\end{array}$$

So $\tilde{\iota}_\gamma^{(2)}$ has a factorization

$$A^{(2)}\big[\tfrac{1}{E}\big]_p^\wedge \to (A^{(2)})_{\mathrm{perf}}\big[\tfrac{1}{E}\big]_p^\wedge \to W(\mathbb{C}_p^\flat).$$

We need to check that $\iota_\tau'^{(2)}$ induces the evaluation map

$$(A^{(2)})_{\mathrm{perf}}\big[\tfrac{1}{E}\big]_p^\wedge \simeq \mathrm{Cont}\big(\hat{G}, W((\hat{L}^{(2)})^\flat)\big)^{H_K^2} \xrightarrow{\tilde{e}_\tau} W(\hat{L}^\flat) \hookrightarrow W(\mathbb{C}_p^\flat).$$

This follows from the isomorphism $(A^{(2)})_{\mathrm{perf}}[\frac{1}{E}]_p^\wedge \simeq W((K_\infty^{(2)})^\flat)$. Then one check directly that for j_1, j_2 defined in (4.2), $\tilde{e}_\gamma \circ j_1 : A_{\mathrm{perf}}[\frac{1}{E}]_p^\wedge \to W(\hat{L}^\flat)$ is equal to the map induced from ι_γ', and $\tilde{e}_\gamma \circ j_2 : A_{\mathrm{perf}}[\frac{1}{E}]_p^\wedge \to W(\hat{L}^\flat)$ is equal to the map induced from ι'. In particular, we have a commutative diagram

$$\begin{array}{ccccccc}
A^{(2)} & & \xrightarrow{\quad\iota_\gamma^{(2)}\quad} & & & & A_{\mathrm{inf}} \\
\big\downarrow & & & & & & \big\downarrow \\
A^{(2)}\big[\tfrac{1}{E}\big]_p^\wedge & \longrightarrow & (A^{(2)})_{\mathrm{perf}}\big[\tfrac{1}{E}\big]_p^\wedge & \xrightarrow{\tilde{e}_\gamma} & W(\hat{L}^\flat) & \hookrightarrow & W(\mathbb{C}_p^\flat)
\end{array} \tag{4.6}$$

Now we can prove Proposition 3.2.2.

Proof of Proposition 3.2.2. First we pick $\gamma = \tilde{\tau}$ that is a preimage of τ under the map $G_K \to \hat{G}$. Then $\gamma(u) - u = Ez$ and $\iota_\gamma^{(2)}$ defined as above is the embedding defined by Remark 2.4.2. In particular, composing the embedding $A^{(2)} \hookrightarrow A_{\mathrm{inf}}$ defined in Section 2.4 with $A_{\mathrm{inf}} \hookrightarrow W(\mathbb{C}_p^\flat)$, one gets the evaluation map

$$(A^{(2)})_{\mathrm{perf}}\big[\tfrac{1}{E}\big]_p^\wedge \simeq \mathrm{Cont}\big(\hat{G}, W((\hat{L}^{(2)})^\flat)\big)^{H_K^2} \xrightarrow{\tilde{e}_\tau} W(\hat{L}^\flat) \hookrightarrow W(\mathbb{C}_p^\flat)$$

restricted to $A^{(2)}$.

Keeping the notations of Section 3.2, let $\mathcal{M}_{A_{\inf}} = W(\mathbb{C}_p^\flat) \otimes_A \mathfrak{M}$ and $\mathcal{M}_A \simeq \mathfrak{M} \otimes_A A[\frac{1}{E}]_p^\wedge$. By Theorems 4.2.11 and 4.2.3 (recall we use $B^{(2)} = A^{(2)}[\frac{1}{E}]_p^\wedge$ and $B_{\mathrm{st}}^{(2)} = A_{\mathrm{st}}^{(2)}[\frac{1}{E}]_p^\wedge$ to simplify our notations), there is a descent datum

$$c : \mathcal{M}_A \otimes_{A[\frac{1}{E}]_p^\wedge, \tilde{\iota}_1} B^{(2)} \to \mathcal{M}_A \otimes_{A[\frac{1}{E}]_p^\wedge, \tilde{\iota}_2} B^{(2)}$$

of $\mathcal{M}_A$ over $B^{(2)}$ that corresponds to the representation T. And the semilinear action of $\gamma = \tilde{\tau}$ on $\mathcal{M}_{A_{\inf}}$ is given by the evaluation c_τ, that is, we have the linearization of the $\tilde{\tau}$-action defined by

$$c_\tau : W(\mathbb{C}_p^\flat) \otimes_{\tilde{\iota}_\gamma, A[\frac{1}{E}]_p^\wedge} \mathcal{M}_A \simeq W(\mathbb{C}_p^\flat) \otimes_{\tilde{\iota}, A[\frac{1}{E}]_p^\wedge} \mathcal{M}_A.$$

By base change c along $B^{(2)} \to B^{(2)}[\frac{1}{p}]$, we get a $B^{(2)}[\frac{1}{p}]$-linear φ-equivariant morphism

$$c' : \mathcal{M}_A \otimes_{A[\frac{1}{E}]_p^\wedge, \tilde{\iota}_1} B^{(2)}[\tfrac{1}{p}] \to \mathcal{M}_A \otimes_{A[\frac{1}{E}]_p^\wedge, \tilde{\iota}_2} B^{(2)}[\tfrac{1}{p}].$$

On the other hand, from the discussions after Proposition 3.2.2, the $\tilde{\tau}$-action also defines a φ-equivariant morphism

$$f_{\tilde{\tau}} : \mathfrak{M} \otimes_{A, \iota_{\tilde{\tau}}} A_{\mathrm{st}}^{(2)}[\tfrac{1}{p}] \simeq \mathfrak{M} \otimes_A A_{\mathrm{st}}^{(2)}[\tfrac{1}{p}].$$

We will see in Proposition 4.3.1 below that $f_{\tilde{\tau}}$ actually descends to a $B^{(2)}[\frac{1}{p}]$-linear morphism. Assuming this fact, if we base change $f_{\tilde{\tau}}$ along $A^{(2)}[\frac{1}{p}] \to W(\mathbb{C}_p^\flat)[\frac{1}{p}]$, we will have $f_{\tilde{\tau}} \otimes W(\mathbb{C}_p^\flat)[\frac{1}{p}] = c_\tau$ since the way we define $f_{\tilde{\tau}}$ is by taking the $\tilde{\tau}$-action.

We claim that $f_{\tilde{\tau}} = c'$ as a $B^{(2)}[\frac{1}{p}]$-linear isomorphism between $\mathcal{M}_A \otimes_{A[1/E]_p^\wedge, \tilde{\iota}_1} B^{(2)}[\frac{1}{p}]$ and $\mathcal{M}_A \otimes_{A[1/E]_p^\wedge, \tilde{\iota}_2} B^{(2)}[\frac{1}{p}]$. Using the notations from Lemma 4.2.16, to show this claim, it is enough to prove $(f_{\tilde{\tau}})_{\tau^m} = c_{\tau^m}$ for all $m \in \mathbb{N}_{>0}$. We fix a basis $\{e_i\}$ of $\mathfrak{M}$ as in Section 3.2, and for all $m \in \mathbb{N}_{>0}$, let $X(m)$ and $Y(m)$ be the matrices defined by $(f_{\tilde{\tau}})_{\tau^m}(e_1, \dots, e_d) = (e_1, \dots, e_d)X(m)$ and $c_{\tau^m}(e_1, \dots, e_d) = (e_1, \dots, e_d)Y(m)$. The discussion in the previous paragraph implies that $X(1) = Y(1)$. Since c satisfies the cocycle condition by definition, proving the claim is equivalent to proving $X(m) = X(1)\tau(X(m-1))$. Since φ is injective on $A_{\inf}[\frac{1}{p}]$, it is sufficient to prove that $\varphi(X(m))$ satisfies the above cocycle condition for $m > 0$.

Comparing the above definitions with the one used in Section 3.2, we have $X(1) = X_\tau$, and by (3.2), it satisfies

$$\varphi(X(1)) = \sum_{i=0}^{\infty} N(i)\gamma_i\left(\log\left(\frac{\tau(u)}{u}\right)\right),$$

where each term is viewed as a matrix with coefficients in $A_{\mathrm{st,max}}^{(2)}[\frac{1}{p}]$. Here, $N(i)$ is defined by

$$N_{\mathfrak{D}}^i(e_1^*, \dots, e_d^*) = (e_1^*, \dots, e_d^*)N(i),$$

with e_i^* as used in Section 3.2. In particular, $N(i)$ has coefficients in $S[\frac{1}{p}]$. By a similar argument to the proof of Proposition 2.4.3, for any $\gamma \in \hat{G}$, the map $\iota_{\gamma,\mathrm{st}}^{(2)} \colon A_{\mathrm{st}}^{(2)} \to A_{\inf}$ [4]

[4]This map is defined similarly to $\iota_\gamma^{(2)}$ for general γ, utilizing the logarithmic prismatic site in Section 5. In particular, it requires the universal property described in Lemma 5.0.12.

uniquely extends to a map

$$\iota^{(2)}_{\gamma,\mathrm{st}}\colon A^{(2)}_{\mathrm{st},\max}\left[\tfrac{1}{p}\right] \to \mathbb{A}_{\max}\left[\tfrac{1}{p}\right]$$

that is φ-equivariant. By definition, $X(m)$ is the base change of $X(1)$ along $\iota^{(2)}_{\tau^m,\mathrm{st}}$, and in particular

$$\varphi(X(m)) = \iota^{(2)}_{\tau^m,\mathrm{st}}(\varphi(X(1))) = \sum_{i=0}^{\infty} N(i)\gamma_i\left(\log\left(\frac{\tau^m(u)}{u}\right)\right).$$

Comparing this formula with (3.2) again, we find that $\varphi(X(m))$ is the matrix of τ^m acting on the basis $\{e_1^*, \dots, e_d^*\}$. In particular, it satisfies the cocycle condition for $m > 0$, as claimed.

By the discussion before Proposition 3.2.2, the matrix $X_{\tilde{\tau}} = X(1)$ has coefficients in $A^{(2)}_{\mathrm{st}}[\frac{1}{p}]$. The matrix $Y(1)$ has coefficients in $B^{(2)} \subset B^{(2)}_{\mathrm{st}}$ because c' is defined by the $B^{(2)}$-linear map c. Hence, by Lemma 2.3.2, $X_{\tilde{\tau}} = X(1) = Y(1)$ has coefficients in $A^{(2)}_{\mathrm{st}}$. The same argument shows that when T is crystalline, $X_{\tilde{\tau}}$ has coefficients in $A^{(2)}$. ■

Proposition 4.3.1. *Base change along $B^{(2)} \to A^{(2)}_{\mathrm{st}}[\frac{1}{E}]^{\wedge}_p$ defines an equivalence of categories of étale φ-modules over $B^{(2)}$ and $A^{(2)}_{\mathrm{st}}[\frac{1}{E}]^{\wedge}_p$ and an equivalence of categories of étale φ-modules over $B^{(2)}[\frac{1}{p}]$ and $A^{(2)}_{\mathrm{st}}[\frac{1}{E}]^{\wedge}_p[\frac{1}{p}]$.*

Proof. By [35, Theorem 4.6], we just need to show the same result after perfections; we will show $(A^{(2)})_{\mathrm{perf}} = (A^{(2)}_{\mathrm{st}})_{\mathrm{perf}}$ in Lemma 5.0.13 using the logarithmic prismatic site. ■

Now, let us prove Theorem 4.1.10 by first producing a functor $\mathcal{T}$ from prismatic F-crystals in finite $\mathcal{O}_{\Delta}$-modules to lattices inside a crystalline representation. For a prism A, we use $i_k : A \to A^{(2)}$ or $A^{(3)}$ for the natural map from A to the k-th factor of $A^{(2)}$ or $A^{(3)}$. The notation $i_{kl} : A^{(2)} \to A^{(3)}$ has a similar meaning.

By Corollary 4.1.9, given a prismatic F-crystal $\mathfrak{M}_{\Delta}$, we obtain a Kisin module $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ of height h together with a descent datum $f : \mathfrak{M} \otimes_{A,i_1} A^{(2)} \to \mathfrak{M} \otimes_{A,i_2} A^{(2)}$ satisfying the cocycle condition $i_{13} \otimes f = (i_{23} \otimes f) \circ (i_{12} \otimes f)$, where $i_{kl} \otimes f$ is the base change of f along i_{kl}, and f is also compatible with the φ-structure on both sides of f. Note that the existence of f follows from the crystal property of $\mathfrak{M}_{\Delta}$:

$$f : \mathfrak{M} \otimes_{A,i_1} A^{(2)} \simeq \mathfrak{M}_{\Delta}((A^{(2)}, (E))) \simeq \mathfrak{M} \otimes_{A,i_2} A^{(2)}. \tag{4.7}$$

We let $\mathcal{M} = \mathfrak{M} \otimes_A A[\frac{1}{E}]^{\wedge}_p$ and $c = f \otimes_{A^{(2)}} B^{(2)}$. Then $(\mathcal{M}, c)$ is an étale φ-module with descent datum, which corresponds to a $\mathbb{Z}_p$-representation of G_K. Moreover the semilinear action of G_K on $\mathfrak{M} \otimes_A W(\mathbb{C}_p^{\flat})$ comes from $\{c_\gamma\}_{\gamma \in G_K}$ using the evaluation maps. If we define

$$f_\gamma : A_{\mathrm{inf}} \otimes_{\iota_\gamma, A} \mathfrak{M} \to A_{\mathrm{inf}} \otimes_{\iota, A} \mathfrak{M}$$

as the base change of f along $\iota^{(2)}_\gamma$, then by (4.6), we have $c_\gamma = f_\gamma$. The semilinear G_K-action commutes with φ, because f does. For any $\gamma \in G_K$, we have $\gamma(A) \subset W(k)[\![u, \epsilon - 1]\!] \subset A^{(2)}_{\mathrm{st}} \subset A_{\mathrm{inf}}$. Therefore, the G_K-action on $A_{\mathrm{inf}} \otimes_A \mathfrak{M}$ defined above factors through $A_{\mathrm{st}}^{(2)} \otimes_A \mathfrak{M}$. We claim that the G_K-action on $\widehat{\mathfrak{M}} := A_{\mathrm{st}}^{(2)} \otimes_A \mathfrak{M}$ defines a $(\varphi, \hat{G})$-module which corresponds to a crystalline representation.

First, for $\gamma \in G_\infty$, we have $\gamma(A) = A$ in A_{inf}, and we conclude $\iota_\gamma^{(2)} : A^{(2)} \to A_{\mathrm{inf}}$ satisfies $\iota_\gamma^{(2)} \circ i_1 = \iota_\gamma^{(2)} \circ i_2$. In particular, for any $\gamma \in G_\infty$ and $j = 1, 2$, using (4.7) and the crystal property of $\mathfrak{M}_\Delta$, f_γ comes from the base change of (4.7) along $\iota_\gamma^{(2)} : A^{(2)} \to A_{\mathrm{inf}}$; in particular,

$$f_\gamma : \mathfrak{M} \otimes_{A, \iota_\gamma^{(2)} \circ i_1} A_{\mathrm{inf}} \simeq \mathfrak{M}_\Delta((A_{\mathrm{inf}}, \operatorname{Ker} \theta)) \simeq \mathfrak{M} \otimes_{A, \iota_\gamma^{(2)} \circ i_2} A_{\mathrm{inf}}.$$

Since $\iota_\gamma^{(2)} \circ i_1 = \iota_\gamma^{(2)} \circ i_2$, we have $f_\gamma = \mathrm{id}$, which means $\mathfrak{M} \subset (\widehat{\mathfrak{M}})^{G_\infty}$. Similarly, G_K acts on $\widehat{\mathfrak{M}}/I_+$ corresponding to the base change of f along

$$A^{(2)} \xrightarrow{\iota_\gamma^{(2)}} A_{\mathrm{inf}} \to W(\bar{k}),$$

where the last arrow is reduction modulo $W(\mathfrak{m})$ ($\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{\mathbb{C}_p}^\flat$). One can check that for all $\gamma \in G_K$ and $j = 1, 2$, the maps

$$A \xrightarrow{i_j} A^{(2)} \xrightarrow{\iota_\gamma^{(2)}} A_{\mathrm{inf}} \to W(\bar{k})$$

are all equal to $A \to W(k) \hookrightarrow W(\bar{k})$ with the first arrow given by $u \mapsto 0$. The above map induces a morphism of prisms $(A, (E)) \to (W(k), (p))$; then using (4.7) and the crystal condition of $\mathfrak{M}_\Delta$, we can similarly prove that G_K acts on $\widehat{\mathfrak{M}}/I_+$ trivially, so $(\mathfrak{M}, \varphi_{\mathfrak{M}}, G_K)$ is a $(\varphi, \hat{G})$-module. Furthermore, $\hat{T}(\widehat{\mathfrak{M}})$ is crystalline by Corollary 3.3.4 and Theorem 3.2.1.

Remark 4.3.2. In Section 5, we will consider a category consisting of modules with descent data, and similar arguments about the triviality of the Galois actions can be conducted directly using the cocycle condition of the descent datum. We summarize this in the following easy fact.

Lemma 4.3.3. *Let $q : (A^{(2)}, (E)) \to (B, J)$ be a map of prisms satisfying $q \circ i_1 = q \circ i_2$. Then for any descent data f over $A^{(2)}$, the base change of f along q is the identity map.*

To show the fully faithfulness of this functor, let $(\mathfrak{M}, f)$, $(\mathfrak{M}', f')$ be two Kisin modules with descent data, and assume there exists a map $\alpha : \mathcal{T}((\mathfrak{M}, f)) \to \mathcal{T}((\mathfrak{M}', f'))$ of lattices in crystalline representations. Then from our construction of $\mathcal{T}$ and Theorem 3.3.3, α is induced from a map $\hat{\alpha} : (\mathfrak{M}, \varphi_{\mathfrak{M}}, \hat{G}_{\mathfrak{M}}) \to (\mathfrak{M}', \varphi_{\mathfrak{M}'}, \hat{G}_{\mathfrak{M}'})$ between $(\varphi, \hat{G})$-modules. The faithfulness of $\mathcal{T}$ follows from the fact that $A \to A[\frac{1}{E}]_p^\wedge$ induces a fully faithful functor between Kisin modules over A and étale φ-modules over $A[\frac{1}{E}]_p^\wedge$ from [25, Proposition 2.1.12]. On the other hand, $\hat{\alpha}$ gives morphisms $\hat{\alpha}_1 : \mathfrak{M} \otimes_{A, i_1} A^{(2)} \to \mathfrak{M}' \otimes_{A, i_1} A^{(2)}$ and $\hat{\alpha}_2 : \mathfrak{M} \otimes_{A, i_2} A^{(2)} \to \mathfrak{M}' \otimes_{A, i_2} A^{(2)}$. If we view A and $A^{(2)}$ as subrings of A_{inf} using diagram (4.5), then the following diagram commutes because $\hat{\alpha} : \widehat{\mathfrak{M}} \to \widehat{\mathfrak{M}}'$

is compatible with the τ-action:

$$\begin{array}{ccc} \mathfrak{M} \otimes_{A,i_1} A^{(2)} & \xrightarrow{f} & \mathfrak{M} \otimes_{A,i_2} A^{(2)} \\ \downarrow{\scriptstyle \hat{\alpha}_1} & & \downarrow{\scriptstyle \hat{\alpha}_2} \\ \mathfrak{M}' \otimes_{A,i_1} A^{(2)} & \xrightarrow{f'} & \mathfrak{M}' \otimes_{A,i_2} A^{(2)} \end{array}$$

Thus we produce a morphism between $(\mathfrak{M}, f)$ and $(\mathfrak{M}', f')$, i.e., $\mathcal{T}$ is also full.

It remains to show the functor $\mathcal{T}$ is essentially surjective. Given a lattice T in a crystalline representation of G_K, let $\mathfrak{M}$ be the corresponding Kisin module. It suffices to construct a descent datum of $\mathfrak{M}$ over $A^{(2)}$. We have shown in our proof of Proposition 3.2.2 that if we view $A^{(2)}$ as a subring of A_{inf} via $\iota^{(2)}_{\tilde{\tau}}$, then $X_{\tilde{\tau}}$ defines a φ-equivariant isomorphism $f : \mathfrak{M} \otimes_{A,i_1} A^{(2)} \simeq \mathfrak{M} \otimes_{A,i_2} A^{(2)}$ of $A^{(2)}$-modules. We also show that the base change of f along $A^{(2)} \to B^{(2)}$ is equal to the descent datum c of the étale φ-module $\mathcal{M}_A = \mathfrak{M} \otimes_A A[\frac{1}{E}]^\wedge_p$ that corresponds to the G_K-action on T. In particular, $c : \mathfrak{M} \otimes_{A,i_1} B^{(2)} \simeq \mathfrak{M} \otimes_{A,i_2} B^{(2)}$ satisfies the cocycle condition. By Lemma 2.3.2, $A^{(2)}$ (resp. $A^{(3)}$) injects into $B^{(2)}$ (resp. $B^{(3)}$), so f also satisfies the cocycle condition. In particular, $(\mathfrak{M}, f)$ produces a prismatic F-crystal in finite free $\mathcal{O}_{\Delta}$-modules by Corollary 4.1.9.

Remark 4.3.4. Given an étale φ-module $(\mathcal{M}_A, \varphi_{\mathcal{M}_A}, c)$ over $A[\frac{1}{E}]^\wedge_p$ with descent datum c, we say $(\mathcal{M}_A, \varphi_{\mathcal{M}_A}, c)$ is *of finite E-height* if $\mathcal{M}_A$ is of finite E-height, i.e., there is a finite free Kisin module $(\mathfrak{M}, \varphi_{\mathfrak{M}})$ of finite height and defined over A such that $\mathfrak{M} \otimes_A A[\frac{1}{E}]^\wedge_p \simeq \mathcal{M}_A$ as φ-modules. Since $(\mathcal{M}_A, \varphi_{\mathcal{M}_A})$ is the étale φ-module for $T|_{G_\infty}$, our definition of finite E-height is compatible with the one given by Kisin under the equivalence in Theorem 4.2.11 (1).

We expect that the same arguments as in the proof of Proposition 3.2.2 can be used to study representations of finite E-height. A similar result has been obtained using the theory of (φ, τ)-modules by Caruso. For example, in [12, proof of Lemma 2.23], Caruso shows that for representations of finite E-height, the τ-action descends to $\mathfrak{S}_{u\text{-np},\tau}$, which is a subring of A_{inf} closely related to $\tilde{\iota}^{(2)}_{\tilde{\tau}}(B^{(2)}) \cap A_{\mathrm{inf}}$, where $\tilde{\tau}$ is a preimage of τ in G_K.

Remark 4.3.5. We can also establish the compatibility of our Theorem 4.1.10, the theory of Kisin and [9, Theorem 1.2]. Given a lattice T in a crystalline representation of G_K with nonnegative Hodge–Tate weight, let $\mathfrak{M}$ be the Kisin module corresponding to T in [25], and let $\mathfrak{M}_{\Delta}$ (resp. $\mathfrak{M}'_{\Delta}$) be the prismatic F-crystal corresponding to $T^\vee$ under [9, Theorem 1.2] (resp. T under Theorem 4.1.10). Note that we need to take $T^\vee$ since in the work of Bhatt–Scholze, the equivalence is covariant. By our construction of $\mathfrak{M}'_{\Delta}$, we have $\mathfrak{M}'_{\Delta}((A, (E))) \simeq \mathfrak{M}$. By [9, Remark 7.11], $\mathfrak{M}_{\Delta}((A, (E))) \simeq \mathfrak{M}$. Next we need to show that the respective descent data over $A^{(2)}$ are the same. By Corollary 2.4.5, we just need to show they are the same as descent data of étale φ-modules over $A^{(2)}[\frac{1}{E}]^\wedge_p$; but this is so by our τ-evaluation criteria in Lemma 4.2.12.

5. Logarithmic prismatic F-crystals and semistable representations

In this section, we will propose a possible generalization of Theorem 4.1.10 to semistable representations using the absolute logarithmic prismatic site. The main reference for this subsection is [27]. We will restrict ourselves to the base ring $R = \mathcal{O}_K$, a complete DVR with perfect residue field. And we give R the log structure associated to the prelog structure $\alpha : \mathbb{N} \to R$ such that $\alpha(1) = \varpi$ is a uniformizer in R, i.e., letting $D = \{\varpi = 0\}$, the log structure on $X = \mathrm{Spf}(R)$ is defined by

$$M_X = M_D \hookrightarrow \mathcal{O}_X \quad \text{where} \quad M_D(U) := \{f \in \mathcal{O}_X(U) \mid f|_{U\backslash D} \in \mathcal{O}^{\times}(U\backslash D)\}.$$

Let us introduce the absolute logarithmic site over (X, M_X).

Definition 5.0.1 ([27, Definitions 2.2 and 3.3]). (1) A $\delta_{\log}$-*ring* is a tuple $(A, \delta, \alpha : M \to A, \delta_{\log} : M \to A)$, where (A, δ) is a δ-pair and α is a prelog structure on A, and $\delta_{\log}$ satisfies

- $\delta_{\log}(e) = 0$,
- $\delta(\alpha(m)) = \alpha(m)^p \delta_{\log}(m)$,
- $\delta_{\log}(mn) = \delta_{\log}(m) + \delta_{\log}(n) + p\delta_{\log}(m)\delta_{\log}(n)$ for all $m, n \in M$.

We will simply denote the $\delta_{\log}$-ring by (A, M) if there is no confusion. Morphisms are morphisms of δ-pairs that are compatible with the prelog structure and $\delta_{\log}$-structure.

(2) A $\delta_{\log}$-*triple* is (A, I, M) such that (A, I) is a δ-pair and (A, M) is a $\delta_{\log}$-ring.

(3) A $\delta_{\log}$-triple (A, I, M) is a *prelog prism* if (A, I) is a prism, and it is *bounded* if (A, I) is bounded.

(4) A *bounded* prelog prism is a log prism that is (p, I)-adically log-affine (cf. [27, Definition 3.3]).

(5) A bounded (pre)log prism is *integral* if M is an integral monoid.

(6) A $\delta_{\log}$-triple (A, I, M) is said to be *over* $(R, \mathbb{N})$ if A/I is an R-algebra and there is a map $M \to \mathbb{N}$ of monoids such that the following diagram commutes:

$$\begin{array}{ccccc} M & & \longrightarrow & & A \\ \downarrow & & & & \downarrow \\ \mathbb{N} & \longrightarrow & R & \longrightarrow & A/I \end{array}$$

All $\delta_{\log}$-triples over $(R, \mathbb{N})$ form a category. Similarly, we can define the category of prelog prisms over $(R, \mathbb{N})$ and the category of bounded log prisms over $(R, \mathbb{N})^a$.

Remark 5.0.2. If A is an integral domain, or more generally if $\alpha(M)$ consists of non-zero-divisors, then $\delta_{\log}$ is uniquely determined by δ if it exists. In particular, morphisms between such $\delta_{\log}$-rings are just morphisms of δ-rings.

Remark 5.0.3. Note that in this paper, for a δ-pair (A, I), we always assume A is (p, I)-adic-complete, but in [27], non-(p, I)-adic-complete $\delta_{\log}$-triples are also studied. By [27, Lemma 2.10], we can always take the (p, I)-adic completion of the δ-pair (A, I) and the $\delta_{\log}$-structure will be inherited.

Proposition 5.0.4 ([27, Corollary 2.15]). *Given a bounded prelog prism (A, I, M), one can associate to it a log prism*

$$(A, I, M)^a = (A, I, M^a).$$

Remark 5.0.5. When we deal with a log prism in this paper, we will always take it as the log prism associated with some prelog prism. And by the above proposition, taking the associated log prism does not change the underlying δ-pair. Moreover, it is a general fact that $(A, I, M)^a$ is integral if (A, I, M) is integral.

Definition 5.0.6. The *absolute logarithmic prismatic site* $(X, M_X)_{\Delta_{\log}}$ is the opposite of the category whose objects are

(1) bounded log prisms (A, I, M_A) with *integral* log structure,

(2) maps of formal schemes $f_A : \mathrm{Spf}(A/IA) \to X$,

(3) the map
$$(\mathrm{Spf}(A/IA), f_A^* M_X) \to (\mathrm{Spf}(A), M_A)^a$$
defines an exact closed immersion of log formal schemes.

A morphism $(A, I, M_A) \to (B, I, M_B)$ is a cover if and only if $A \to B$ is (p, I)-complete faithfully flat and the pullback induces an isomorphism of log structure. We will also use $\mathcal{O}_{\Delta}$ (resp. $\mathcal{I}_{\Delta}$) to denote the structure sheaf (resp. ideal sheaf of Hodge–Tate divisor) on $(X, M_X)_{\Delta_{\log}}$ by $(A, I, M_A) \mapsto A$ (resp. $(A, I, M_A) \mapsto I$).

There is a variant of the above definition that we will also use in this subsection: we define $(X, M_X)_{\Delta_{\log}}^{\mathrm{perf}}$ to be the full subcategory of $(X, M_X)_{\Delta_{\log}}$ whose objects are (A, I, M_A) with A perfect.

Remark 5.0.7. Our definition of the absolute logarithmic prismatic site is different from [27, Definition 4.1]. First, we need to consider the absolute prismatic site, not the relative one. Furthermore, we use the (p, I)-complete faithfully flat topology instead of the (p, I)-complete étale topology. Also we require the log structures to be integral.

Proposition 5.0.8. $(X, M_X)_{\Delta_{\log}}$ *forms a site.*

Proof. Similar to [8, Corollary 3.12], we need to show that for a given diagram

$$(C, I, M_C) \xleftarrow{c} (A, I, M_A) \xrightarrow{b} (B, I, M_B)$$

in $(X, M_X)_{\Delta_{\log}}$ such that b is a cover, the pushout of b along c is a covering. From the argument in [8], we know that for the underlying prisms, the pushout of b along c is the (p, I)-completed tensor product $D = C \widehat{\otimes}_A B$, and (D, I) is a bounded prism that covers (C, I) in the (p, I)-complete faithful flat topology. And we give D the log structure M_D defined by viewing $\mathrm{Spf}(D)$ as the fiber product via [31, Proposition 2.1.2]. Then

$(C, M_C) \to (D, M_D)$ is a strict morphism by [31, Remark 2.1.3], so in particular M_D is integral since M_C is. For the same reason,

$$(\mathrm{Spf}(D/ID), f_D^* M_X) \to (\mathrm{Spf}(D), M_D)^a$$

is strict since it is the base change of a strict morphism. It is an exact closed immersion since pushout of a surjective map of monoids is again surjective. ■

Example 5.0.9 ([27, Example 3.4]). (1) Let $(A, (E))$ be the Breuil–Kisin prism. Then we can define a perlog structure on $(A, (E))$ by $\mathbb{N} \to A, n \mapsto u^n$. Then $(A, (E), \mathbb{N})^a$ is in $(X, M_X)_{\Delta_{\log}}$, where (3) in Definition 5.0.6 follows from the fact that the prelog structures $\mathbb{N} \to R \to A/(E)$ and $\mathbb{N} \to A \to A/(E)$ induce the same log structure.

(2) Any prism (B, J) over $(A, (E))$ has a natural prelog structure $\mathbb{N} \to A \to B$, and similar to (1), $(B, J, \mathbb{N})^a$ is in $(X, M_X)_{\Delta_{\log}}$.

(3) A special case of (2) is that (B, J) equals $(A_{\mathrm{perf}}, (E))$, the perfection of $(A, (E))$. The prelog structure in (2) can be directly defined as $1 \mapsto [\varpi^\flat]$. And $(A, (E), \mathbb{N})^a \to (B, J, \mathbb{N})^a$ is a covering of log prisms in $(X, M_X)_{\Delta_{\log}}$.

Actually, all logarithmic structures of log prisms in $(X, M_X)_{\Delta_{\log}}$ is the log structure associated with a prelog structure defined by $\mathbb{N}$. We thank Teruhisa Koshikawa for letting us know the following lemma.

Lemma 5.0.10. *For any log prism (B, J, M_B) inside $(X, M_X)_{\Delta_{\log}}$, $(B, M_B)^a$ admits a chart $\mathbb{N} \to B$ defined by $n \mapsto u_B^n$ for some $u_B \in B$ satisfying $u_B = \varpi \bmod J$.*

Proof. For any log prism (B, J, M_B) inside $(X, M_X)_{\Delta_{\log}}$, the map

$$(\mathrm{Spf}(B/J), f_B^* M_X) \to (\mathrm{Spf}(B), M_B)^a$$

defines an exact closed immersion of log formal schemes. So by [27, proof of Proposition 3.7], if we let $N_{B/J}^a := \Gamma(\mathrm{Spf}(B/J), \underline{\mathbb{N}}^a)$ for the prelog structure $\mathbb{N} \to \mathcal{O}_K \to B/J$ induced from the given prelog structure on $\mathcal{O}_K$, then the fiber product $M_B \times_{N_{B/J}^a} \mathbb{N}$ is a chart for $(B, M_B)^a$. Moreover, since we assume that M_B is integral, it follows that $(\mathrm{Spf}(B/J), f_B^* M_X) \to (\mathrm{Spf}(B), M_B)^a$ is a log thickening with ideal J in the sense of [31, Definition 2.1.1], and one can show $M_B \times_{N_{B/J}^a} \mathbb{N} \simeq \mathbb{N} \times (1 + J)$. Now $1 + J = (1 + J)^\times$, so

$$\mathbb{N} \to \mathbb{N} \times (1 + J) \simeq M_B \times_{N_{B/J}^a} \mathbb{N} \to B$$

is also a chart for $(B, M_B)^a$. And the prelog structure given by $n \mapsto u_B^n$ for some $u_B \in B$ is such that the image of u_B in B/J coincides with the image of ϖ under $\mathcal{O}_K \to B/J$. ■

In the rest of this subsection, we will try to generalize the results we proved in Sections 4.1–4.3 to the logarithmic prismatic site.

Lemma 5.0.11. (1) *For $(A, I_A, M_A)^a, (B, I_B, M_B)^a \in (X, M_X)_{\Delta_{\log}}$ such that M_A, M_B are integral and $(A, M_A) \to (A/I_A, \mathbb{N})$ and $(B, M_B) \to (B/I_B, \mathbb{N})$ are exact surjective, there is a prelog prism (C, I_C, M_C) with integral log structure that is universal in the sense that the diagram*

$$(A, I_A, M_A) \to (C, I_C, M_C) \leftarrow (B, I_B, M_B)$$

is initial in the category of diagrams

$$(A, I_A, M_A) \to (D, I_D, M_D) \leftarrow (B, I_B, M_B)$$

of prelog prisms over $(R, \mathbb{N})$, and $(D, M_D) \to (D/I_D, \mathbb{N})$ is an exact surjective.

(2) *If (C, I_C) in (1) is bounded, then $(C, I_C, M_C)^a$ is the product of $(A, I_A, M_A)^a$ and $(B, I_B, M_B)^a$ inside $(X, M_X)_{\Delta_{\log}}$.*

(3) *If both $(A, I_A, M_A)^a$ and $(B, I_B, M_B)^a$ in (1) are inside $(X, M_X)^{\mathrm{perf}}_{\Delta_{\log}}$, let (C_{perf}, I_C) be the perfection of (C, I_C) defined in (1). Let $(C_{\mathrm{perf}}, I_C, M_C)$ be the prelog prism with prelog structure induced from C. Then $(C_{\mathrm{perf}}, I_C, M_C)^a$ is isomorphic to the product of $(A, I_A, M_A)^a$ and $(B, I_B, M_B)^a$ in $(X, M_X)^{\mathrm{perf}}_{\Delta_{\log}}$.*

Proof. Let $(A, I_A, M_A), (B, I_B, M_B) \in (X, M_X)_{\Delta_{\log}}$, define C_0 to be the (p, I_A, I_B)-adic completion of $A \otimes_{W(k)} B$ and let J be the kernel of

$$C_0 \to A/I_A \mathbin{\widehat{\otimes}}_R B/I_B.$$

Then $(C_0, J, M_A \times M_B)$ is a $\delta_{\log}$-triple over (A, I_A, M_A). Furthermore, $(C_0, J, M_A \times M_B) \to (C_0/J, \mathbb{N})$ is surjective. Then we can apply [27, Proposition 3.6] to get a universal prelog prism (C, I_C, M_C) over (A, I_A, M_A) and (B, I_B, M_B) and $(C, M_C) \to (C/J, \mathbb{N})$ is exact surjective. Just as in [27, proof of Proposition 3.6], we first construct a $\delta_{\log}$-triple (C', J', M'_C) which is universal in the sense that it is a $\delta_{\log}$-triple over both (A, I_A, M_A) and (B, I_B, M_B) such that C'/J' is over A/I_A and B/I_B as an R-algebra and $(C', M'_C) \to (C'/J', \mathbb{N})$ is exact surjective. Then we take the prismatic envelope with respect to $(A, I_A) \to (C', J')$ to get (C, I_C). Next we can check that (C, I_C, M_C) satisfies the universal property. For (2), when (C, I_C) is bounded, the fact that $(C, I_C, M_C)^a$ is the product of $(A, I_A, M_A)^a$ and $(B, I_B, M_B)^a$ inside $(X, M_X)_{\Delta_{\log}}$ follows from [27, Proposition 3.7]. For (3), (C_{perf}, I_C) is automatically bounded, and one can check (C_{perf}, I_C) is universal using exactly the same proof of [27, Proposition 3.7]. ∎

We thank Koji Shimizu for the following lemma about $A^{(2)}_{\mathrm{st}}$.

Lemma 5.0.12. *Let $(A, I, \mathbb{N})^a$ be the Breuil–Kisin prism defined in Example 5.0.9 (1). Then the self-product (resp. self-triple-product) of $(A, I, \mathbb{N})^a$ in $(X, M_X)_{\Delta_{\log}}$ exists. Moreover, if we let $(A^{\langle 2\rangle}, I, M^2)^a$ (resp. $(A^{\langle 3\rangle}, I, M^3)^a$) be the self-product (resp. self-triple-product) of $(A, I, \mathbb{N})^a$, then $A^{\langle i\rangle} \simeq A^{(i)}_{\mathrm{st}}$ for $i = 2, 3$.*

Proof. By our construction in Lemma 5.0.11, $(A^{\langle 2\rangle}, I, M)$ is the prelog prismatic envelope (C, I_C, M_C) with respect to

$$(A, (E), \mathbb{N}) \to (C_0, J, \mathbb{N}^2) \quad \text{and} \quad (C_0/J, \mathbb{N}^2) \to (R, \mathbb{N})$$

where $C_0 = W[\![u, v]\!]$, $J = (E(u), u - v)$ with the prelog structure given by $\beta : (1, 0) \mapsto u, (0, 1) \mapsto v$. The prelog prismatic envelope is constructed using the technique of exactification: consider $\pi : (C_0, \mathbb{N}^2) \to (R = C/J, \mathbb{N})$ where the map between log structures is given by $\pi_{\log} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $(m, n) \mapsto m + n$; here $\pi_{\log}$ is surjective but not exact, so to construct the exactification of $\pi : (C, \mathbb{N}^2) \to (R, \mathbb{N})$ (cf. [27, Construction 2.18]), first we note that the (complete) exactification of $\pi_{\log}$ is

$$\alpha : M^2 \to \mathbb{N} \quad \text{given by} \quad (m, n) \mapsto m + n,$$

where $M^2 = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid m + n \in \mathbb{N}\}$. Since M^2 is generated by $(-1, 1)$, $(1, -1)$, $(0, 1)$ and $(1, 0)$, the exactification of π is

$$\big(W(k)[\![u, v]\!][\tfrac{v}{u}, \tfrac{u}{v}]^{\wedge}_{(p,J')}, J', M^2; \alpha : (1, 0) \mapsto u, (0, 1) \mapsto v, (\pm 1, \mp 1) \mapsto \pm\tfrac{u}{v}\big),$$

where $J' := \ker(W(k)[\![u, v]\!][\frac{v}{u}, \frac{u}{v}] \to R)$.

The (p, J')-adic completion of $W(k)[\![u, v]\!][\frac{v}{u}, \frac{u}{v}]$ is $W(k)[\![u, \frac{v}{u} - 1]\!]$. Then take the prismatic envelope of $(A, (E)) \to (W(k)[\![u, \frac{v}{u} - 1]\!], (E, \frac{v}{u} - 1))$. One can check

$$W(k)\left[\!\!\left[u, \frac{v}{u} - 1\right]\!\!\right]\left\{\frac{v/u - 1}{E(u)}\right\}^{\wedge}_{\delta} \simeq A^{(2)}_{\mathrm{st}}$$

directly from the definition of $A^{(2)}_{\mathrm{st}}$.

Similarly, we can show $A^{\langle 3\rangle} \simeq A^{(3)}_{\mathrm{st}}$, which is also bounded. ∎

The following is one of our key observations.

Lemma 5.0.13. *We have* $(A^{\langle 2\rangle})_{\mathrm{perf}} \simeq (A^{(2)})_{\mathrm{perf}}$.

Proof. Let u_1, u_2 be the images of u under the two natural maps $i_j : A_{\mathrm{perf}} \to (A^{(2)})_{\mathrm{perf}}$ for $j = 1, 2$. We claim that u_2/u_1 is inside $(A^{(2)})_{\mathrm{perf}}$.

Firstly, we have already shown $A_{\mathrm{perf}} \simeq W(\widehat{\mathcal{O}}^{\flat}_{K_\infty})$ and $u = [\varpi^\flat]$, where $\varpi^\flat = (\varpi_n)$ with $\{\varpi_n\}_{n\geq 0}$ being a compatible system of p^n-th roots of ϖ inside $\mathcal{O}_{\widehat{K}_\infty}$, and $(\varpi_n) \in \mathcal{O}^{\flat}_{\widehat{K}_\infty}$ via the identification $\mathcal{O}^{\flat}_{\widehat{K}_\infty} \simeq \lim_{x\mapsto x^p} \mathcal{O}_{\widehat{K}_\infty}$. Let $S = (A^{(2)})_{\mathrm{perf}}/(E)$. This is an integral perfectoid ring over $\mathcal{O}_K$ in the sense of [5]. We have $S^\flat \simeq (A^{(2)})_{\mathrm{perf}}/(p)$. For $j = 1, 2$, define $\varpi_j^\flat = u_j \bmod p \in S^\flat$. Then we have $u_j = [\varpi_j^\flat]$ for $j = 1, 2$.

Recall from Section 2.1 that $z = \frac{y-x}{E(x)}$ in $A^{(2)}$. Since $E(x) = x^e \bmod p$, we have $x(1 + x^{e-1}z) = y \bmod p$. If we denote by $\iota : A^{(2)} \to (A^{(2)})_{\mathrm{perf}}$ the natural map, then $\iota(x) = u_1$ and $\iota(y) = u_2$ in our definition, and $u_1(1 + u_1^{e-1}\iota(z)) = u_2 \bmod p$ inside $S^\flat = A^{(2)}_{\mathrm{perf}}/(p)$. This is the same as $\varpi_1^\flat \mu = \varpi_2^\flat$ with $\mu = (1 + u_1^{e-1}\iota(z)) \bmod p$ in $S^\flat$. So $[\mu]u_1 = [\mu][\varpi_1^\flat] = [\varpi_2^\flat] = u_2$, which proves our claim.

Now by symmetry, u_1/u_2 is also inside $(A^{(2)})_{\mathrm{perf}}$, so u_1/u_2 is a unit in $(A^{(2)})_{\mathrm{perf}}$. Hence we can give $(A^{(2)})_{\mathrm{perf}}$ a prelog structure

$$\alpha : M^2 \to (A^{(2)})_{\mathrm{perf}} \quad \text{with} \quad (1,-1) \mapsto \frac{u_1}{u_2}, (-1,1) \mapsto \frac{u_2}{u_1}, (1,0) \mapsto u_1, (0,1) \mapsto u_2$$

with the monoid M^2 defined as in the proof of Lemma 5.0.12. Then $((A^{(2)})_{\mathrm{perf}}, (E), M^2)^a$ is in $X^{\mathrm{perf}}_{\mathbb{\Delta}_{\log}}$.

One can check that the maps $i_1, i_2 : (A, (E)) \to (A^{(2)}, (E)) \to ((A^{(2)})_{\mathrm{perf}}, (E))$ induce maps $i_1, i_2 : (A_{\mathrm{perf}}, (E), \mathbb{N}) \to ((A^{(2)})_{\mathrm{perf}}, (E), M^2)$ of prelog prisms. So by Lemma 5.0.12, there is a unique map $(A^{\langle 2\rangle}, I, M^2) \to ((A^{(2)})_{\mathrm{perf}}, (E), M^2)$ that will factor through $((A^{\langle 2\rangle})_{\mathrm{perf}}, (E), M^2)$. Let $((A^{\langle 2\rangle})_{\mathrm{perf}}, (E), M^2) \to ((A^{(2)})_{\mathrm{perf}}, (E), M^2)$ be the induced map in $X^{\mathrm{perf}}_{\mathbb{\Delta}_{\log}}$. On the other hand, by the universal property of $A^{(2)}$, there is a map $(A^{(2)})_{\mathrm{perf}} \to (A^{\langle 2\rangle})_{\mathrm{perf}}$ fitting into the coproduct diagram in $X^{\mathrm{perf}}_{\mathbb{\Delta}}$, which is the full subcategory of $X_{\mathbb{\Delta}}$ containing perfect prisms.

The composition $\eta : ((A^{(2)})_{\mathrm{perf}}, (E)) \to ((A^{\langle 2\rangle})_{\mathrm{perf}}, (E)) \to ((A^{(2)})_{\mathrm{perf}}, (E))$ satisfy $\eta \circ i_j = i_j \circ \eta$ for $i_1, i_2 : (A_{\mathrm{perf}}, (E)) \to ((A^{(2)})_{\mathrm{perf}}, (E))$. Such a map is unique inside $X^{\mathrm{perf}}_{\mathbb{\Delta}}$, so $\eta = \mathrm{id}_{((A^{(2)})_{\mathrm{perf}}, (E))}$.

On the other hand, the composition

$$\eta' : ((A^{\langle 2\rangle})_{\mathrm{perf}}, (E), M^2)^a \to ((A^{(2)})_{\mathrm{perf}}, (E), M^2)^a \to ((A^{\langle 2\rangle})_{\mathrm{perf}}, (E), M^2)^a$$

satisfies $\eta \circ i'_j = i'_j \circ \eta$ for $i'_1, i'_2 : (A_{\mathrm{perf}}, (E), \mathbb{N})^a \to ((A^{\langle 2\rangle})_{\mathrm{perf}}, (E), M^2)^a$ induced from $i'_1, i'_2 : (A, (E), \mathbb{N}) \to (A^{\langle 2\rangle}, (E), M^2)$. Such a map is also unique inside $X^{\mathrm{perf}}_{\mathbb{\Delta}_{\log}}$, so $\eta' = \mathrm{id}_{((A^{\langle 2\rangle})_{\mathrm{perf}}, (E), M^2)^a}$. Hence in particular $(A^{\langle 2\rangle})_{\mathrm{perf}} \simeq (A^{(2)})_{\mathrm{perf}}$. ■

Combining the above lemma and our discussions in Section 4.2, we get the following logarithmic variant of prismatic (φ, τ)-module theory.

Theorem 5.0.14. *The category of étale φ-modules over $A[\frac{1}{E}]^{\wedge}_p$ with descent data over $A^{(2)}_{\mathrm{st}}[\frac{1}{E}]^{\wedge}_p$ is equivalent to the category of lattices in representations of G_K. Moreover, for all $\gamma \in \hat{G}$, we can define the evaluation map*

$$e_\gamma : A^{(2)}_{\mathrm{st}}[\tfrac{1}{E}]^{\wedge}_p \to W(\hat{L}^\flat)$$

such that Lemma 4.2.12 is still valid. Moreover, the $\mathbb{Q}$-isogeny version of this theorem also holds.

Remark 5.0.15. The above theorem should be related to the étale comparison theorem in the logarithmic prismatic settings, which has not been studied in [27].

Moreover, the logarithmic version of Lemma 4.1.8 holds. We thank Teruhisa Koshikawa for hints on the following result.

Proposition 5.0.16. *The sheaf represented by $(A, (E), \mathbb{N})^a$ covers the final object $*$ in* $\mathrm{Shv}((X, M_X)_{\mathbb{\Delta}_{\log}})$.

Proof. For any log prism (B, J, M_B), by Lemma 5.0.10, we can assume $(B, J, M_B)^a = (B, J, \mathbb{N})^a$, with prelog structure defined by $n \mapsto u_B^n$ with $u_B = \varpi \bmod J$.

By deformation theory, there is a unique $W(k)$-algebra structure for B, and we define $C = B[\![u]\!][\frac{u_B}{u}, \frac{u}{u_B}]\{\frac{u_B/u-1}{J}\}_\delta^\wedge$, where the completion is taken for the (p, J)-adic topology. Much as in the proof of Lemma 5.0.12, $(C, JC, \mathbb{N})^a$ is the product of $(A, (E), \mathbb{N})^a$ and $(B, J, \mathbb{N})^a$ inside $(X, M_X)_{\Delta_{\log}}$. Moreover, $B \to C$ is (p, J)-complete flat by [8, Proposition 3.13]. It remains to show that $(B, J) \to (C, J)$ is a covering, i.e., $B \to C$ is (p, J)-complete faithfully flat. Let

$$C^{\mathrm{nc}} := B[\![u]\!]\left[\frac{u_B}{u}, \frac{u}{u_B}\right]\left\{\frac{u_B/u-1}{J}\right\}_\delta$$

be the noncomplete version of C so that the (p, J)-adic completion of C^{nc} is C. Now we just need to show that the flat ring map $B/(p, J) \to C/(p, J) = C^{\mathrm{nc}}/(p, J)$ is also faithful.

We claim that $C/(p, J)$ is free over $B/(p, J)$. One has $JC = E(u)C$, and $(p, J) = (p, E) = (p, J, E)$ in C. So $C/(p, J) = C^{\mathrm{nc}}/(p, J)$ is equal to

$$B[\![u]\!]\left[\frac{u_B}{u}, \frac{u}{u_B}\right][\delta^i(z), i \geq 0]\Big/\left(p, J, E, \delta^i\left(\frac{u_B}{u} - 1\right) = \delta^i(Ez), i \geq 0\right).$$

After quotienting modulo (p, J), the above is the direct limit of

$$B/(p, J)[\delta^i(z)]\Big/\left(\delta^i\left(\frac{u_B}{u} - 1\right) = \delta^i(Ez) \bmod (p, E, J)\right)$$

for $i \geq 0$.

Now we use Lemma 2.2.5 to compute $\delta^i(\frac{u_B}{u} - 1) = \delta^i(Ez) \bmod (p, E, J)$. We keep the notations of the lemma. By induction, we have $b_n = 0 \bmod (p, E)$. Using $a_p^{(j)} \in A_0^\times$, $\delta^i(\frac{u_B}{u} - 1) = \delta^i(Ez) \bmod (p, E, J)$ gives a relation $(z_{i-1})^p = \sum_{j=0}^{p-1} \tilde{a}_j^{(i)}(z_{i-1})^j$ where $z_i = \mathfrak{z}_i \bmod (p, J, E)$ and $\tilde{a}_j^{(i)} \in B/(p, J)[z_0, z_1, \ldots, z_{i-2}]$. In summary, we have

$$C/(p, J) = B/(p, J)[z_i, i \geq 0]\Big/\left((z_i)^p - \sum_{j=0}^{p-1} \tilde{a}_j^{(i)}(z_i)^j, i \geq 1\right),$$

which is free over $B/(p, J)$. ■

Definition 5.0.17. (1) A *prismatic crystal over* $(X, M_X)_{\Delta_{\log}}$ (in finite locally free $\mathcal{O}_\Delta$-modules) is a finite locally free $\mathcal{O}_\Delta$-module $\mathfrak{M}_\Delta$ such that every morphism $f : (A, I, M_A) \to (B, J, M_B)$ of log prisms over $(X, M_X)_{\Delta_{\log}}$ induces an isomorphism

$$f^*\mathfrak{M}_{\Delta,A} := \mathfrak{M}_\Delta((A, I, M_A)) \otimes_A B \simeq \mathfrak{M}_{\Delta,B} := \mathfrak{M}_\Delta((B, J, M_B)).$$

(2) A prismatic F-crystal over $(X, M_X)_{\Delta_{\log}}$ of height h (in finite locally free $\mathcal{O}_\Delta$-modules) is a prismatic crystal $\mathfrak{M}_\Delta$ in finite locally free $\mathcal{O}_\Delta$-modules together with a φ_Δ-semilinear endomorphism $\varphi_{\mathfrak{M}_\Delta}$ of $\mathfrak{M}_\Delta$ whose linearization $\varphi_\Delta^*\mathfrak{M}_\Delta \to \mathfrak{M}_\Delta$ has cokernel killed by $\mathcal{I}_\Delta^h$.

In particular, with the help of Theorem 5.0.14 and Proposition 5.0.16, a direct translation of proofs in Section 4.3 with $A^{(2)}$ replaced by $A^{(2)}_{\mathrm{st}}$ shows the following theorem.

Theorem 5.0.18. *The category of prismatic F-crystals over $(X, M_X)_{\Delta_{\log}}$ of height h is equivalent to the category of lattices in semistable representations of G_K with Hodge–Tate weights between* 0 *and h.*

Appendix A. Some discussions on base rings

In this appendix, we show that our base ring assumed at the beginning of Section 2 covers many situations of base rings used in [11, 24].

Let K be a complete DVR with perfect residue field k, and let $K_0 = W[\frac{1}{p}]$ with $W = W(k)$. Fix a uniformizer $\varpi \in \mathcal{O}_K$ and a minimal polynomial $E(u) \in W[u]$ of ϖ over K_0. Let R be a normal domain with the property that R is a p-complete flat $\mathcal{O}_K$-algebra that is complete with respect to the J-adic topology for an ideal $J = (\varpi, t_1, \dots, t_d)$ of R containing ϖ. We also assume $\overline{R} = R/(\varpi)$ is a finitely generated k-algebra with *finite p-basis* discussed in [14, Section 1.1].

Lemma A.0.1 ([24, Lemmas 2.3.1 and 2.3.4]). (1) *In the above setting, there is a p-adic formally smooth flat W-algebra R_0 equipped with a Frobenius lift φ_0 such that $\overline{R} := R_0/(p)$. Moreover, let J_0 be the preimage of $\overline{J}$ inside R_0. Then R_0 is J_0-adically complete, and under this topology, R_0 is formally smooth.*

(2) *$R_0/(p) \xrightarrow{\sim} R/(\varpi)$ lifts to a W-algebra morphism $R_0 \to R$, and the induced $\mathcal{O}_K$-algebra morphism $\mathcal{O}_K \otimes_W R_0 \to R$ is an isomorphism. Moreover, this isomorphism is continuous with respect to the J_0-adic topology.*

Let (R_0, φ_{R_0}) denote a flat W-lift of $R/(\varpi)$ obtained from the above lemma. We have $J_0 = (p, t_1, \dots, t_d) \in R_0$, and we write $\overline{J} = (\overline{t_1}, \dots, \overline{t_d}) \subset \overline{R}$.

Definition A.0.2. Let R_0 be a p-complete $\mathbb{Z}_p$-algebra. We say R_0 satisfies the *refined almost étaleness* assumption, or simply RAE assumption, if $\hat{\Omega}_{R_0} = \bigoplus_{i=1}^m R_0 dT_i$ with $T_i \in R_0^\times$, where $\hat{\Omega}_{R_0}$ is the module of p-adically continuous Kähler differentials.

The following are examples of R_0 and R which satisfy the assumptions of Lemma A.0.1 and the RAE assumption.

Example A.0.3. (1) If $R/(\varpi)$ is a completed Noetherian regular local ring with residue field k, then the Cohen structure theorem implies $R/(\varpi) = k[\![\overline{x_1}, \dots, \overline{x_d}]\!]$. In this case, $R_0 = W[\![x_1, \dots, x_d]\!]$ and $J_0 = (p, x_1, \dots, x_d)$. Then $R = W[\![x_1, \dots, x_d]\!][u]/E$, with $E \in W[u]$ an Eisenstein polynomial.

(2) Let $R_0 = W(k)\langle t_1^{\pm 1}, \dots, t_m^{\pm 1}\rangle$ and $J_0 = (p)$. In this example, $\overline{R} = k[\overline{t}_1^{\pm 1}, \dots, \overline{t}_m^{\pm 1}]$ is not local.

(3) An unramified complete DVR (R_0, p) with residue field k so that $[k : k^p] < \infty$.

(4) Note that the Frobenius lifting in Lemma A.0.1 is not unique. In (2), we can choose $\varphi_{R_0}(t_i) = t_i^p$. In (1), we can choose $\varphi_{R_0}(x_i) = x_i^p$ or $\varphi_{R_0}(x_i) = (x_i + 1)^p - 1$.

Let R_0 be a p-complete algebra which satisfies the RAE assumption. We set $\breve{R}_0 = W\langle t_1, \ldots, t_m\rangle$ and define $f : \breve{R}_0 \to R_0$ by sending t_i to T_i.

Proposition A.0.4. *Assume that R_0 is a p-complete integral domain which has a finite p-basis and satisfies the RAE assumption. Then f is formally étale p-adically.*

Proof. We thank Wansu Kim for providing the following proof. By standard technique using [21, Chapitre III, Corollaire 2.1.3.3] (e.g., see [24, proof in Lemma 2.3.1]), it suffices to show that the cotangent complex $\mathbb{L}_{R_0/\breve{R}_0}$ is acyclic. Since both R_0 and $\breve{R}_0$ are $\mathbb{Z}_p$-flat, it suffices to show that $\mathbb{L}_{R_1/\breve{R}_1}$ is acyclic, where $R_1 = R_0/pR_0$ and $\breve{R}_1 = \breve{R}_0/p\breve{R}_0$. Since R_0 has a finite p-basis, by [14, Lemma 1.1.2], $\mathbb{L}_{R_1/k} \simeq \Omega_{R_1/k}$. Note that the maps $k \to \breve{R}_1 \to R_1$ induce a fiber sequence

$$\mathbb{L}_{\breve{R}_1/k} \otimes^{\mathbb{L}}_{\breve{R}_1} R_1 \to \mathbb{L}_{R_1/k} \to \mathbb{L}_{R_1/\breve{R}_1}.$$

Since $\mathbb{L}_{\breve{R}_1/k} \simeq \Omega_{\breve{R}_1/k}$ and $\Omega_{\breve{R}_1/k} \simeq \Omega_{R_1/k}$ by the RAE condition, we conclude that $\mathbb{L}_{R_1/\breve{R}_1} = 0$ as required. ∎

Let us end with a discussion about our base rings and the base rings used in [11]. As explained at the beginning of [11, Chapter 2], the base ring R_0 in [11] is obtained from $W\langle t_1^{\pm 1}, \ldots, t_m^{\pm 1}\rangle$ by a finite number of iterations of certain operations and is also assumed to satisfy certain properties. By [11, Proposition 2.0.2], R_0 has a finite p-basis and satisfies the RAE assumption. So the base ring R_0 in [11] also satisfies the requirement that $f : W\langle t_1, \ldots, t_m\rangle \to R_0$ is formally étale by Proposition A.0.4.

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