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Horosymmetric limits of Kähler–Ricci flow on Fano G-manifolds

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Abstract. We prove that on a Fano $\mathbf{G}$-manifold (M, J), the Gromov–Hausdorff limit of the Kähler–Ricci flow with initial metric in $2\pi c_1(M)$ must be a $\mathbb{Q}$-Fano horosymmetric variety M_∞ which admits a singular Kähler–Ricci soliton. Moreover, M_∞ is the limit of a $\mathbb{C}^*$-degeneration of M induced by an element in the Lie algebra of the Cartan torus of $\mathbf{G}$. A similar result can be proved for Kähler–Ricci flows on any Fano horosymmetric manifolds. As an application, we generalize our previous result about the existence of type II singularities of Kähler–Ricci flows on Fano $\mathbf{G}$-manifolds to Fano horosymmetric manifolds.

Keywords: $\mathbf{G}$-manifolds, Kähler–Ricci soliton, Kähler–Ricci flow, horosymmetric space.

1. Introduction

It has been known that the existence of Kähler–Einstein, abbreviated by KE, metrics on a Fano manifold M, that is, a compact Kähler manifold with positive first Chern class, is equivalent to K-stability (see [8, 12, 30, 32, 44, 45], etc.). Since not every Fano manifold is K-stable, there are Fano manifolds which do not admit KE metrics and we are led to studying the problem of optimal deformations of such Fano manifolds which admit a canonical metric. The Kähler–Ricci (KR) flow provides an approach to solving this problem by geometric-analytic methods, more precisely, we have the following conjecture, referred to as the Hamilton–Tian (HT) conjecture (see [27, 44]):

Any sequence of metrics $(M, \omega(t))$ in the KR flow contains a subsequence converging to a length space $(M_\infty, \omega_\infty)$ in the Gromov–Hausdorff (GH) topology and $(M_\infty, \omega_\infty)$ is a smooth KR soliton outside a closed subset S, called the singular set, of codimension at least 4*. Moreover, this subsequence of $(M, \omega(t))$ converges to $(M_\infty, \omega_\infty)$ in the Cheeger–Gromov topology.*

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This conjecture has been confirmed (see [3, 14, 44, 48, 54]). Actually, the existence of GH limits follows from Perelman's noncollapsing result [27,43] and Q. Zhang's non-expanding result [60]. Moreover, $(M_\infty, \omega_\infty)$ is unique (see [13,25,55]).

Recall that a *KR soliton* on a Fano manifold M is a pair (X, ω), where X is a holomorphic vector field (HVF) on M and ω is a Kähler metric on M satisfying

$$\mathrm{Ric}(\omega) - \omega = L_X(\omega), \tag{1.1}$$

where L_X denotes the Lie derivative along X. If $X = 0$, the KR soliton becomes a KE metric. The uniqueness theorem in [49,50] states that a KR soliton on a compact complex manifold, if it exists, must be unique modulo $\mathrm{Aut}(M)$.[1] Furthermore, X lies in the center of the Lie algebra of the reductive part of $\mathrm{Aut}(M)$.

On a Fano manifold M, we usually consider the following normalized KR flow:

$$\frac{\partial \omega(t)}{\partial t} = -\mathrm{Ric}(\omega(t)) + \omega(t), \quad \omega(0) = \omega_0, \tag{1.2}$$

where ω_0 and $\omega(t)$ denote the Kähler forms of a given Kähler metric g_0 and the solutions of the Ricci flow, respectively. It is proved in [11] that (1.2) has a global solution $\omega(t)$ for all $t \geq 0$ whenever ω_0 represents $2\pi c_1(M)$. It is a natural problem to study the limiting behavior of $\omega(t)$ as $t \to \infty$ as well as its limiting structure.

The purpose of this paper is to solve the above problem in the case of Fano **G**-manifolds, where **G** is a complex reductive Lie group (i.e., the complexification of a compact group **K**). By a **G**-*manifold*, we mean a (bi-equivariant) compactification of **G** which admits a holomorphic $\mathbf{G} \times \mathbf{G}$-action and has an open and dense orbit isomorphic to **G** as a $\mathbf{G} \times \mathbf{G}$-homogeneous space. A special case is toric manifolds when **G** is a torus.

The problem of existence of KE metrics and KR solitons on **G**-manifolds has been extensively studied (see [16, 17, 19, 36, 38], etc.). A criterion has been found for their existence in terms of the barycenter of the moment polytope associated to the Cartan torus subgroup of **G**. By using this criterion, one can construct many examples of **G**-manifolds which admit KE metrics or KR solitons, as well as examples of **G**-manifolds which admit neither KE metrics nor KR solitons (see [16,64]). Furthermore, we have recently proved the following result for the KR flow on **G**-manifolds [37,65].

Theorem 1.1. *Let (M, J) be a Fano* **G***-manifold which admits no KR soliton. Then any solution $\omega(t)$ of (1.2) with initial metric $\omega_0 \in 2\pi c_1(M, J)$ will develop a singularity of type II, that is, the curvature of $\omega(t)$ must blow up as $t \to \infty$.*

Using Theorem 1.1, we found that there are two $\mathrm{SO}_4(\mathbb{C})$-manifolds and one $\mathrm{Sp}_4(\mathbb{C})$-manifold on which the KR flow develops singularities of type II [37]. The result provides the first example of a Fano manifold on which the KR flow develops a singularity of type II (see [39]).

Theorem 1.1 implies that the GH limit $(M_\infty, \omega_\infty)$ from the HT conjecture must be singular [3, 55]. Also we have shown that the limit may not be a Fano **G**-variety [36,

[1] In the case of KE metrics, this uniqueness theorem is due to Bando–Mabuchi [4].

Section 7]. Nevertheless, we hope that $(M_\infty, \omega_\infty)$ will still keep some symmetries and can be classified. Our first main theorem is the following:

Theorem 1.2. *The GH limit of the KR flow* (1.2) *on a Fano* $\mathbf{G}$*-manifold* (M, J) *is a* $\mathbb{Q}$*-Fano horosymmetric variety which admits a singular KR soliton. More precisely, if the initial metric* ω_0 *in* (1.2) *is* $\mathbf{K} \times \mathbf{K}$*-invariant, the solution* $\omega(t)$ *converges locally smoothly in the Cheeger–Gromov topology to a KR soliton* ω_{KS} *on a horosymmetric space whose completion is the GH limit of the KR flow* (1.2) *with the structure of a* $\mathbb{Q}$*-Fano horosymmetric variety* (M_∞, J_∞) *as the limit of a* $\mathbb{C}^*$*-degeneration of* (M, J) *induced by an element in the Lie algebra of the Cartan torus. Moreover, if* ω_{KS} *is not a KE metric, then* M_∞ *coincides with the limit of the* $\mathbb{C}^*$*-degeneration of* (M, J) *induced by the soliton HVF of* ω_{KS}.

A homogeneous space $\mathbf{G}/\mathbf{H}$ is called *horosymmetric* if it is a fibration over a generalized flag manifold whose fibers are symmetry spaces (see [18]). A horosymmetric variety $M = \overline{\mathbf{G}/\mathbf{H}}$ with group action $\mathbf{G}$ is simply a compactification of a horosymmetric space $\mathbf{G}/\mathbf{H}$; if it is smooth, we call it a *horosymmetric manifold* (see [2, 18]). Theorem 1.2 means that any Fano $\mathbf{G}$-manifold has a degeneration to a $\mathbb{Q}$-Fano horosymmetric variety on which there is a singular KR soliton. Furthermore, this degeneration can be realized by a $\mathbb{C}^*$-degeneration induced by an element in the Lie algebra of the Cartan subgroup of $\mathbf{G}$ (see Theorem 3.2). Namely, we have

Corollary 1.3. *Any Fano* $\mathbf{G}$*-manifold* (M, J) *admits a* $\mathbf{G} \times \mathbf{G}$*-equivariant* $\mathbb{C}^*$*-degeneration induced by an element in the Lie algebra of the Cartan torus of* $\mathbf{G}$ *such that its central fiber in the* $\mathbb{C}^*$*-degeneration is* (*modified*) K*-stable relative to the group* $\mathbf{G} \times \mathbf{G}$.

The (modified) K-stability in Corollary 1.3 follows from the fact that (M_∞, J_∞) in Theorem 1.2 admits a singular KR soliton (see [9, 24, 53]).

Remark 1.4. The $\mathbf{G} \times \mathbf{G}$-equivariant $\mathbb{C}^*$-degeneration in Corollary 1.3 is referred to as an optimal degeneration for (M, J). Since the soliton HVF on M_∞ in Theorem 1.2 induces a $\mathbf{G} \times \mathbf{G}$-equivariant $\mathbb{C}^*$-degeneration on M which attains the minimum of the H-invariant for special degenerations [20, 55], we can find a unique degeneration by minimizing the H-invariant for the $\mathbf{G} \times \mathbf{G}$-equivariant $\mathbb{C}^*$-degenerations induced by a valuation cone in the Lie algebra of the Cartan torus, defined by Delcroix [17]. Thus the algebraic variety structure of M_∞ in Theorem 1.2 can be classified in this way if M is not K-stable. A detailed computation of the H-invariant on such degenerations has recently been given by Li–Li [34] and Wang–Zhu [57].

Since the limit of the KR flow (1.2) is unique, we need to prove Theorem 1.2 only for a $\mathbf{K} \times \mathbf{K}$-invariant initial metric. Then the flow can be reduced to a parabolic equation of Monge–Ampère (MA) type for a class of convex functions as in the case of toric manifolds [63]. The main difference here is that the convex functions should be invariant under the Weyl subgroup of $\mathbf{G}$, but the convex functions induced by the Cartan torus transformations will not preserve the Weyl invariant in general. In particular, we cannot get global Kähler potentials on a $\mathbf{G}$-orbit by the induced convex functions. Our method is to do local C^k-estimates for those induced convex functions on a Euclidean cone $\mathfrak{a}_+$ and prove that the limit convex function on a new cone $\mathfrak{a}'_+$ will define a KR soliton on a horosymmetric space (see Sections 3–6).

We will divide our proof of Theorem 1.2 into three cases according to concentration points x_t of convex functions w_t on the cone $\mathfrak{a}_+$ (see Theorem 3.2). Those points induce a family of torus translations for convex functions ψ_t (or ψ'_t) which will determine the soliton HVF on the limit space (M_∞, J_∞) (see Proposition 5.1 and Lemma 5.2). We mention that similar arguments have been used to get local estimates for Kähler potentials in the KR flow in general by using partial C^0-estimates in [54].

The horosymmetric structure $\overline{\mathbf{N}}$ on M_∞ in Theorem 1.2 can be explicitly constructed via the $\mathbf{G}$-equivariant $\mathbb{C}^*$-degeneration induced by an element in the Lie algebra of the Cartan subgroup of $\mathbf{G}$ (see Example 2.4 and Remark 3.5). For general constructions of those $\mathbb{C}^*$-degenerations on horosymmetric varieties, we refer the reader to recent papers by Delcroix [17] and Li–Li [34].

Theorem 1.2 can be generalized to any horosymmetric manifolds. Namely, we can also prove

Theorem 1.5. *The GH limit of the KR flow* (1.2) *on a Fano horosymmetric manifold* $M = \overline{\mathbf{G}/\mathbf{H}}$ *is a $\mathbb{Q}$-Fano horosymmetric variety which admits a singular KR soliton. More precisely, if the initial metric ω_0 in* (1.2) *is K-invariant, the solution $\omega(t)$ after Cartan torus transformations of M converges in the Cheeger–Gromov topology to a KR soliton on a $\mathbb{Q}$-Fano horosymmetric variety whose completion is the GH limit of the KR flow* (1.2) *with the structure of a $\mathbb{Q}$-Fano horosymmetric variety M_∞ as the limit of a $\mathbb{C}^*$-degeneration of (M, J) induced by an element in the Lie algebra of the Cartan torus. Moreover, if ω_{KS} is not a KE metric, then M_∞ coincides with the limit of the $\mathbb{C}^*$-degeneration of (M, J) induced by the soliton HVF of ω_{KS}.*

As an application, we generalize Theorem 1.1 to the case of Fano horosymmetric manifolds as follows.

Theorem 1.6. *Let (M, J) be a Fano horosymmetric manifold which admits no KR-soliton. Then any solution $\omega(t)$ of* (1.2) *with initial metric $\omega_0 \in 2\pi c_1(M, J)$ will develop a singularity of type II.*

The proof of Theorem 1.5 is almost identical to that of Theorem 1.2, so we will only sketch it at the end of paper.

The organization of the paper is as follows. In Section 2, we review some previous results on horosymmetric geometry, in particular from Delcroix's works [16–18]. In Section 3, we reduce the KR flow (1.2) to a parabolic equation (3.1) of MA type and derive several estimates related to the C^0-estimate for equation (3.1) (see Proposition 3.1 and Lemmas 3.4 and 3.6–3.8). Section 4 is devoted to the C^2-estimate for (3.1) by controlling the norm of HVFs (see Proposition 4.2). We note that the estimate is completely different from the classical (global) C^2-estimate of Calabi–Yau for complex MA equations (cf. [11, 51, 58], etc.). Theorem 1.2 will be proved by completing Theorem 3.2 in Section 5 while we give a sketch proof of Theorem 1.5 as well as Theorem 1.6 in Section 7. In

Section 6, we prove the local convergence of Kähler metrics associated to solutions of (3.1) as well as (5.8) (see Proposition 6.1). Proposition 6.1 will be used in the proof of Theorem 3.2 in Section 5. Section 8 is an appendix, where we prove an existence result for singular KR solitons on $\mathbb{Q}$-Fano horosymmetric varieties (see Theorem 8.1).

2. K-invariant metrics on horosymmetry spaces

In this paper, we always assume that $\mathbf{G}$ is a reductive Lie group which is the complexification of a compact Lie group $\mathbf{K}$. Let $\mathbf{T}^{\mathbb{C}}$ be an r-dimensional maximal complex torus of $\mathbf{G}$ with Lie algebra $\mathfrak{t}^{\mathbb{C}}$, and $\mathfrak{M}$ the group of characters of $\mathfrak{t}^{\mathbb{C}}$. Denote by Φ the root system of $(\mathbf{G}, \mathbf{T}^{\mathbb{C}})$ in $\mathfrak{M}$ and choose a set Φ_+ of positive roots. Then each element in Φ can be regarded as an element of $\mathfrak{a}^*$, where $\mathfrak{a}^*$ is the dual of the noncompact part $\mathfrak{a}$ of $\mathfrak{t}^{\mathbb{C}}$.

2.1. $\mathbf{K} \times \mathbf{K}$-*invariant metrics on* $\mathbf{G}$-*manifolds*

By the $\mathbf{K} \times \mathbf{K}$-invariance of the Kähler metric ω on $\mathbf{G}$, the restriction of ω to $\mathbf{T}^{\mathbb{C}}$ is an open toric Kähler metric. Thus, it induces a strictly convex function ψ on $\mathfrak{a}$ (see also Lemma 2.2 below) such that

$$\omega = \sqrt{-1}\,\partial\bar{\partial}\psi \quad \text{on } \mathbf{T}^{\mathbb{C}}. \tag{2.1}$$

On the other hand, by the KAK-decomposition [28, Theorem 7.39], for any $g \in \mathbf{G}$, there are $k_1, k_2 \in \mathbf{K}$ and $x \in \mathfrak{a}$ such that $g = k_1 \exp(x) k_2$. Here x is uniquely determined up to a Weyl action. This means that x is unique in $\overline{\mathfrak{a}_+}$, where $\overline{\mathfrak{a}_+}$ is the closed cone of the Weyl chamber $\mathfrak{a}_+$ defined by

$$\mathfrak{a}_+ = \{x \in \mathfrak{a} \mid \alpha(x) = \langle \alpha, x \rangle > 0,\ \forall \alpha \in \Phi_+\}.$$

Thus there is a bijection between $\mathbf{K} \times \mathbf{K}$-invariant functions ϕ on $\mathbf{G}$ and Weyl-invariant functions ψ on $\mathfrak{a}$ which is given by

$$\phi_\psi(\mathbf{K} \cdot \exp(x) \cdot \mathbf{K}) = \psi(x), \quad \forall x \in \mathfrak{a}_+. \tag{2.2}$$

Hence, (2.1) can be extended on $\mathbf{G}$ in such a way that

$$\omega_\psi = \sqrt{-1}\,\partial\bar{\partial}\phi_\psi \quad \text{on } \mathbf{G}. \tag{2.3}$$

With no risk of confusion, we will not distinguish between ψ and ϕ_ψ, and call ϕ_ψ (or ψ) *convex* on $\mathbf{G}$ if ψ is Weyl-invariant convex on $\mathfrak{a}$.

The following KAK-integration formula can be found in [28, Proposition 5.28].

Proposition 2.1. *Let $dV_{\mathbf{G}}$ be a Haar measure on $\mathbf{G}$ and dx the Lebesgue measure on $\mathfrak{a}$. Then there exists a constant $C_H > 0$ such that for any $\mathbf{K} \times \mathbf{K}$-invariant $dV_{\mathbf{G}}$-integrable function ψ on $\mathbf{G}$,*

$$\int_{\mathbf{G}} \psi(g)\, dV_{\mathbf{G}} = C_H \int_{\mathfrak{a}_+} \psi(x) J(x)\, dx,$$

where

$$J(x) = \prod_{\alpha \in \Phi_+} \sinh^2 \alpha(x). \tag{2.4}$$

Without loss of generality, we may normalize $C_H = 1$ for simplicity.

Next we recall a local holomorphic coordinate system on $\mathbf{G}$ used in [16]. By the standard Cartan decomposition, we can decompose $\mathfrak{g}$ as

$$\mathfrak{g} = (\mathfrak{t} \oplus \mathfrak{a}) \oplus \bigoplus_{\alpha \in \Phi} V_\alpha,$$

where $\mathfrak{t}$ is the Lie algebra of $\mathbf{T}$ and

$$V_\alpha = \{X \in \mathfrak{g} \mid \mathrm{ad}_H(X) = \alpha(H)X,\ \forall H \in \mathfrak{t} \oplus \mathfrak{a}\}$$

is the root space with respect to α of complex dimension 1. By [26, Chapter VI, Lemma 3.1], one can choose $X_\alpha \in V_\alpha$ such that $X_{-\alpha} = -\iota(X_\alpha)$, $[X_\alpha, X_{-\alpha}] = \alpha$ and X_α generates the complex line V_α, where ι is the Cartan involution. Let

$$E_\alpha = X_\alpha - X_{-\alpha}, \quad E_{-\alpha} = J(X_\alpha + X_{-\alpha}). \tag{2.5}$$

Denote by $\mathfrak{k}_\alpha$, $\mathfrak{k}_{-\alpha}$ the real lines spanned by E_α, $E_{-\alpha}$, respectively. Then we get the Cartan decomposition of the Lie algebra $\mathfrak{k}$ of $\mathbf{K}$,

$$\mathfrak{k} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_+} (\mathfrak{k}_\alpha \oplus \mathfrak{k}_{-\alpha}).$$

Choose a real basis $\{E_1^0, \ldots, E_r^0\}$ of $\mathfrak{t}$, where r is the dimension of $\mathbf{T}$. Then $\{E_1^0, \ldots, E_r^0\}$ together with $\{E_\alpha, E_{-\alpha}\}_{\alpha \in \Phi_+}$ forms a real basis of $\mathfrak{k}$, which can be reindexed as $\{E_1, \ldots, E_n\}$. We can also regard $\{E_1, \ldots, E_n\}$ as a complex basis of $\mathfrak{g}$. For any $g \in \mathbf{G}$, we define local coordinates $\{z^i_{(g)}\}_{i=1,\ldots,n}$ on a neighborhood of g by

$$(z^i_{(g)}) \to \exp(z^i_{(g)} E_i) g. \tag{2.6}$$

It is easy to see that $\theta^i|_g = dz^i_{(g)}|_g$, where the dual θ^i of E_i is a right-invariant holomorphic 1-form. Thus

$$dV_{\mathbf{G}}|_g := \textstyle\bigwedge_{i=1}^n (dz^i_{(g)} \wedge d\bar{z}^i_{(g)})|_g, \quad \forall g \in \mathbf{G}, \tag{2.7}$$

is also a right-invariant (n, n)-form, which defines a Haar measure.

For a $\mathbf{K} \times \mathbf{K}$-invariant function ϕ as in (2.2), Delcroix computed the Hessian of ϕ in the above local coordinates as follows [16, Theorem 1.2].

Lemma 2.2. *Let ϕ_ψ be a $\mathbf{K} \times \mathbf{K}$-invariant function on $\mathbf{G}$ associated to a Weyl-invariant convex ψ on $\mathfrak{a}$. Then for any $x \in \mathfrak{a}_+$, the complex Hessian matrix of ϕ_ψ in the above coordinates is block diagonal, and equals*

$$\mathrm{Hess}_{\mathbb{C}}(\phi_\psi)(\exp(x)) = \begin{pmatrix} \frac{1}{4}\mathrm{Hess}_{\mathbb{R}}(\psi)(x) & 0 & & 0 \\ 0 & M_{\alpha_{(1)}}(x) & & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & & M_{\alpha_{(\frac{n-r}{2})}}(x) \end{pmatrix}, \tag{2.8}$$

where $\Phi_+ = \{\alpha_{(1)}, \dots, \alpha_{(\frac{n-r}{2})}\}$ *is the set of positive roots and*

$$M_{\alpha_{(i)}}(x) = \frac{1}{2}(\alpha_{(i)}, \nabla\psi(x)) \begin{pmatrix} \coth\alpha_{(i)}(x) & \sqrt{-1} \\ -\sqrt{-1} & \coth\alpha_{(i)}(x) \end{pmatrix}.$$

Here $(\,,\,)$ *is the Killing inner product on the dual space of the Lie algebra* $\mathfrak{t}$.

By (2.8), ψ induced by a $\mathbf{K}\times\mathbf{K}$-invariant Kähler form ω is convex on $\mathfrak{a}$. The complex MA measure is given by

$$\omega_\psi^n = (\sqrt{-1}\,\partial\bar\partial\phi_\psi)^n = \mathrm{MA}_{\mathbb{C}}(\phi_\psi)\, dV_{\mathbf{G}}, \tag{2.9}$$

where

$$\mathrm{MA}_{\mathbb{C}}(\phi_\psi)(\exp(x)) = \frac{1}{2^{r+n}}\mathrm{MA}_{\mathbb{R}}(\psi)(x)\frac{1}{J(x)}\prod_{\alpha\in\Phi_+}(\alpha, \nabla\psi(x))^2. \tag{2.10}$$

In particular, by Proposition 2.1,

$$\mathrm{vol}(M, \omega_\psi) = \int_M \omega_\psi^n = C_0\int_{2P_+}\pi\, dy\ = C_0V_0 = V, \tag{2.11}$$

where P_+ is the quotient space of the moment polytope P by the Weyl group W, and

$$\pi(y) = \prod_{\alpha\in\Phi_+}(\alpha, y)^2. \tag{2.12}$$

For a $\mathbf{K}\times\mathbf{K}$-invariant KE metric defined by a convex function ψ as in Lemma 2.2, by (2.10), it is easy to reduce the KE equation to the following real MA-type equation on the cone $\mathfrak{a}_+$ (see [16]):

$$\pi(\nabla\psi)\det(\nabla^2\psi) = J(x)e^{-\psi}. \tag{2.13}$$

2.2. *Geometry of horosymmetric spaces*

Let $\mathbf{H}$ be a subgroup of $\mathbf{G}$. A homogeneous space $\mathbf{G}/\mathbf{H}$ is called *horosymmetric* if there is a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$ such that $\mathbf{P}/\mathbf{H}$ is a symmetric space and $\mathbf{G}/\mathbf{H}$ is a $\mathbf{P}/\mathbf{H}$-fibration over the generalized flag manifold $\mathbf{G}/\mathbf{P}$ [18]. In particular, if the Lie algebra of $\mathbf{H}$ contains all X_α with $\alpha\in\Phi_+$, the homogeneous space $\mathbf{G}/\mathbf{H}$ is called *horospherical*.

According to [18], there is a Levi subgroup $\mathbf{L}$ of $\mathbf{P}$ and an involution σ of $\mathbf{L}$ such that the following is true for the Lie algebras of $\mathbf{P}$ and $\mathbf{H}$:

(1) The Lie algebra of $\mathbf{P}$ has the decomposition

$$\mathfrak{p} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\alpha\in\Phi_{\mathbf{L}}} X_\alpha \oplus \sum_{\beta\in\Phi_u} X_{-\beta}, \tag{2.14}$$

where $\Phi_{\mathbf{L}}$ is a root system of $\mathbf{L}$, which is a subset of the root system Φ of $\mathbf{G}$, and $\Phi_u = \Phi_+ \setminus (\Phi_+\cap\Phi_{\mathbf{L}})$ is a subset of the positive root system Φ_+ of $\mathbf{G}$, called the set of roots of the unipotent radical of $\mathbf{P}$.

(2) The Lie algebra of $\mathbf{H}$ can be represented as

$$\mathfrak{h} = \mathfrak{t}_0^{\mathbb{C}} \oplus \sum_{\alpha' \in \Phi_{\mathbf{L}^\sigma}} X_{\alpha'} \oplus \sum_{\beta \in \Phi_u} X_{-\beta}, \tag{2.15}$$

where $\mathfrak{t}_0$ is a subtorus of $\mathfrak{t}$ which is fixed by σ and $\Phi_{\mathbf{L}^\sigma}$ is a root system of the fixed subgroup $\mathbf{L}^\sigma$ of $\mathbf{L}$. Clearly, $\mathbf{G}/\mathbf{H}$ is horospherical if and only if $\Phi_{\mathbf{L}} = \Phi_{\mathbf{L}^\sigma}$.

On a horosymmetric spaces $\mathbf{G}/\mathbf{H}$, we can consider $\mathbf{K}$-invariant metrics (see [18]). Let

$$\Phi_s^+ = \Phi_+ \setminus \Phi_{\mathbf{L}^\sigma} \tag{2.16}$$

and

$$\rho_u = \frac{1}{2} \sum_{\beta \in \Phi_u} \beta.$$

Then for a $\mathbf{K}$-invariant metric, there is a convex function ψ on $\mathfrak{a}'$, where $\mathfrak{a}'$ is the real part of $\mathfrak{t}'$ $(\cong \mathfrak{t}/\mathfrak{t}_0)$ with dimension r_0 such that the metric can be divided into three parts as follows (see [18]):

$$\begin{aligned}\omega_\psi((k_1 \exp\{x\} k_2) \cdot \mathbf{H}) = {} & \frac{\sqrt{-1}}{4} \sum_{i,j=1}^{r_0} (\mathrm{Hess}(\psi))_{i\bar{j}} dz^i \wedge d\bar{z}^j \\ & + \frac{\sqrt{-1}}{2} \sum_{\alpha \in \Phi_s^+} \frac{(\alpha, \nabla\psi)}{\sinh\langle\alpha, \cdot\rangle} dz^{i_\alpha} \wedge d\bar{z}^{i_\alpha} \\ & + \frac{\sqrt{-1}}{2} \sum_{\beta \in \Phi_u} e^{-\langle\beta, x\rangle} (\beta, \nabla\psi) dz^{i_\beta} \wedge d\bar{z}^{i_\beta}.\end{aligned} \tag{2.17}$$

Since ψ is invariant under the restricted Weyl group associated to Φ_s^+ [18], $x \in \mathfrak{a}'$ can be restricted to the cone $\mathfrak{a}'_+$ in $\mathfrak{a}'$ given by

$$\mathfrak{a}'_+ = \{x \in \mathfrak{a}' \mid \langle\alpha, x\rangle > 0,\ \forall \alpha \in \Phi_s^+\}. \tag{2.18}$$

Example 2.3. Every reductive Lie group $\mathbf{G}$ can be regarded as a horosymmetric space

$$\mathbf{G} = \hat{\mathbf{G}}/\mathrm{diag}(\mathbf{G} \times \mathbf{G})$$

with the parabolic subgroup $\hat{\mathbf{P}} = \hat{\mathbf{G}} = \mathbf{G} \times \mathbf{G}$. With $\sigma(g_1, g_2) = (g_2, g_1)$, the set Φ_s^+ in (2.16) associated to $\mathbf{G}$ is given by

$$\hat{\Phi}_s^+ = \{(\alpha, 0) \cup (0, -\alpha') \mid \alpha, \alpha' \in \Phi_+\}.$$

By (2.17), we have

$$\begin{aligned}\omega_\psi((\hat{k}_1 \exp\{x\} \hat{k}_2) \cdot \mathbf{H}) = {} & \frac{\sqrt{-1}}{4} \sum_{i,j=1}^{r} (\mathrm{Hess}(\psi))_{i\bar{j}} dz^i \wedge d\bar{z}^j \\ & + \frac{\sqrt{-1}}{2} \sum_{\alpha \in \Phi_+} \frac{(\alpha, \nabla\psi)}{\sinh\langle\alpha, \cdot\rangle} (dz^{i_\alpha} \wedge d\bar{z}^{i_\alpha} + dz^{i_{-\alpha}} \wedge d\bar{z}^{i_{-\alpha}}),\end{aligned} \tag{2.19}$$

where $\hat{k}_1, \hat{k}_2 \in \hat{\mathbf{K}} = \mathbf{K} \times \mathbf{K}$. Then by (2.5), it is easy to see that (2.19) coincides with (2.8), where the holomorphic coordinates $(z^{i_\alpha}, z^{i_{-\alpha}})$ correspond to the basis $\{X_\alpha, X_{-\alpha}\}$, $\alpha = r+1, \ldots, (n+r)/2$.

Let $M = \overline{\mathbf{G}/\mathbf{H}}$ be the Fano compactification of a horosymmetric space such that $\mathbf{G}$ can be holomorphically extended to M. Let Y be a rational vector in $\mathfrak{a}'$ and m an integer. Then by [17, Theorem 3.30], the pair (Y, km) for some integer $k > 0$ will induce a special $\mathbf{G}$-equivariant $\mathbb{C}^*$-degeneration $\mathcal{M}$ of M which can be regarded as a compactification of the homogeneous space

$$\mathbf{G} \times \mathbb{C}^*/\mathbf{H} \times \mathbf{e}.$$

Moreover, the center fiber of $\mathcal{M}$ is a $\mathbb{Q}$-Fano spherical variety $\overline{\mathbf{N}}$ with a homogeneous open orbit $\mathbf{N} = \mathbf{G}/\mathbf{H}_0$ for some subgroup $\mathbf{H}_0$ of $\mathbf{G}$. In the case of the Fano $\mathbf{G}$-manifold M being a compactification of the homogeneous space of Example 2.3, the homogeneous open orbit of center fiber $\overline{\mathbf{N}}$ is horosymmetric, which has been described in [34] as follows.

Example 2.4 (Center fiber $\overline{\mathbf{N}}$). Let $\{\alpha_1, \ldots, \alpha_{r_0}\}$ be a subset of Φ_+ which consists of all simple roots in Φ_+ such that

$$\langle \alpha_i, Y \rangle = 0, \quad i = 1, \ldots, r_0. \tag{2.20}$$

Let $\{\alpha_{r_0+1}, \ldots, \alpha_{r_1}\}$ be the other simple roots in Φ_+ which do not satisfy (2.20). Then there is a subset of Φ_+, $\{\alpha_{r_1+1}, \ldots, \alpha_{r_2}\}$, each of whose elements is a linear combination of some elements in $\{\alpha_1, \ldots, \alpha_{r_0}\}$. Denote by $\{\alpha_{r_2+1}, \ldots, \alpha_{r_3}\}$ the remaining elements of Φ_+ which do not lie in the above three subsets of Φ_+.

We define a subgroup $\mathbf{H}_0$ of $\hat{\mathbf{G}} = \mathbf{G} \times \mathbf{G}$ with Lie subalgebra $\mathfrak{h}$ generated by a basis

$$\begin{aligned}
&\big\{(Y, Y), \mathrm{diag}\{Y^\perp\};\\
&(X_\alpha, X_\alpha), (X_{-\alpha}, X_{-\alpha}),\ \alpha \in \Phi_+ \setminus \Phi_u = \{\alpha_1, \ldots, \alpha_{r_0}, \alpha_{r_1+1}, \ldots, \alpha_{r_2}\};\\
&(X_\beta, 0), (0, X_{-\beta}),\ \beta \in \Phi_u = \{\alpha_{r_0+1}, \ldots, \alpha_{r_1}, \alpha_{r_2+1}, \ldots, \alpha_{r_3}\}\big\}.
\end{aligned} \tag{2.21}$$

Then $\mathbf{N} = \hat{\mathbf{G}}/\mathbf{H}_0$ is a horosymmetric space, which is the open orbit of center $\overline{\mathbf{N}}$ of a special $\mathbf{G}$-equivariant $\mathbb{C}^*$-degeneration of M induced by Y. The parabolic Lie subgroup $\mathbf{P}$ of $\hat{\mathbf{G}}$ associated to $\mathbf{N}$ is generated by a basis

$$\begin{aligned}
&\{(E_i, E_j),\ i, j = 1, \ldots, r;\\
&(X_\alpha, X_{\alpha'}), (X_{-\alpha}, X_{-\alpha'}),\ \alpha, \alpha' \in \Phi_+ \setminus \Phi_u;\\
&(X_\beta, 0), (0, X_{-\beta}),\ \beta \in \Phi_u\}.
\end{aligned} \tag{2.22}$$

Moreover, the sets Φ_s^+ and Φ_u associated to $\mathbf{N}$ are given respectively by

$$\hat{\Phi}_s^+ = \{(\alpha, 0) \cup (0, -\alpha) \mid \alpha \in \Phi_+ \setminus \Phi_u\}$$

and

$$\hat{\Phi}_u = \{(\beta, 0) \cup (0, -\beta) \mid \beta \in \Phi_u\}. \tag{2.23}$$

Thus any $\mathbf{K}\times\mathbf{K}$-invariant metric on $\mathbf{N}$ is determined by a convex function ψ on $\mathfrak{a}$ such that

$$\omega_\psi((\hat{k}_1\exp\{x\}\hat{k}_2)\cdot\mathbf{H}_0)=\frac{\sqrt{-1}}{4}\sum_{i,j=1,\dots,r}(\mathrm{Hess}(\psi))_{i\bar{j}}dz^i\wedge d\bar{z}^j$$
$$+\frac{\sqrt{-1}}{2}\sum_{\alpha\in\Phi_+\setminus\Phi_u}\frac{(\alpha,\nabla\psi)}{\sinh\langle\alpha,\cdot\rangle}(dz^{i_\alpha}\wedge d\bar{z}^{i_\alpha}+dz^{i_{-\alpha}}\wedge d\bar{z}^{i_{-\alpha}})$$
$$+\frac{\sqrt{-1}}{2}\sum_{\beta\in\Phi_u}e^{-\langle\beta,x\rangle}(\beta,\nabla\psi)(dz^{i_\beta}\wedge d\bar{z}^{i_\beta}+dz^{i_{-\beta}}\wedge d\bar{z}^{i_{-\beta}}). \qquad (2.24)$$

The above $x\in\mathfrak{a}$ can be restricted to the cone $\mathfrak{a}'_+$ in $\mathfrak{a}$ associated to the root subsystem $\hat{\Phi}_s^+$,

$$\mathfrak{a}'_+=\{x\in\mathfrak{a}\mid\alpha(x)=\langle\alpha,x\rangle>0,\ \forall\alpha\in\Phi_+\setminus\Phi_u\}. \qquad (2.25)$$

2.3. *Reduced KR soliton equation on horosymmetric manifolds*

As in (2.4), we introduce a function on $\mathfrak{a}'_+$ by

$$J_0(x)=\prod_{\alpha\in\Phi_s^+}\sinh\alpha(x). \qquad (2.26)$$

Then, analogously to (2.13), by (2.17), the KE equation on a horosymmetric manifold can be reduced to the following real MA-type equation on $\mathfrak{a}'_+$ (see also [19, Theorem 4.3]):

$$\pi_0(\nabla\psi+2\rho_u)\det(\nabla^2\psi)=J_0(x)e^{-\psi}, \qquad (2.27)$$

where

$$\pi_0(y)=\prod_{\alpha\in\Phi_s^+\cup\Phi_u}(\alpha,y). \qquad (2.28)$$

We note that in the case of toroidal horospherical manifolds there is a constant $B>0$ such that for any convex function ψ in (2.17) (see [42]),

$$\prod_{\beta\in\Phi_u}(\nabla\psi(x)+2\rho_u,\beta)\ge B,\quad x\in\mathfrak{a}'_+. \qquad (2.29)$$

Similarly, for a KR soliton with respect to a HVF on M induced by an element X in $\mathfrak{a}$, we get a reduced KR soliton equation on $\mathfrak{a}'_+$,

$$\pi_0(\nabla\psi+2\rho_u)\det(\nabla^2\psi)=J_0(x)e^{-\psi-X(\psi)}. \qquad (2.30)$$

By [51], X should lie in the center part $\mathfrak{a}_c$ of $\mathfrak{a}$. Thus in Example 2.4, by (2.24), equation (2.30) becomes[2]

$$\pi(\nabla\psi+4\rho_u)\det(\nabla^2\psi)=J'(x)e^{-\psi-X(\psi)}, \qquad (2.31)$$

[2]Here $2\rho_u$ in (2.30) is replaced by $4\rho_u$ by the notation $\hat{\Phi}_u$ in (2.23).

where $\pi(y)$ is defined by (2.12) and

$$J'(x) = \prod_{\alpha \in \Phi_+ \setminus \Phi_u} \sinh^2 \alpha(x). \tag{2.32}$$

(2.31) is defined on the cone $\mathfrak{a}'_+$ given by (2.25).

3. Reduced Kähler–Ricci flow

By (2.13), for a $\mathbf{K} \times \mathbf{K}$-invariant initial metric in $2\pi c_1(M)$, the KR flow (1.2) on the Fano **G**-manifold M can be reduced to the following parabolic equation of MA type on $\mathfrak{a}_+$:

$$\frac{\partial \psi}{\partial t} = \log[\pi(\nabla \psi) \det(\nabla^2 \psi)] + (\psi + j), \tag{3.1}$$

where

$$j(x) = -\log J(x), \quad x \in \mathfrak{a}_+,$$

is a strictly convex function on $\mathfrak{a}_+$ (see [16, Lemma 3.7]). The solution $\psi = \psi_t$ is a family of Weyl-invariant functions on $\mathfrak{a}$ and $-\frac{\partial \psi}{\partial t}$ is a Ricci potential of the metric ω_ψ. By a result of Perelman (see [43, 51, 54]), there is a constant c_t such that

$$h_t = -\frac{\partial \psi}{\partial t} + c_t$$

is uniformly bounded. In fact, c_t can be normalized by the identity

$$\int_{\mathfrak{a}_+} e^{h_t} \pi(\nabla \psi_t) \det(\nabla^2 \psi_t)\, dx = \int_{2P_+} \pi \, dy = V_0. \tag{3.2}$$

For simplicity, we let $w_t = \psi_t + j$ as a convex function on $\mathfrak{a}_+$. Then by replacing ψ_t with $\psi_t - c_t$ (without confusion, we still use the same ψ_t), (3.1) can be rewritten as a family of MA-type equations

$$\pi(\nabla \psi_t) \det(\nabla^2 \psi_t) = e^{-h_t} e^{-w_t}. \tag{3.3}$$

Note that $j(x) \to \infty$ as $x \to W_\alpha$, where $W_\alpha = \{x \in \mathfrak{a} \mid \langle \alpha, x \rangle = 0\}$ is a Weyl well associated to the positive root α. Moreover, one can show that w_t goes to infinity at infinity of $\mathfrak{a}_+$ by using the fact that $\omega_{\psi_t} \in 2\pi c_1(M)$, as in the proof of [16, Lemma 3.8]. Thus there is a unique $x_t \in \mathfrak{a}_+$ and a number m_t such that

$$\inf_{\mathfrak{a}_+} w_t = w_t(x_t) = m_t. \tag{3.4}$$

As in [16], for any nonnegative integer k, we set

$$A_k(x_t) = \{x \in \mathfrak{a}_+ \mid m_t + k \leq w_t(x) \leq m_t + k + 1\}.$$

Then each $U_k(x_t)=\bigcup_{i=0}^k A_i(x_t)=\{w_t\le m_t+k+1\}$ is a convex set. Moreover, for any $k\ge 0$, $A_k(x_t)$ is a bounded set and the minimum m_t is attained at some point in A_0.

Since

$$\nabla(-j)(x_t)=\nabla\psi_t(x_t)\in P_+,$$

where P_+ is the positive part of the moment polytope P of $(2\pi c_1(M),\mathbb{T}^{\mathbb{C}})$ in $\mathfrak{a}^*$ (see [16, 36, 38]), $\nabla(-j)(x_t)$ is uniformly bounded. Thus there is a uniform constant δ_0 such that

$$\langle\alpha,x_t\rangle\ge\delta_0,\quad \forall\alpha\in\Phi_+. \tag{3.5}$$

This implies that there is a ball $B_\kappa(x_t)\subset A_0$ with a fixed radius κ such that

$$\mathrm{dist}(B_\kappa(x_t),W_\alpha)\ge\delta_0',\quad \forall\alpha\in\Phi_+. \tag{3.6}$$

Due to [16], we also have the following estimates for equation (3.3).

Proposition 3.1. *The following estimates hold:*

$$(1)\ |m_t|\le C\quad \textit{and}\quad (2)\ \mathrm{diam}(U_k(x_t))\le D_k. \tag{3.7}$$

Proof. A version of estimate (1) for m_t has been obtained for a family of MA-type equations considered for the existence problem of KE metrics via the continuity method on toric manifolds and **G**-manifolds in [56, Lemma 3.2] and [16, Proposition 4.4], respectively. Thus following the argument for the KR flow on toric manifolds [63], we can also get (1). We leave the detailed proof to the reader.

Estimate (2) follows from (1) and the gradient estimate of $w_t(x)$ (or $j(x)$)) together with equation (3.3) with the normalization (3.2) via the volume. ∎

To prove Theorem 1.2, we establish the following explicit result.

Theorem 3.2. *Let (M,J) be a Fano **G**-manifold. Then only three cases for the KR flow (3.1) are possible:*

Case 1: *There is a sequence t_i such that*

$$|x_{t_i}|\le C.$$

Then (3.1) converges to a KE metric in the sense of Kähler potentials. As a consequence, (M,J) is a KE manifold.

Case 2: *$|x_t|\to\infty$ as $t\to\infty$ and there is a sequence t_i such that*

$$|Pj(x_{t_i})|\le C,$$

where Pj is the projection $(\mathfrak{a}_c,\mathfrak{a}')\to\mathfrak{a}'$ with $\mathfrak{a}_c$ the center part of $\mathfrak{a}$. Then (3.1) converges to a KR soliton in the sense of Kähler potentials. As a consequence, (M,J) admits a KR soliton.

Case 3: $|Pj(x_t)| \to \infty$ *as* $t \to \infty$*. Then there are two subcases:*

Case 3.1: *For any* $\alpha \in \Phi_+$,

$$\langle \alpha, x_t \rangle \to \infty \quad as\ t \to \infty. \tag{3.8}$$

Then (3.1) *converges locally smoothly in the Cheeger–Gromov topology to a KR soliton* ω_{KS} *on a horospherical space whose completion is the GH limit of the KR flow* (1.2) *with the structure of a* $\mathbb{Q}$*-Fano horospherical variety* M_∞ *as the limit of a* $\mathbb{C}^*$*-degeneration of* (M, J) *induced by an element in the Lie algebra of the Cartan torus of* **G**. *Moreover, if* ω_{KS} *is not a KE metric, then* M_∞ *coincides with the limit of the* $\mathbb{C}^*$*-degeneration of* (M, J) *induced by the soliton HVF of* ω_{KS}.

Case 3.2: (3.8) *does not hold. Then there is a subset* Φ_u *of* Φ_+ *such that*

$$\langle \beta, x_t \rangle \to \infty \quad as\ t \to \infty,\ \forall \beta \in \Phi_u, \tag{3.9}$$

and

$$\delta_0 \le \langle \alpha', x_t \rangle \le A, \quad \forall \alpha' \in \Phi_+ \setminus \Phi_u. \tag{3.10}$$

Moreover, as in Case 3.1, (3.1) *converges locally smoothly in the Cheeger–Gromov topology to a KR soliton* ω_{KS} *on a horosymmetric space, whose completion is the GH limit of the KR flow* (1.2) *with the structure of a* $\mathbb{Q}$*-Fano horosymmetric variety* M_∞ *as the limit of a* $\mathbb{C}^*$*-degeneration of* (M, J) *induced by an element in the Lie algebra of the Cartan torus of* **G**; *if* ω_{KS} *is not a KE metric, then* M_∞ *coincides with the limit of the* $\mathbb{C}^*$*-degeneration of* (M, J) *induced by the soliton HVF of* ω_{KS}.

Remark 3.3. (1) If **G** is semisimple, Case 2 in Theorem 3.2 does not happen.

(2) In Case 2, when (M, J) is a toric manifold, the result was proved in [63, Theorem 1.1].

(3) Case 3.1 is just Case 3.2 when $\Phi_u = \Phi_+$. The lower bound of (3.10) comes from (3.5).

(4) In Case 3, Φ_u will not be empty and so the curvature along the flow must blow up according to [37, proof of Lemma 4.4, Case 1, p. 14] (see also [65, Lemma 7.4]).[3] In particular, (M, J) does not admit any KR soliton [20, 47]. Thus together with the results in Cases 1 and 2, Theorem 3.2 also implies Theorem 1.1.

Theorem 3.2 will be proved in Section 5. In the rest of this section, we prove several fundamental lemmas for Case 3.2, which will be used in the next two sections.

Lemma 3.4. *Suppose that* $|Pj(x_t)| \to \infty$ *as* $t \to \infty$ *in Case* 3.2. *Let*

$$X_t = \frac{x_t}{|x_t|}. \tag{3.11}$$

[3] In fact, the flow must blow up if there are $\beta \in \Phi_+$ and a sequence $\{x_{t_i}\}$ such that (3.9) holds.

Then there is a subset Φ_u of Φ_+ such that

$$\lim_t \langle \nabla \log \sinh(\langle \beta, x\rangle)(x_t), X\rangle = \langle \beta, X\rangle, \quad \forall \beta \in \Phi_u,\ X \in \mathfrak{a};$$
$$\langle \alpha', X_\infty\rangle = 0, \quad \forall \alpha' \in \Phi_+ \setminus \Phi_u, \tag{3.12}$$

where X_∞ is the limit of any sequence X_{t_i}.

Proof. By (3.5), it is easy to see that there are a subset Φ_u of Φ_+ and a sequence t_i such that (3.9) and (3.10) are both satisfied. Since there are finitely many choices of Φ_u, and x_t is continuous in t (see (5.11) below), (3.9) must hold uniformly in t. As a consequence, (3.10) also holds uniformly in t. In the case of (3.9), for any vector $X \in \mathfrak{a}$ we have

$$\lim_t \langle \nabla \ln \sinh(\langle \beta, x\rangle)(x_t), X\rangle = \langle \beta, X\rangle, \quad \forall \beta \in \Phi_u.$$

In the case of (3.10), we get

$$\langle \alpha', X_\infty\rangle = \lim_i \frac{1}{|x_{t_i}|}\langle \alpha', x_{t_i}\rangle = 0, \quad \forall \alpha' \in \Phi_+ \setminus \Phi_u,$$

since $|x_{t_i}| \to \infty$. ∎

Remark 3.5. Φ_u is the same as in (2.21); it actually corresponds to the set $\hat{\Phi}_u$ of roots of the unipotent radical in (2.23) for the horosymmetric space $\mathbf{N} = \hat{\mathbf{G}}/\mathbf{H}_0$ in Example 2.4. In fact, we may modify X_∞ to be a rational vector in $\mathfrak{a}$ so that (2.20) is satisfied. Then $\mathbf{N}$ is a horosymmetric space with a parabolic subgroup $\mathbf{P}$ given by (2.22).

Lemma 3.6. *Under the assumption of Lemma 3.4, we have*

$$\operatorname{dist}(U_k(x_t), W_\alpha) \geq \delta_k > 0, \quad \forall \alpha \in \Phi_+. \tag{3.13}$$

Proof. By (2) in Proposition 3.1, we see that

$$|\psi(x) - \psi(x_t)| \leq C_1, \quad \forall x \in U_k(x_t).$$

Thus, it suffices to verify (3.13) for $\alpha \in \Phi_+ \setminus \Phi_u$. In fact, by (3.9) and (3.10), we have

$$\begin{aligned}|w_k(x) - w_k(x_t)| &\geq -C_1 + |j(x) - j(x_t)| \\ &\geq -C_2 + \Big| - \prod_{\alpha \in \Phi_+ \setminus \Phi_u} \ln \sinh(\langle \alpha, x\rangle) + \prod_{\alpha \in \Phi_+ \setminus \Phi_u} \ln \sinh(\langle \alpha, x_t\rangle)\Big| \\ &\geq -C_3 + \Big| - \prod_{\alpha \in \Phi_+ \setminus \Phi_u} \ln \sinh(\langle \alpha, x\rangle)\Big|.\end{aligned}$$

Then

$$\Big| - \prod_{\alpha \in \Phi_+ \setminus \Phi_u} \ln \sinh(\langle \alpha, x\rangle)\Big| \leq C_k$$

and so

$$\langle \alpha, x\rangle \geq \epsilon_k, \quad \forall \alpha \in \Phi_+ \setminus \Phi_u,\ x \in U_k(x_t).$$

This implies (3.13). ∎

Lemma 3.7. *Let*

$$\rho_u = \frac{1}{2}\sum_{\beta\in\Phi_u}\beta. \tag{3.14}$$

Then

$$|\psi_t(x_t) - \langle 4\rho_u, x_t\rangle| \le C. \tag{3.15}$$

Proof. By (3.9) and (3.10), it is easy to see that $|j(x_t) + \langle 4\rho_u, x_t\rangle| \le C$. Then by (3.7), we get (3.15). ∎

By (3.10), we see that for each sequence $\{x_{t_i}\} \to \infty$ there is a subsequence of $\{x_{t_i}\}$ (not relabeled) such that the corresponding sequence of functions $\sinh^2(\langle\alpha', x_{t_i}\rangle)$ $(\alpha' \in \Phi_+ \setminus \Phi_u)$ has a limit. Thus the limit function

$$\hat{J}'(x) = \lim_i \prod_{\alpha'\in\Phi_+\setminus\Phi_u} \sinh^2(\langle\alpha', x + x_{t_i}\rangle) \tag{3.16}$$

is well-defined.

Lemma 3.8. *Let* $\{x_{t_i}\} \to \infty$ *be a sequence in* (3.16). *Then there is a vector* $\mathbf{a}_0 \in \mathfrak{a}_+$ *such that*

$$\lim_i \sinh(\langle\alpha', x_{t_i}\rangle) = \sinh(\langle\alpha', \mathbf{a}_0\rangle), \quad \forall\alpha' \in \Phi_+ \setminus \Phi_u. \tag{3.17}$$

As a consequence,

$$\hat{J}'(x) = \prod_{\alpha'\in\Phi_+\setminus\Phi_u} \sinh^2(\langle\alpha', x + \mathbf{a}_0\rangle). \tag{3.18}$$

Proof. Choose a set $\{\alpha_1, \ldots, \alpha_{r_0}\}$ of simple roots in $\Phi_+ \setminus \Phi_u$. Then there are numbers $\delta_{\alpha'}$ such that

$$\lim_i \sinh(\langle\alpha', x_{t_i}\rangle) = \delta_{\alpha'}, \quad \forall\alpha' \in \{\alpha_1, \ldots, \alpha_{r_0}\}.$$

Let $M_{\alpha'}$ be the hyperplane given by

$$M_{\alpha'} = \{x \in \mathfrak{a} \mid \langle\alpha', x\rangle = \delta_{\alpha'}\}.$$

Thus there is a vector $\mathbf{a}_0 \in \bigcap_{\alpha'\in\{\alpha_1,\ldots,\alpha_{r_0}\}} M_{\alpha'}$ such that

$$\sinh(\langle\alpha', \mathbf{a}_0\rangle) = \delta_{\alpha'}, \quad \forall\alpha' \in \{\alpha_1, \ldots, \alpha_{r_0}\}.$$

This implies

$$\sinh(\langle\alpha', \mathbf{a}_0\rangle) = \delta_{\alpha'}, \quad \forall\alpha' \in \Phi_+ \setminus \Phi_u,$$

and so (3.17) and (3.18) hold. ∎

4. Local C^2-estimate

In this section, we first give an upper bound for the Hessian of ψ through the norm estimate of torus vector fields on M. We prove the following general result.

Lemma 4.1. *Let (M, ω_0) be a Fano manifold with $\omega_0 \in 2\pi c_1(M)$. Let $g = g(t)$ be a family of solutions of the KR flow (1.2) with initial metric ω_0. Then for any Hamitonian HVF X which satisfies $L_{\mathrm{Im}(X)}\omega_0 = 0$, we have*

$$|\nabla\theta_X|_g = |X|_g \le C(|X|_{\omega_0}, \omega_0), \tag{4.1}$$

where $\theta_X = \theta_X(g)$ is a real-valued potential of X associated to g.

Proof. We need to recall Perelman's result for the KR flow again (see [43, 50, 54]). Namely, for the flow (1.2), there is a uniform constant A_0 depending on the initial metric ω_0 such that

$$|h| \le A_0, \quad |\nabla h|_g \le A_0, \quad |R_g| \le A_0, \tag{4.2}$$

where $h = h_g$ is the Ricci potential of ω_g, which is normalized as in (3.2) by

$$\int_M e^h \omega_g^n = \int_M \omega_0^n.$$

Moreover, by a result of Zhang [59], the Sobolev constant $C_s(\omega_g)$ of ω_g is uniformly bounded and it depends only on ω_0.

After adding a constant, θ_X satisfies the following equation (see [23,51]):

$$\Delta_g\theta_X + \theta_X + (\partial\theta_X, \partial h)_g = 0. \tag{4.3}$$

Note that θ_X is uniformly bounded (see [61,62]). Actually,

$$\sup_M \theta_X(g) = \sup_M \theta_X(\omega_0), \quad \inf_M \theta_X(g) = \inf_M \theta_X(\omega_0).$$

Then the function $F = e^h\theta_X$ is also uniformly bounded by the first relation of (4.2). Moreover, by (4.3), we have

$$\Delta_g F = -F + \theta_X \Delta_g e^h = -F + e^h\theta_X(R_g - n + |\nabla h|_g^2). \tag{4.4}$$

Clearly, $e^h\theta_X(R_g - n + |\nabla h|_g^2)$ is uniformly bounded by (4.2). Thus integrating (4.4) multiplied by $-F$, we see that $\int_M |\nabla F|_g^2 \omega_g^n$ is uniformly bounded. Hence, by the gradient estimate in [47, Lemma 7.2], we obtain

$$|\nabla F|_g \le C(\|\nabla F\|_{L^2(M,g)}, C_s(\omega_g), |\nabla h|_g).$$

As a consequence, we deduce (4.1) from (4.2). The lemma is proved. ∎

We apply Lemma 4.1 to the family of $\mathbf{K} \times \mathbf{K}$-invariant metrics $\omega_{\psi_t} = \sqrt{-1}\,\partial\bar\partial\phi_{\psi_t}$ in the KR flow (3.1) on the Fano G-manifold (M, ω_0). Let $\{e_1, \ldots, e_r\}$ be r vector fields generated by the basis $\{E_1^0, \ldots, E_r^0\}$ of $\mathfrak{t}$. Then on $T^{\mathbb{C}}$,

$$e_A = \frac{\partial}{\partial x_A} + \sqrt{-1}\, J\frac{\partial}{\partial x_A} = \frac{\partial}{\partial z^A}, \quad A = 1, \ldots, r.$$

It follows that

$$|e_A|^2_{\omega_{\psi_t}} = \omega_{\psi_t}\left(\frac{\partial}{\partial z^A}, \overline{\frac{\partial}{\partial z^A}}\right) = \psi_{t,AA}, \quad A = 1, \ldots, r, \tag{4.5}$$

Proposition 4.2.

$$\mathrm{Hess}(\nabla^2\psi_t) \leq C.$$

Proof. Let

$$X = \sum_A a_A e_A$$

for some real numbers a_A, $A = 1, \ldots, r$. Then by Lemma 4.1,

$$|X|^2_{\omega_{\psi_t}} \leq C(a_1, \ldots, a_r).$$

Namely by (4.5),

$$\sum a_A a_B \psi_{t,AB} \leq C'(a_1, \ldots, a_r).$$

Thus the proposition is true since a_A can be chosen arbitrary. ∎

Lemma 4.3. *On each $U_k(x_t)$,*

$$(\nabla\psi_t, \alpha) \geq C_k > 0, \quad \forall\alpha \in \Phi_+. \tag{4.6}$$

Proof. By Proposition 4.2 and (3.3), we have

$$\pi(\nabla\psi_t) \geq C'_k > 0 \quad \text{on } U_k(x_t).$$

Note that $(\nabla\psi_t, \alpha)$ is uniformly bounded. Thus we get (4.6) immediately. ∎

Proposition 4.4. *Let*

$$\hat{\psi}_t = \psi_t(\cdot + x_t) - \psi_t(x_t).$$

Then $\hat{\psi}_t$ is uniformly $C^{l,\alpha}$-bounded for any $l \geq 0$ and $\alpha \in (0, 1)$ on each $U_k(x_t) - \{x_t\}$.

Proof. Since $(\nabla\psi_t, \alpha)$ is uniformly bounded, by (3.3) and the first relation of (3.7) we have

$$\det(\nabla^2\psi_t) \geq B_k \quad \text{on } U_k(x_t), \tag{4.7}$$

where the constant B_k depends only on k. Then by Proposition 4.2,

$$B_k \leq \mathrm{Hess}(\hat{\psi}_t) \leq C \quad \text{on } U_k(x_t) - \{x_t\}.$$

Thus, by the regularity of the parabolic MA equation (3.1) together with Proposition 3.1 and Lemmas 3.6 and 4.3, we conclude that $\hat{\psi}_t(\cdot)$ is locally uniformly $C^{l,\alpha}$-bounded for any $l \geq 0$ and $\alpha \in (0, 1)$. ∎

Proposition 4.5. *Under the assumption of Lemma 3.4, the set $U_k(x_t) - \{x_t\}$ converges as $t \to \infty$ to a cone $\mathfrak{a}'_+ \supset \mathfrak{a}_+$ in $\mathfrak{a}$, defined by*

$$\mathfrak{a}'_+ = \{x \in \mathfrak{a} \mid \langle\alpha', x\rangle > 0,\ \forall\alpha' \in \Phi_+ \setminus \Phi_u\}. \tag{4.8}$$

Proof. By Lemmas 3.7 and 3.8, we have

$$\hat{w}_t(x) = w_t(x + x_t) = \psi_t(x + x_t) - \psi_t(x_t) - \langle\rho_u, x\rangle - \log\hat{J}'(x + \mathbf{a}_0) + O(1). \tag{4.9}$$

Let B_r be an r-ball in $\mathfrak{a}$. Then the leading term of $\hat{w}_t(x)$ in $B_r \cap \mathfrak{a}'_+$ is $-\log\hat{J}'(x + \mathbf{a}_0)$. Thus the set $\{x \in \mathfrak{a} \mid \hat{w}_t(x) \leq k\}$ will exhaust $\mathfrak{a}'_+$, and the proposition is proved. ∎

5. Proof of Theorem 3.2

In this section, we prove Theorem 3.2 and so get Theorem 1.2. We first establish the existence of a limit solution of the reduced KR flow (3.1).

5.1. Smooth KR soliton solution on $\mathbf{N}$

Let $\mathbf{N}$ be the horosymmetric space constructed in Example 2.4 associated to the vector $X_\infty \in \mathfrak{a}$ in Lemma 3.4 (see Remark 3.5). Let $\mathfrak{a}'_+$ be the cone in $\mathfrak{a}$ defined by (4.8). We need to prove the following.

Proposition 5.1. *Let ρ_u be the vector in $\mathfrak{a}'_+$ defined in (3.14) and $J'(x)$ the function defined on $\mathfrak{a}'_+$ by (2.32). Then there is a smooth function ϕ on $\mathfrak{a}'_+$ which satisfies the equation*

$$\pi(\nabla\phi + 4\rho_u)\det(\nabla^2\phi) = J'(x)e^{-\phi - Y(\phi)}, \tag{5.1}$$

where the vector $Y \in \mathfrak{a}'_+$ satisfies

$$\langle\alpha', Y\rangle = 0, \quad \forall\alpha' \in \Phi_+ \setminus \Phi_u. \tag{5.2}$$

Moreover, the metric ω_ϕ of the form (2.24) associated to ϕ is a $\mathbf{K}\times\mathbf{K}$-invariant KR soliton with respect to the HVF e_Y on $\mathbf{N}$ induced by Y.

By Proposition 4.4, $\hat\psi_t$ is uniformly $C^{l,\alpha}$-bounded on each $U_k(x_t) - \{x_t\}$. Then there is a sequence $\hat\psi_{t_i}$ which converges locally uniformly to a smooth function ψ_∞ on $\mathfrak{a}'_+$. Without loss of generality, we may assume that t_i is chosen as in (3.16) with the function $\hat J'(x)$. On the other hand, by (3.3),

$$\begin{aligned}\pi(\nabla\hat\psi_{t_i})\det(\nabla^2\hat\psi_{t_i}) = e^{-h_{t_i}} &\prod_{\alpha'\in\Phi_+\setminus\Phi_u}\sinh^2(\langle\alpha', x + x_{t_i}\rangle)e^{-(\hat\psi_{t_i} + \psi_{t_i}(x_{t_i}) - \langle 4\rho_u, x_{t_i}\rangle)}\\ &\times e^{-\langle 4\rho_u, x_{t_i}\rangle}\prod_{\beta\in\Phi_u}\sinh^2(\langle\beta, x + x_{t_i}\rangle).\end{aligned} \tag{5.3}$$

By Lemmas 3.7 and 3.8, on each set $U_k - \{x_{t_i}\}$ we have respectively

$$e^{\psi_{t_i}(x_{t_i}) - \langle 4\rho_u, x_{t_i}\rangle} \times e^{-\langle 4\rho_u, x_{t_i}\rangle}\prod_{\beta\in\Phi_u}\sinh^2(\langle\beta, x + x_{t_i}\rangle) \to A_0 e^{\langle 4\rho_u, x\rangle} \tag{5.4}$$

for some constant A_0, and

$$\prod_{\alpha'\in\Phi_+\setminus\Phi_u}\sinh^2(\langle\alpha', x + x_{t_i}\rangle) \to \hat J'(x). \tag{5.5}$$

Moreover, the convergences are both uniform on a fixed domain $U \subset \mathfrak{a}'_+$ which is contained in $U_k(x_t) - \{x_t\}$ for $k \gg 1$.

Lemma 5.2. *Let*
$$\hat{h}_t(\cdot) = h_t(\cdot + x_t).$$
Then there are $Y \in \mathfrak{a}'_+$ and a constant c such that
$$\lim_t \hat{h}_t(\cdot) = Y(\psi_\infty) + c, \tag{5.6}$$
where $Y(\psi_\infty) = \langle Y, \nabla\psi_\infty\rangle$. The convergence of $\hat{h}_t(\cdot)$ is locally uniform in $C^{l,\alpha}$ as also is the convergence of $\hat{\psi}_t$. Moreover, Y satisfies (5.2).

Proof. We will modify the argument in [63, Lemma 4.1] for **G**-manifolds. The difficulty is that $\hat{\psi}_t$ has only local convergence at present. The proof is divided into four steps.

Step 1. Recall x_t is the critical point of w_t in (3.4). We claim that there is an $\epsilon_0 > 0$ such that
$$|x_t - x_{t+\epsilon_0}| \le C, \quad \forall t > 0, \tag{5.7}$$
where C is a uniform constant.

For any time $t > 0$, we consider equation (3.1) on $[t, t+\epsilon_0)$,
$$\frac{\partial\psi'}{\partial s} = \log[\pi(\nabla\psi')\det(\nabla^2\psi')] + (\psi' + j), \quad s \in [t, t+\epsilon_0). \tag{5.8}$$
By Perelman's result, we may assume
$$\left|\frac{\partial\psi'}{\partial s}(t,\cdot)\right| = h_t \le A, \tag{5.9}$$
where h_t is chosen as in (3.2). Then the solution ψ'_s just differs by a constant from ψ_s in (3.3). By the maximum principle for $\frac{\partial\psi'}{\partial s}$, we get
$$|\psi'_t - \psi'_s| \le C_1\epsilon_0, \quad \left|\frac{\partial\psi'}{\partial s}\right| \le 2A, \quad \forall s \in [t, t+\epsilon_0). \tag{5.10}$$

Let
$$w' = w'_s = \psi'_s + j.$$
Then the minimal point of w'_s coincides with the minimal point x_s of w_s in (3.4). By Proposition 4.4, we see that
$$C_2^{-1} \le \mathrm{Hess}(\nabla^2\psi'_t) \le C_2 \quad \text{on } B_\kappa(x_t),$$
where the κ-ball $B_\kappa(x_t)$ is chosen as in (3.6). It follows that
$$(C'_2)^{-1} \le \mathrm{Hess}(\nabla^2 w'_t) \le C'_2 \quad \text{on } B_\kappa(x_t).$$
Thus
$$|w'_t(x_t) - w'_t(x)| \ge 4C_1\epsilon_0, \quad \forall x \in \partial B_\kappa(x_t),$$
if ϵ_0 is chosen small enough. As a consequence,
$$w'_t(x) \ge w'_t(x_t) + 4C_1\epsilon_0, \quad \forall x \in \partial B_\kappa(x_t).$$

Hence, by (5.10),

$$w_s'(x) \geq w_s'(x_t) + 2C_1\epsilon_0 \geq \inf_{x' \in B_\kappa(x_t)} w_s'(x') + 2C_1\epsilon_0, \quad \forall x \in \partial B_\kappa(x_t).$$

Therefore, there is a critical point x_s' of w_s' in $B_\kappa(x_t)$ such that

$$\nabla w_s'(x_s') = 0.$$

By the convexity of w_s', x_s' is the global minimal point of w_s'. Thus $x_s' = x_s$. Moreover,

$$|x_s - x_t| \leq \kappa, \quad \forall s \in [t, t+\epsilon_0).$$

In particular,

$$|x_{t+\epsilon_0} - x_t| \leq \kappa, \quad \forall t > 0. \tag{5.11}$$

Hence, we have proved (5.7). Actually, from the above argument one can deduce that for any small κ there exists ϵ_0 such that (5.11) holds. This implies that $x_t \in \mathfrak{a}_+$ is uniformly continuous in t.

Step 2. We prove the uniqueness of ψ_∞, which is independent of the choice of the sequence $\{\hat{\psi}_{t_i}\}$.

By (5.7), we see that for any integer $l > 0$ we have

$$|x_{l\epsilon_0} - x_{(l+1)\epsilon_0}| \leq C.$$

Then we can use a trick as in [15,51,63] to choose a smooth family of modified points y_t in $\mathfrak{a}_+$ such that

$$|x_t - y_t| \leq C \quad \text{and} \quad \left|\frac{dy_t}{dt}\right| \leq C. \tag{5.12}$$

Moreover, we may assume that the vector $\frac{dy_t}{dt}$ in $\mathfrak{a}_+$ has a limit $Y \in \mathfrak{a}_+'$.

Let $\tilde{\psi}'(\cdot) = \tilde{\psi}_t'(\cdot) = \psi_t'(y_t + \cdot)$ and

$$w_t'(x) = \tilde{\psi}_t'(x) + j(x + y_t).$$

Then

$$m_t' = \inf_{\mathfrak{a}_+} w_t'(x) = w_t'(x_t)$$

also satisfies (1) in Proposition 3.1. Since $\nabla\psi'$ is uniformly bounded, analogously to the function $\hat{\psi}_t$ in Proposition 4.4,

$$\psi_t(x_t + \cdot) - \psi_t(x_t) = \psi_t'(x_t + \cdot) - \psi_t'(x_t),$$

and

$$\hat{\psi}_t' = \tilde{\psi}_t' - \psi_t'(y_t) \tag{5.13}$$

is also uniformly $C^{l,\alpha}$-bounded on each $U_k'(x_t) = \{x \in \mathfrak{a}_+ \mid w_t'(x) - m_t' < k+1\}$. As in Proposition 4.5, $U_k'(x_t) - \{x_t\}$ converges to the cone $\mathfrak{a}_+' \supset \mathfrak{a}_+$. Hence, any sequence

$\{\hat{\psi}'_{t_i}\}$ will converge to a convex function ψ'_∞ on $\mathfrak{a}'_+$, like $\{\hat{\psi}_{t_i}\}$ to ψ_∞. We need to show that ψ'_∞ modulo a constant is independent of the choice of the sequence $\{\hat{\psi}'_{t_i}\}$.

In fact, by Proposition 6.1 below, $\omega_{\psi'_{t_i}}$ converges locally to $(\mathbf{N}, \omega_{\psi'_\infty})$ in the Cheeger–Gromov topology. Thus, by the HT conjecture [3,54], $\omega_{\psi'_\infty}$ is actually a KR soliton ω_{KS} on $\mathbf{N}$ which is a regular open set of the GH limit space M_∞ of the KR flow (1.2). By the uniqueness of ω_{KS} [13], we conclude that ψ'_∞ modulo a constant is independent of the sequence $\{\hat{\psi}'_{t_i}\}$. Therefore, ψ_∞ is also independent of the sequence $\{\hat{\psi}_{t_i}\}$ under the normalization (3.2).

Step 3. We now prove (5.6). Note that

$$\frac{\partial \psi'}{\partial s} = \frac{\partial \tilde{\psi}'}{\partial s} - e_s(\hat{\psi}'), \quad \forall s \in [l\epsilon_0, (l+1)\epsilon_0), \tag{5.14}$$

where $e_s(\hat{\psi}') = \frac{dy_s}{ds}(\hat{\psi}') = \langle \frac{dy_s}{ds}, \nabla \hat{\psi}' \rangle$. Then by (5.8), $\frac{\partial \psi'}{\partial s}$ is uniformly bounded in the $C^{l,\alpha}$-norm on $U_k(x_t) - \{x_t\}$, and so is $\frac{\partial \tilde{\psi}'}{\partial s}$. Thus $\frac{\partial \tilde{\psi}'}{\partial s}$ is convergent as long as $\frac{\partial \psi'}{\partial s}$ is. On the other hand, by the uniqueness of the soliton HVF e_∞ associated to the limits of KR solitons along the flow (1.2) [55, Corollary 3.3] (see also [13]), $\frac{\partial \psi'}{\partial s}$ converges to a potential θ_{e_∞} of e_∞ [54, Lemma 4.2]. Hence there is a smooth θ on $\mathfrak{a}'_+$ and a family of constants A_s such that

$$\lim_s \left(\frac{\partial \hat{\psi}'}{\partial s} - A_s \right) = \theta,$$

since $\frac{\partial \hat{\psi}'}{\partial s}$ just differs by a constant from $\frac{\partial \tilde{\psi}'}{\partial s}$. By the normalization (5.9), it is easy to see that the A_s are uniformly bounded.

We claim that θ is a constant. Suppose that it is not. Then there are two points x_∞, y_∞ in $\mathfrak{a}'_+$, limits of sequences $\{x_l\}, \{y_l\}$ in $U'_k(x_t) - \{x'_t\}$, such that

$$|\theta(x_\infty) - \theta(y_\infty)| \geq 2a_0.$$

Thus, as $l \gg 1$, we may assume

$$\frac{\partial \hat{\psi}'}{\partial s}(x_l) - \frac{\partial \hat{\psi}'}{\partial s}(y_l) \geq a_0, \quad \forall s \in [l\epsilon_0, (l+1)\epsilon_0). \tag{5.15}$$

On the other hand, there is a sequence of constants c_l such that

$$\lim_{s \to l\epsilon_0^-} \frac{\partial \psi'}{\partial s} = \left. \frac{\partial \psi'}{\partial s} \right|_{l\epsilon_0} + c_l,$$

where $\left. \frac{\partial \psi'}{\partial s} \right|_{l\epsilon_0} = h_{l\epsilon_0}$ is normalized as in (5.9). Then we also have

$$\lim_{s \to l\epsilon_0^-} \frac{\partial \hat{\psi}'}{\partial s} = \left. \frac{\partial \hat{\psi}'}{\partial s} \right|_{l\epsilon_0} + c'_l.$$

In particular,

$$\lim_{s \to l\epsilon_0^-} \frac{\partial \hat{\psi}'}{\partial s}(x_l) - \lim_{s \to l\epsilon_0^-} \frac{\partial \hat{\psi}'}{\partial s}(y_l) = \left. \frac{\partial \hat{\psi}'}{\partial s} \right|_{l\epsilon_0}(x_l) - \left. \frac{\partial \hat{\psi}'}{\partial s} \right|_{l\epsilon_0}(y_l).$$

Thus the function $\frac{\partial \hat{\psi}'}{\partial s}(x_l) - \frac{\partial \hat{\psi}'}{\partial s}(y_l)$ is continuous in s. Hence, for any integers l, k with $l < k$, we get

$$[\hat{\psi}'|_{k\epsilon_0}(x_l) - \hat{\psi}'|_{l\epsilon_0}(x_l)] - [\hat{\psi}'|_{k\epsilon_0}(y_l) - \hat{\psi}'|_{l\epsilon_0}(y_l)] \\ = \int_{l\epsilon_0}^{k\epsilon_0} \left[\frac{\partial \hat{\psi}'}{\partial s}(x_l) - \frac{\partial \hat{\psi}'}{\partial s}(y_l)\right] ds. \tag{5.16}$$

But this is impossible since the left side of (5.16) is always bounded by (5.13), while the right side of (5.16) goes to infinity as $k \to \infty$ by (5.15). Hence, θ must be a constant.

Recall that ψ'_∞ is a C^∞ limit of $\hat{\psi}'_{t_i}$ and $\mathbf{b}$ a limit of the vectors $x_{t_i} - y_{t_i}$. Then it is easy to see that

$$\psi'_\infty(x) = \psi_\infty(x - \mathbf{b}) + c \tag{5.17}$$

for some constant c. Thus by (5.14) and the above claim, we have

$$\hat{h}_\infty = \lim_t \hat{h}_t(\cdot) = \lim_t h_t(\cdot + x_t) = -\lim_t \frac{\partial \tilde{\psi}'}{\partial s}\bigg|_{s=t} + c_t = \langle Y, \nabla \psi_\infty \rangle + c.$$

This proves (5.6).

Step 4. We prove (5.2). We may assume that Y is not zero and take a sequence x_{t_i} in (3.16) with $t_i = t_i\epsilon_0$. By the first relation in (5.12), with no loss of generality we may also assume that

$$1 < |y_{t_{i+1}} - y_{t_i}| \le N_0$$

and

$$\lim_i \langle \alpha', y_{t_i} \rangle = \delta_{\alpha'} > 0, \quad \forall \alpha' \in \Phi_+ \setminus \Phi_u.$$

Note that

$$y_{t_{i+1}} - y_{t_i} = |y_{t_{i+1}} - y_{t_i}| \left(\frac{Y}{|Y|} + o(1)\right), \quad \forall \alpha' \in \Phi_+ \setminus \Phi_u.$$

Thus

$$\begin{aligned} 0 &= \lim_i |\langle \alpha', y_{t_{i+1}} - y_{t_i} \rangle| \\ &= \lim_i |y_{t_{i+1}} - y_{t_i}| \, |\langle \alpha', Y/|Y| + o(1) \rangle| \\ &\ge |\langle \alpha', Y/|Y| + o(1) \rangle|. \end{aligned}$$

Hence (5.2) must be true. ∎

Remark 5.3. (1) It seems possible to prove that $Y = aX_\infty$ for some $a \neq 0$, where X_∞ is the limit of X_t in (3.11). In particular, Y will be not zero in Case 3 of Theorem 3.2. This implies that in Case 3 the Fano $\mathbf{G}$-manifold (M, J) must be K-unstable by [54, Propositions 4.16, 4.17]. On the other hand, in Case 1 (or 2) of Theorem 3.2, (M, J) admits a KE metric (or a KR soliton) and so it is K-polystable [5] (or modified K-polystable

[9, 24]). Thus we guess that a Fano **G**-manifold (M, J) is K-polystable (or modified K-polystable) if it is K-semistable (or modified K-semistable) by the above observation. Actually, we will confirm this in the case of modified K-semistability: see Corollary 7.2 below.

(2) Note that ψ'_t just differs by a constant from ψ_t. By (5.17), the limit $\hat{\phi}(x) = \psi'_\infty - \langle \rho_u, x\rangle$ coincides with the solution $\phi(x)$ in (5.1), which is the limit of the sequence

$$\hat{\phi}_{t_i} = \psi_{t_i}(x + x_{t_i} + \mathbf{b}) - \psi_{t_i}(x_{t_i}) - \langle 4\rho_u, x\rangle + c \tag{5.18}$$

for some constant c.

Proof of Proposition 5.1. By Lemma 5.2 together with (5.4) and (5.5), we get from (5.3),

$$\pi(\nabla\psi_\infty)\det(\nabla^2\psi_\infty) = A'_0 J'(x - \mathbf{a}_0)e^{\langle 4\rho_u, x\rangle}e^{-\psi_\infty - Y(\psi_\infty)}$$

for some constant A'_0. The above equation can be defined in the cone $\mathfrak{a}'_+$ by Proposition 4.5. Let

$$\phi(x) = \psi_\infty(x + \mathbf{a}_0) - \langle 4\rho_u, x\rangle. \tag{5.19}$$

Then we obtain

$$\pi(\nabla\phi + 4\rho_u)\det(\nabla^2\phi) = \tilde{A}_0 J'(x)e^{-\phi - Y(\phi)},$$

where

$$J'(x) = \hat{J}'(x - \mathbf{a}_0).$$

Hence, by adding a constant to $\phi(x)$, $\phi(x)$ becomes a solution of (5.1).

By the relation (5.19), $\phi(x)$ is invariant under the restricted Weyl group and so is $Y(\phi)$. Let e_Y be an induced HVF of Y on **N** and ψ_ϕ be a smooth function associated to ϕ on **N** defined by

$$\psi_\phi(\hat{k}_1 \exp\{x\}\hat{k}_2 \cdot \mathbf{H}_0) = \phi(x), \quad \forall x \in \mathfrak{a}'_+.$$

Then the function $Y(\phi)$ on $\mathfrak{a}'_+$ can be extended to a smooth function $e_Y(\psi_\phi)$ on **N**. Recall that ω_ϕ is a $\mathbf{K} \times \mathbf{K}$-invariant metric associated to ϕ on **N** defined as in (2.24). By (5.1), ω_ϕ satisfies the KR soliton equation

$$\mathrm{Ric}(\omega_\phi) - \omega_\phi = \sqrt{-1}\,\partial\bar{\partial}Y(\phi) = L_{e_Y}\omega_\phi. \tag{5.20}$$

That is, we have proved that e_Y is the soliton HVF of ω_ϕ. ∎

By Proposition 5.1 the metrics $\omega_\phi((\hat{k}_1 \exp\{x\}\hat{k}_2) \cdot \mathbf{H}_0)$ of the form (2.24) determined by the solution ϕ of (5.1) define a KR soliton ω_{KS} on **N**. Moreover, $\omega_\phi((\hat{k}_1 \exp\{x\}\hat{k}_2) \cdot \mathbf{H}_0)$ is the local limit of the KR flow $\omega(t) = \omega_{\psi'_t}$ in the Cheeger–Gromov topology with **N** being a smooth part of M_∞ by Proposition 6.1 below. In the remaining part of this section, we shall extend $\omega_\phi((\hat{k}_1 \exp\{x\}\hat{k}_2) \cdot \mathbf{H}_0)$ to a singular KR soliton on $\overline{\mathbf{N}}$ in order to use the modified K-stability theory to complete the proof of Theorem 3.2.

Remark 5.4. When $\Phi_u = \Phi_+$, **P** is just a Borel subgroup of $\mathbf{G} \times \mathbf{G}$. Thus **N** is a horospherical space in this case.

5.2. Singular KR soliton solution on $\overline{\mathbf{N}}$

Proposition 5.5. *Let $Y \in \mathfrak{a}'_+$ be as in Proposition 5.1. Then there exists a singular $\mathbf{K}\times\mathbf{K}$-invariant KR soliton ω'_{KS} with respect to Y on $\overline{\mathbf{N}}$.*[4]

Proof. We will apply Corollary 8.3 in the Appendix. Let a $\mathbf{K}\times\mathbf{K}$-invariant metric ω_0 on M be the initial metric in (1.2).

As in Section 6 below, we let $2P_\infty = \mathrm{Image}(\nabla\phi')(\mathfrak{a}'_+)$. Then by the volume relation in (6.5), we have

$$\mathrm{vol}(\mathbf{N}, \omega_{KS}) = \int_{2P_\infty} \pi(y)\, dy = \int_{2P_+} \pi(y)\, dy = \mathrm{vol}(M, \omega_0). \tag{5.21}$$

Note that any subset of P_∞ can be approximated by the gradient images of a locally convergent sequence $\hat{\psi}_{t_i}$. Thus $P_\infty \subseteq P_+$. By (5.21),

$$P_\infty = P_+. \tag{5.22}$$

As a consequence, $\frac{1}{2}(2P_+ - 4\rho_u)$ is a quotient of the moment polytope P' associated to the torus Lie algebra $\mathfrak{t}'$ and the metric ω_{KS}. Hence, P' satisfies the Delzant condition [1]. In particular, P' is *fine*, which means that each vertex of P' is the intersection of precisely $r\ (= \dim(P'))$ facets (see [22, 36]).

For any admissible metric ω representing $\frac{1}{l}c_1(\overline{\mathbf{N}}, L_{\overline{\mathbf{N}}}^{-l})$ [44], by $\mathbb{C}^*$-degeneration, we have

$$\mathrm{vol}(M, \omega_0) = \mathrm{vol}(\overline{\mathbf{N}}, \omega).$$

Then by (5.21), we get

$$\mathrm{vol}(\mathbf{N}, \omega_{KS}) = \mathrm{vol}(\overline{\mathbf{N}}, \omega),$$

which means that $(\mathbf{N}, \omega_{KS})$ has full mass. On the other hand, since ω is induced from the Fubini–Study metric in a higher-dimensional $\mathbb{C}P^m$, we may assume that it is $\mathbf{K}\times\mathbf{K}$-invariant and smooth on $\mathbf{N}$. By the computation of volume for the metric of the form (2.24), it follows that

$$\mathrm{vol}(\overline{\mathbf{N}}, \omega) = C_0 \int_{2P_{\overline{\mathbf{N}}}^+} \pi(y + 4\rho_u)\, dy,$$

where $P_{\overline{\mathbf{N}}}^+$ is the quotient of the moment polytope $P_{\overline{\mathbf{N}}}$ associated to $\frac{1}{l}c_1(\overline{\mathbf{N}}, L_{\overline{\mathbf{N}}}^{-l})$ by the restricted Weyl group (see [17]). Thus by the first equality in (5.21), we derive

$$\int_{2P'_+} \pi(y + 4\rho_u)\, dy = \int_{2P_\infty - 4\rho_u} \pi(y + 4\rho_u)\, dy = \int_{2P_{\overline{\mathbf{N}}}^+} \pi(y + 4\rho_u)\, dy, \tag{5.23}$$

where $P'_+ = \frac{1}{2}(2P_+ - 4\rho_u)$ is the quotient of the moment polytope P' by the restricted Weyl group. Since $P'_+ \subseteq P_{\overline{\mathbf{N}}}^+$ means $P_\infty \subseteq P_+$, we conclude that

$$P' = P_{\overline{\mathbf{N}}}.$$

Hence, we have proved that $P_{\overline{\mathbf{N}}}$ is fine as so is P', while $(\mathbf{N}, \omega_{KS})$ has full mass in (5.21).

[4] We do not know whether ω_{KS} can be extended to a singular KR soliton on $\overline{\mathbf{N}}$ with a weak Kähler potential in $\mathcal{E}^1(\overline{\mathbf{N}}, -K_{\overline{\mathbf{N}}})$ directly from the solution ϕ of (5.1).

Now we can apply Corollary 8.3 to see that there is a singular $\mathbf{K}\times\mathbf{K}$-invariant KR soliton ω'_{KS} with respect to Y on the $\mathbb{Q}$-Fano horosymmetric variety $\overline{\mathbf{N}}$. ■

Proof of Theorem 3.2. It is clear that in the family of x_t there exists either a sequence $\{t_i\}$ which is uniformly bounded, or one that goes to infinity uniformly. In the latter case, there exists either a sequence $\{t_i\}$ for which $Pj(x_{t_i})$ are uniformly bounded, or one for which $Pj(x_t)$ go to infinity uniformly. Thus we need to prove the theorem in the three cases considered there.

Case 1. The proof in this case is actually the same as in [16,56] by the continuity method. We can let the initial metric ω_0 in (3.1) be a background $\mathbf{K}\times\mathbf{K}$-invariant metric given by a $\mathbf{K}\times\mathbf{K}$-invariant function ψ_0 as in (2.9) such that

$$\omega_0 = \sqrt{-1}\,\partial\bar\partial\phi_{\psi_0}.$$

Then as in the case of toric manifolds, the Kähler potential $\varphi_{\psi_t} = \phi_{\psi_t} - \phi_{\psi_0}$ for the solution ψ_t of (3.1) has a uniform upper bound (see [56, Lemma 3.4], [63, Proposition 3.2]). By the Harnack inequality (see [51, Propositions 3.1, 5.1]), we get a uniform C^0-bound for φ_{ψ_t}. Hence, by the regularity of the KR flow (1.2), we get all $C^{k,\alpha}$-bounds for φ_{ψ_t}. As a consequence, the limit φ_∞ of φ_{ψ_t} will define a KE metric

$$\omega_{KE} = \sqrt{-1}\,\partial\bar\partial(\phi_{\psi_0} + \varphi_\infty)$$

on (M, J). The proof in this case is finished.

Case 2. The proof in this case is the same as in Case 1. Let

$$x_t = a_t + Pj(x_t),$$

where $a_t \in \mathfrak{a}_c$. Then we define a new family of $\mathbf{K}\times\mathbf{K}$-invariant functions by

$$\tilde\psi_t(\cdot) = \psi_t(\cdot + a_t),$$

By (3.1), $\tilde\psi_t$ satisfies

$$\frac{\partial\psi}{\partial t}(\cdot + a_t) = \log[\pi(\nabla\tilde\psi)\det(\nabla^2\tilde\psi)] + (\tilde\psi + j). \tag{5.24}$$

By the same argument as for toric manifolds [63, Proposition 4.1], we can also get a C^0-estimate for Kähler potentials as in Case 1,

$$\varphi_{\tilde\psi_t} = \phi_{\tilde\psi_t} - \phi_{\psi_0}.$$

Moreover, all $C^{k,\alpha}$-norms of $\varphi_{\tilde\psi_t}$ are uniformly bounded. Hence, in this case, the $\mathbf{K}\times\mathbf{K}$-invariant metric

$$\sqrt{-1}\,\partial\bar\partial\phi_{\tilde\psi_t} = \sqrt{-1}\,\partial\bar\partial(\phi_{\psi_0} + \varphi_{\tilde\psi_t}) \tag{5.25}$$

will converge subsequentially to a KR soliton on (M, J) in the sense of Kähler potentials, and $\hat h_t$ in Lemma 5.2 converges to a potential of a soliton HVF (see (5.6) and (5.20)).

In fact, if we let g_t be a family of group elements in the center $\mathbf{T}_c$ associated to a_t, the holomorphic transformations

$$\sigma_t : \mathbf{G} \to g_t\mathbf{G}$$

induce the metrics in (5.25) by $\sigma_t^*\omega_{\psi_t}$.

Case 3. By Remark 3.3 (3) we need to consider Case 3.2. Then both (3.9) and (3.10) hold according to the proof of Lemma 3.4.

We notice that $(\mathbf{N}, \omega_{KS})$ is a smooth part of $(M_\infty, \omega_\infty)$ by Proposition 6.1. Moreover, by (5.21), $(\mathbf{N}, \omega_{KS})$ does not lose the volume of $(M, \omega(t))$. Then by [3, Theorem 1.1], the metric completion of $(\mathbf{N}, \omega_{KS})$ is the limit $(M_\infty, \omega_\infty)$ of the KR flow (1.2) in the GH topology. Also, we know that M_∞ is homeomorphic to a $\mathbb{Q}$-Fano variety $\hat{M}_\infty$ which admits a singular KR soliton $\hat{\omega}_{KS}$ with respect to the HVF Y with Kähler potential in $\mathcal{E}^1(\hat{M}_\infty, -K_{\hat{M}_\infty})$ [54]. It remains to show that $\hat{M}_\infty$ is biholomorphic to $\overline{\mathbf{N}}$. In the following, we assume that $(M_\infty, \omega_\infty)$ is a KE metric or a KR (but not KE) soliton.

Suppose that $(M_\infty, \omega_\infty)$ is a (singular) KE metric. In other words, $Y = 0$ since ω_∞ equals ω_{KS} on $\mathbf{N}$ from the proof of Lemma 5.2. Then the soliton metric $\hat{\omega}_{KS}$ is KE and so $\hat{M}_\infty$ is reductive by [55, Proposition A.1]. By using the technique of partial C^0-estimates [54, 55], there exists a special degeneration from M to $\hat{M}_\infty$ via Luna's lemma. Thus Mabuchi's K-energy on (M, J) is bounded below [29] (see also [55, Proposition 5.2]). As a consequence, (M, J) is K-semistable in the sense of [44] (see [40]). On the other hand, $\overline{\mathbf{N}}$ is also K-polystable just as $\hat{M}_\infty$ since we know that it admits a singular KE metric ω'_{KE} by Proposition 5.5 together with the fact $Y = 0$ [5]. Note that $\overline{\mathbf{N}}$ is the center limit of a special degeneration of M induced by $X_\infty \in \mathfrak{a}$. Therefore, by the uniqueness result of Li–Xu–Wang [33, Theorem 1.3] for K-polystable limits of $\mathbb{C}^*$-degenerations, $\hat{M}_\infty$ must be biholomorphic to $\overline{\mathbf{N}}$.

In the special case that $\overline{\mathbf{N}}$ is smooth, ω'_{KE} is smooth (see [7, Theorem B.1]). This means that the $\mathbb{C}^*$-degeneration on M induced by X_∞ is a smooth deformation with the KE manifold $\overline{\mathbf{N}}$ as its center. By [55, Theorem 1.4] (or [46, Proposition 7.7]) for the special case of KE manifolds, $(\overline{\mathbf{N}}, \omega'_{KE})$ is a smooth limit of the KR flow (1.2). Thus by the uniqueness of limits of the KR flow (independent of the choice of the initial metric $\omega_0 \in 2\pi c_1(M)$) [25, 55], we also infer that $\hat{M}_\infty$ is biholomorphic to $\overline{\mathbf{N}}$.

Suppose now that $(M_\infty, \omega_\infty)$ is not a (singular) KE metric, so $Y \neq 0$. Then by (5.2) we can modify Y to be a rational vector field in $\mathfrak{a}$ as done for X_∞ in Remark 3.5 so that (2.20) or (3.12) is satisfied. Thus, the $\mathbb{Q}$-Fano variety $\overline{\mathbf{N}}$ is also a limit of a $\mathbb{C}^*$-degeneration of (M, J) induced by the soliton HVF Y according to the construction of the subgroup $\mathbf{H}_0$ of $\hat{\mathbf{G}} = \mathbf{G} \times \mathbf{G}$ in Example 2.4 (see also Remark 3.5). By a result of [13], there is another $\mathbb{C}^*$-degeneration relative to Y from $\overline{\mathbf{N}}$ to $\hat{M}_\infty$. Since both $\overline{\mathbf{N}}$ and $\hat{M}_\infty$ admit singular KR solitons with respect to Y, this new $\mathbb{C}^*$-degeneration must be trivial by the modified K-stability for the KR soliton $(\overline{\mathbf{N}}, \omega'_{KS})$ with respect to Y [9, 24, 53]. Hence, $\hat{M}_\infty$ is biholomorphic to $\overline{\mathbf{N}}$. As a consequence, M_∞ can be realized as a $\mathbb{C}^*$-degeneration of (M, J) induced by X_∞. Moreover, ω_{KS} can be extended to a singular KR soliton with respect to Y on $\overline{\mathbf{N}}$ since $\mathbf{N}$ can be regarded as a smooth dense subset of $\hat{M}_\infty$ because ω_{KS} has full mass by (5.21) [54, Theorem 1.1]. The metric completion of $(\mathbf{N}, \omega_{KS})$ is isometric to $(M_\infty, \omega_\infty)$ by [3, Theorem 1.1] with the help of (5.21). Hence, the proof in Case 3 is complete and so Theorem 3.2 is proved. ∎

6. Locally smooth limit of $(M, \omega_{\psi'_{t_i}})$

Let $U_k(x_t)$ be a sequence of subsets of $\mathfrak{a}_+$ as in Proposition 3.1 and

$$W_k(x_t) = \{g = k_1 \exp(x) k_2 \in \mathbf{G} \mid \pi(g) = x \in U_k(x_t)\}.$$

In this section, we prove

Proposition 6.1. *Let $\{\hat{\psi}'_{t_i}\}$ be a subsequence of $\hat{\psi}'_t$ in (5.13) which converges to a convex function ψ'_∞ on $\mathfrak{a}'_+$. Let*[5]

$$\phi'(x) = \psi'_\infty(x + \mathbf{a}_0 + \mathbf{b}), \tag{6.1}$$

where $\mathbf{a}_0$ and $\mathbf{b}$ are two vectors in $\mathfrak{a}$ determined in (3.17) and (5.17), respectively. Then there exist a sequence $k_i \to \infty$ and a family of diffeomorphisms σ_{t_i} from $W_{k_i}(x_{t_i})$ to $\mathbf{N}$ such that $(\sigma_{t_i}^{-1})^* \omega_{\psi'_{t_i}}|_{\sigma_{t_i}(W_{k_i}(x_{t_i}))}$ converges locally to $(\mathbf{N}, \omega_{\phi'})$, where $\omega_{\phi'}$ is a global metric defined on $\mathbf{N}$ associated to ϕ' as in (2.24).

Proof. For any $q_t \in W_k(x_t)$ with $\pi(q_t) = y_t \in U_k(x_t)$, we choose a neighborhood U_{y_t} of y_t such that $U_{y_t} \subset U_k(x_t)$ and $\pi^{-1}(U_{y_t}) = \bigcup_\gamma W^\gamma(y_t) \subset W_k(x_t)$, where each open set $W^\gamma(y_t)$ with holomorphic coordinates z as in (2.6) satisfies

$$B_{\delta_1} \times B'_{\delta_1} \subset \{z(q') \mid q' \in W^\gamma(x_t)\} \subset B_{\delta_2} \times B'_{\delta_2}$$

for some δ_1-ball B_{δ_1} and δ_2-ball B_{δ_2} ($\delta_1 < \delta_2$) in $\mathbb{C}^r$, and δ'_1-ball B'_{δ_1} and δ'_2-ball B'_{δ_2} in $\mathbb{C}^{n-r}$.

For each $\beta \in \Phi_u$, we let

$$(z^{i_\beta}, z^{i_{-\beta}}) = e^{\frac{1}{2}\langle \beta, x_t \rangle} (\hat{z}^{i_\alpha}, \hat{z}^{i_{-\alpha}}). \tag{6.2}$$

Then in the new coordinates $\hat{z} = (z^1, \ldots, z^r, \ldots, z^{i_\alpha}, z^{i_{-\alpha}}, \ldots, \hat{z}^{i_\beta}, \hat{z}^{i_{-\beta}}, \ldots)$ on $W^\gamma(y_t)$, $\omega_{\psi'_{t_i}}$ from (2.19) turns into

$$\begin{aligned}
\omega_{\psi'_{t_i}} = {} & \frac{\sqrt{-1}}{4} \sum_{i,j=1}^{r} (\mathrm{Hess}(\psi'_{t_i}))_{i\bar{j}} dz^i \wedge d\bar{z}^j \\
& + \frac{\sqrt{-1}}{2} \sum_{\alpha \in \Phi_+ \setminus \Phi_u} \frac{(\alpha, \nabla \psi'_{t_i})}{\sinh \langle \alpha, \cdot \rangle} (dz^{i_\alpha} \wedge d\bar{z}^{i_\alpha} + dz^{i_{-\alpha}} \wedge d\bar{z}^{i_{-\alpha}}) \\
& + \frac{\sqrt{-1}}{2} \sum_{\beta \in \Phi_u} \frac{e^{\langle \beta, x_{t_i} \rangle}}{\sinh \langle \beta, \cdot \rangle} (\beta, \nabla \psi'_{t_i}) (d\hat{z}^{i_\beta} \wedge d\bar{\hat{z}}^{i_\beta} + d\hat{z}^{i_{-\beta}} \wedge d\bar{\hat{z}}^{i_{-\beta}}).
\end{aligned} \tag{6.3}$$

[5] ϕ' differs from ϕ in Proposition 5.1 by a linear function; see (5.17) and (5.19).

Recall that ψ_∞ is the limit of $\hat{\psi}_{t_i}$ in Proposition 4.4. Since ψ'_t just differs by a constant from ψ_t, by Proposition 4.4 together with Lemmas 3.4 and 3.8, the metric matrix of $\omega_{\psi'_{t_i}}$ in (6.3) is uniformly positive definite and its elements

$$(\mathrm{Hess}(\psi'_{t_i}))_{i\bar{j}}, \quad \frac{(\alpha, \nabla\psi'_{t_i})}{\sinh\langle\alpha,\cdot\rangle}, \quad \frac{e^{\langle\beta, x_{t_i}\rangle}}{\sinh\langle\beta,\cdot\rangle}(\beta, \nabla\psi'_{t_i})$$

are uniformly $C^{l,\alpha}$-bounded in the coordinates $\hat{z}$. In fact, the metric matrix converges to a form of (2.24) as in Example 2.4,

$$\begin{aligned}
\omega_{\phi'} = {} & \frac{\sqrt{-1}}{4}\sum_{i,j=1}^{r}(\mathrm{Hess}(\phi'))_{i\bar{j}}dz^i\wedge d\bar{z}^j \\
& + \frac{\sqrt{-1}}{2}\sum_{\alpha\in\Phi_+\setminus\Phi_u}\frac{(\alpha,\nabla\phi')}{\sinh\langle\alpha,\cdot\rangle}(dz^{i_\alpha}\wedge d\bar{z}^{i_\alpha} + dz^{i_{-\alpha}}\wedge d\bar{z}^{i_{-\alpha}}) \\
& + \frac{\sqrt{-1}}{2}\sum_{\beta\in\Phi_u}e^{-\langle\beta,x\rangle}(\beta,\nabla\phi')(d\hat{z}^{i_\beta}\wedge d\bar{\hat{z}}^{i_\beta} + d\hat{z}^{i_{-\beta}}\wedge d\bar{\hat{z}}^{i_{-\beta}}). \qquad (6.4)
\end{aligned}$$

Since ϕ' is globally well-defined on $\mathfrak{a}'_+$ and ω_ϕ is invariant under the restricted Weyl group, $\omega_{\phi'}$ actually defines a global metric $\omega_{\phi'}$ on $\mathbf{N}$.

By the above convergence of the metric matrix of $\omega_{\psi'_{t_i}}$, the curvature of $\omega_{\psi'_{t_i}}$ on $\pi^{-1}(U_{y_{t_i}})$ is uniformly bounded. Thus by the volume noncollapse result of Perelman for the KR flow (1.2) (see [43]), the injectivity radius is uniformly bounded below on $\pi^{-1}(U_{y_{t_i}})$ (see [41, Lemma 51]). By the compactness theorem (see [41, Theorems 72, 74]), we see that $(\pi^{-1}(U_{y_{t_i}}), \omega_{\psi'_{t_i}})$ converges smoothly to an open Riemannian manifold $(W_\infty, \omega_\infty)$ in the Cheeger–Gromov (abbreviated by CG) topology. We need to show that $(W_\infty, \omega_\infty)$ is isometric to an open subset of $(\mathbf{N}, \omega_{\phi'})$.

Let x'_0 be a limit of $y_{t_i} \in U_k(x_{t_i}) - \{x_{t_i}\}$ in $\mathbb{R}^r$ and $q'_0 = \hat{k}_1\exp\{x'_0\}\hat{k}_2\cdot\mathbf{H}_0 \in \mathbf{N}$. Then in the new coordinates $\hat{z}$, there is a Euclidean ball $\hat{D}$ in $\mathbb{C}^n$ centered at the origin and a geodesic ϵ'_0-ball $B_{\epsilon'_0}(\omega_{\phi'}) \subset \mathbf{N}$ centered at q'_0 such that

$$\hat{z}(B_{\epsilon'_0}(\omega_{\phi'})) \subset \hat{D}.$$

On the other hand, for any curve Γ starting from q_{t_i} such that $\pi(\Gamma)\cap U_{y_{t_i}} \neq \emptyset$, we see that by the Hessian estimate of ψ'_{t_i} above there exists a uniformly small constant ϵ such that

$$\mathrm{length}(\Gamma) \geq \epsilon.$$

Moreover, for any minimal geodesic ray Γ starting from q_{t_i} such that $\pi(\Gamma)\cap U_{y_{t_i}} = \emptyset$, its length is greater than the injectivity radius of q_{t_i}, which is uniformly bounded below. Thus we conclude that there is a uniform constant $\epsilon_0 > 0$ such that each set $\pi^{-1}(U_{y_{t_i}})$ contains a geodesic ϵ_0-ball $B_{\epsilon_0}(\omega_{\psi'_{t_i}})$ centered at q_{t_i}. We may assume $\epsilon_0 < \epsilon'_0$. Note that by the coordinate transformation in (6.2), it is easy to see that for any fixed index γ the set $W^\gamma(y_{t_i}) \cap B_{\epsilon_0}(\omega_{\psi'_{t_i}})$ converges to a subset in $B_{\epsilon'_0}(\omega_\phi)$ with zero volume in the CG topology. Hence, by the smoothness of $\omega_{\psi'_t}$ in the geodesic coordinates, there is a sequence $\gamma_i \to \infty$ such that the open Riemannian manifold $(\bigcup_{\gamma=1}^{\gamma_i} W^{\gamma}(y_{t_i}) \cap B_{\epsilon_0}(\omega_{\psi'_{t_i}}), \omega_{\psi'_{t_i}})$ converges to an open subset of $B_{\epsilon'_0}(\omega_{\phi})$ in the CG topology. As a consequence, $(W_k(x_{t_i}) \cap B_{\epsilon_0}(\omega_{\psi'_{t_i}}), \omega_{\psi'_{t_i}})$ converges to an open subset of $B_{\epsilon'_0}(\omega_{\phi'})$ in the CG topology.

Summarizing the above, for a fixed k and any sequence $q_{t_i} \in W_k(x_{t_i})$ there are geodesic ϵ-balls $B_\epsilon(\omega_{\psi'_{t_i}})$ in $(\mathbf{G}, \omega_{\psi'_{t_i}})$ which converge subsequentially to a geodesic ϵ-ball $B_\epsilon(\omega_{\phi'})$ in $(\mathbf{N}, \omega_{\phi'})$ in the CG topology. Then by the standard covering argument (cf. [41, proof of Theorem 74]), we conclude that $(W_k(x_{t_i}), \omega_{\psi'_{t_i}})$ (taking a subsequence of $\{t_i\}$ if necessary) converges to an open subset in $(\mathbf{N}, \omega_{\phi'})$ in the CG topology. By the diagonal method, $(\mathbf{G}, \omega_{\psi'_{t_i}})$ converges locally and subsequentially to an open set in $(\mathbf{N}, \omega_{\phi'})$. Thus, there exist a sequence $k_i \to \infty$ and a family of diffeomorphisms σ_{t_i} (maybe for a subsequence of $\{t_i\}$) from $W_{k_i}(x_{t_i})$ to $\mathbf{N}$ such that $((\sigma_{t_i}^{-1})^* \omega_{\psi'_{t_i}}, \sigma_{t_i}(W_{k_i}(x_{t_i})))$ converges locally uniformly to an open subset W_∞ in $\mathbf{N}$.

Next we prove that the open set $W'_{k_i} = \sigma_{t_i}(W_{k_i}(x_{t_i}))$ exhausts $\mathbf{N}$. We claim that

$$\lim_i \mathrm{vol}(W_{k_i}(x_{t_i}), \omega_{\psi'_{t_i}}) = \mathrm{vol}(\mathbf{N}, \omega_{\phi'}). \tag{6.5}$$

We notice that similarly to the case of toric manifolds [56, 63], from the argument for the upper bound estimate of m_t in Proposition 3.1 (see also [16, Proposition 4.4]), we can get an estimate

$$\int_{\mathfrak{a}_+ \setminus U_k(x_t)} e^{w_t}\, dx \le \frac{C}{e^{m_t}} \sum_{l \ge k+1} \frac{(l+1)^n}{e^l},$$

which goes to zero by Proposition 3.1 as $k \to \infty$. Since e^{-h_t} is uniformly bounded, we obtain

$$\lim_{k,t\to\infty} \int_{\mathfrak{a}_+ \setminus U_k(x_t)} e^{-h_t + w_t}\, dx = 0.$$

By (3.3), it follows that

$$\lim_{k,t\to\infty} \int_{\mathfrak{a}_+ \setminus U_k(x_t)} \pi(\nabla \psi_t) \det(\nabla^2 \psi_t)\, dx = 0. \tag{6.6}$$

Thus by the volume formula (2.11), we derive

$$\begin{aligned} &\lim_i \mathrm{vol}(W_{k_i}(x_{t_i}), \omega_{\psi'_{t_i}}) \\ &\quad = C_0 \int_{2P_+} \pi(y)\, dy - C_0 \lim_i \int_{\mathfrak{a}_+ \setminus U_{k_i}(x_{t_i})} \pi(\nabla \psi'_{t_i}) \det(\nabla^2 \psi'_{t_i})\, dx \\ &\quad = C_0 \int_{2P_+} \pi(y)\, dy = \mathrm{vol}(M, \omega_0). \end{aligned} \tag{6.7}$$

On the other hand, if we let $2P_\infty = \mathrm{Image}(\nabla \phi')(\mathfrak{a}'_+)$, then by the similar volume formula (2.11) for the metric form (6.4) and the local convergence of $\hat{\psi}'_{t_i}$, we have

$$
\begin{aligned}
\operatorname{vol}(\mathbf{N}, \omega_{\phi'}) &= C_0 \int_{2P_\infty} \pi(y)\, dy \\
&= C_0 \lim_i \int_{\mathrm{Image}(\nabla\psi'_{t_i})(U_{k_i}(x_{t_i}))} \pi(y)\, dy \\
&= C_0 \lim_i \int_{U_{k_i}(x_{t_i})} \pi(\nabla\psi'_{t_i}) \det(\nabla^2\psi'_{t_i})\, dx.
\end{aligned}
$$

Thus similarly to (6.7), we also get

$$
\begin{aligned}
\operatorname{vol}(\mathbf{N}, \omega_{\phi'}) &= C_0 \int_{2P_+} \pi(y)\, dy - C_0 \lim_i \int_{\mathfrak{a}_+ \setminus U_{k_i}(x_{t_i})} \pi(\nabla\psi'_{t_i}) \det(\nabla^2\psi'_{t_i})\, dx \\
&= C_0 \int_{2P_+} \pi(y)\, dy = \operatorname{vol}(M, \omega_0). \qquad (6.8)
\end{aligned}
$$

Hence, combining (6.7) and (6.8), we obtain (6.5).

By the volume noncollapse result of Perelman together with (6.7), the CG limit $(W'_\infty, \omega_\infty)$ $(= (W_\infty, \omega_{\phi'}))$ of $(W_{k_i}(x_{t_i}), \omega_{\psi'_{t_i}})$ is a smooth dense subset of the global GH limit M_∞ of $(M, \omega_{\psi'_{t_i}})$. On the other hand, by [3, Theorem 1.1], the metric completion of this dense subset is just the metric space M_∞. Thus, ω_∞ can be extended on M_∞, that is,

$$
\overline{(W'_\infty, \omega_\infty)} = (M_\infty, \omega_\infty).
$$

Since $W'_{k_i} \subset \mathbf{N}$, we have $\overline{(W'_{k_i}, \omega_{\phi'})} \subseteq \overline{(\mathbf{N}, \omega_{\phi'})}$. It follows that

$$
\overline{(W'_\infty, \omega_\infty)} \subseteq \overline{(\mathbf{N}, \omega_{\phi'})}.
$$

Thus by (6.5), we get

$$
\overline{(W'_\infty, \omega_{\phi'})} = \overline{(\mathbf{N}, \omega_{\phi'})}.
$$

Hence, W'_{k_i} must exhaust $\mathbf{N}$. The proof is complete. ■

7. Case of horosymmetric manifolds

In this section, we prove Theorem 1.5. By (2.27), as in the case of Fano **G**-manifolds, the KR flow (1.2) with a **K**-invariant initial metric on a Fano horosymmetric manifold (M, J) can be reduced to the following parabolic equation of MA type:

$$
\frac{\partial\psi}{\partial t} = \log[\pi_0(\nabla\psi + 2\rho_u)\det(\nabla^2\psi)] + (\psi + j_0) \quad \text{in } \mathfrak{a}'_+, \qquad (7.1)
$$

where $\mathfrak{a}'_+$ is the cone in $\mathfrak{a}'$ defined in (4.8), and

$$
j_0(x) = -\log J_0(x) \quad \text{in } \mathfrak{a}'_+.
$$

As in Section 3, we let

$$
w'_t(x) = (\psi + j_0)(x).
$$

Then as in (3.4), there is a family $\{x_t\}$ such that

$$\inf_{\alpha'_+} w'_t = w'_t(x_t) = m_t, \quad |m_t| \le C. \tag{7.2}$$

Analogously to Theorem 3.2, we can establish a convergence result for the flow (7.1) for horosymmetric manifolds.

Theorem 7.1. *Let (M, J) be a Fano horosymmetric manifold. Let x_t be as in (7.2). Then for the flow (7.1), only three cases can occur:*

Case 1: *There is a sequence t_i such that*

$$|x_{t_i}| \le C.$$

Then (7.1) converges to a KE metric in the sense of Kähler potentials. As a consequence, (M, J) is a KE manifold.

Case 2: $|x_t| \to \infty$ *as* $t \to \infty$ *and there is a sequence t_i such that*

$$|Pj(x_{t_i})| \le C.$$

Then (7.1) converges to a KR soliton in the sense of Kähler potentials. As a consequence, (M, J) admits a KR soliton.

Case 3: $|Pj(x_t)| \to \infty$ *as* $t \to \infty$*. Then there are two subcases:*

Case 3.1: *For any $\alpha \in \Phi_s^+$, we have*

$$\langle \alpha, x_t \rangle \to \infty \quad \text{as } t \to \infty. \tag{7.3}$$

Then (7.1) converges locally smoothly in the Cheeger–Gromov topology to a KR soliton ω_{KS} on a horospherical space whose completion is the GH limit of the KR flow (1.2) with the structure of a $\mathbb{Q}$-Fano horospherical variety M_∞ as the limit of a $\mathbb{C}^*$-degeneration of (M, J) induced by an element in the Lie algebra of the Cartan torus of M. Moreover, if ω_{KS} is not a KE metric, then M_∞ coincides with the limit of the $\mathbb{C}^*$-degeneration of (M, J) induced by the soliton HVF of ω_{KS}.

Case 3.2: (7.3) *does not hold. Then there is a subset Φ_0 of Φ_s^+ such that*

$$\langle \beta, x_t \rangle \to \infty \quad \text{as } t \to \infty, \ \forall \beta \in \Phi_0, \tag{7.4}$$

and

$$\delta_0 \le \langle \alpha', x_t \rangle \le A, \quad \forall \alpha' \in \Phi_s^+ \setminus \Phi_0. \tag{7.5}$$

Moreover, as in Case 3.1, (7.1) *converges locally smoothly in the Cheeger–Gromov topology to a KR soliton ω_{KS} on a horosymmetric space whose completion is the GH limit of the KR flow (1.2) with the structure of a $\mathbb{Q}$-Fano horosymmetric variety M_∞ as the limit of a $\mathbb{C}^*$-degeneration of (M, J) induced by an element in the Lie algebra of the Cartan torus of M; if ω_{KS} is not a KE metric, then M_∞ coincides with the limit of the $\mathbb{C}^*$-degeneration of (M, J) induced by the soliton HVF of ω_{KS}.*

The proof Theorem 7.1 is almost the same as that of Theorem 3.2 and we only need to consider Case 3.2.

We define a sequence of convex sets in $\mathfrak{a}'_+$ by

$$U_k(x_t) = \{x \in \mathfrak{a}'_+ \mid w_t(x) < m'_t + k + 1\}$$

and a family of convex functions on $U_k(x_t)$ by

$$\hat{\psi}'_t(x) = \psi'_t(x + x_t) - \psi'_t(x_t).$$

Then analogously to Proposition 4.4, $\hat{\psi}'_t$ is uniformly $C^{l,\alpha}$-bounded on each $U_k(x_t)$. Moreover, the set $U_k(x_t) - \{x_t\}$ converges as $t \to \infty$ to the cone $\mathfrak{a}''_+ \supset \mathfrak{a}'_+$ in $\mathfrak{a}'$ defined by

$$\mathfrak{a}''_+ = \{x \in \mathfrak{a} \mid \langle \alpha', x\rangle > 0,\ \forall \alpha' \in \Phi_s^+ \setminus \Phi''_u\},$$

where

$$\Phi''_u = \Phi_u \cup \Phi_0. \tag{7.6}$$

We introduce a function on $\mathfrak{a}''_+$ by

$$J''(x) = \prod_{\alpha \in \Phi_s^+ \setminus \Phi''_+} \sinh \alpha(x). \tag{7.7}$$

and a vector

$$\rho'_u = \frac{1}{2} \sum_{\beta \in \Phi''_u} \beta.$$

Then as in the proof of Proposition 5.1 (see also Remark 5.3), there are $\mathbf{a}_0 \in \mathfrak{a}'_+$ and a constant c_t such that the convex function

$$\psi_t(x + x_t + \mathbf{a}_0) - \psi_t(x_t) - \langle 2\rho'_u, x\rangle + c_t$$

converges subsequentially to a solution ϕ of

$$\pi_0(\nabla\phi + 2\rho'_u)\det(\nabla^2\phi) = J''(x)e^{-\phi - Y(\phi)}, \tag{7.8}$$

where $Y \in \mathfrak{a}''_+$ and π_0 is the function defined by (2.28). Moreover, Y satisfies

$$\langle \alpha', Y\rangle = 0, \quad \forall \alpha' \in \Phi_s^+ \setminus \Phi''_u. \tag{7.9}$$

Analogously to (5.1), the solution of (7.8) will define a KR soliton ω_{KS} of the form (2.17) on a new horosymmetric space $\mathbf{N}' = \mathbf{G}/\mathbf{H}'$ of a $\mathbb{Q}$-Fano horosymmetric variety $\overline{\mathbf{N}'}$, which is induced by a $\mathbb{C}^*$-degeneration via an element X_∞ in the Lie algebra of the Cartan subgroup of $\mathbf{G}$ as in Example 2.4. In fact, X_∞ is a limit of $x_t/|x_t|$ in (7.4) as in Lemma 3.4. Moreover, ω_{KS} has full mass on $\overline{\mathbf{N}'}$ by (5.21), and the moment polytope associated to $\frac{1}{l}c_1(\overline{\mathbf{N}'}, K^{-l}_{\overline{\mathbf{N}'}})$ is fine. Thus by Corollary 8.3, $\overline{\mathbf{N}'}$ admits a singular KR soliton with respect to Y analogous to the $\mathbb{Q}$-Fano horosymmetric variety $\overline{\mathbf{N}}$ in the proof of Theorem 3.2.

The Lie algebra of $\mathbf{H}'$ can be constructed as follows. By (7.9), we can construct the Lie algebra of a new parabolic subgroup $\mathbf{P}'$ of $\mathbf{G}$ as in (2.14),

$$\mathfrak{p}' = \mathfrak{t} + \sum_{\alpha\in\Phi_{\mathbf{L}'}} X_\alpha + \sum_{\alpha'\in\Phi'_u} X_{-\alpha'}, \tag{7.10}$$

where $\Phi'_u = \Phi''_u$ and $\Phi_{\mathbf{L}'} = \Phi_{\mathbf{L}} \setminus (\Phi''_u \cup (-\Phi''_u))$. The first two parts determine a Levi subgroup $\mathbf{L}'^{\sigma}$ of $\mathbf{P}'$. Then there is a fixed subgroup $\mathbf{L}'^{\sigma}$ of $\mathbf{L}'$ by the involution σ with root system given by

$$\Phi_{\mathbf{L}'^{\sigma}} = \{\alpha \in \Phi_{\mathbf{L}'} \mid \sigma(\alpha) = \alpha \text{ or } -\alpha\}.$$

Thus the Lie algebra of $\mathbf{H}'$ can be represented as

$$\mathfrak{h} = \mathfrak{t}'_0 + \sum_{\beta\in\Phi_{\mathbf{L}'^{\sigma}}} X_\beta + \sum_{\alpha'\in\Phi'_u} X_{-\alpha'}, \tag{7.11}$$

where $\mathfrak{t}'_0$ is a subtorus of $\mathfrak{t}$ fixed by σ.

Completion of proof of Theorem 7.1. By the argument above, we know that the metric of the form (2.17) determined by the solution ϕ of (7.8) defines a KR soliton ω_{KS} on $\mathbf{N}'$, where the root subsystems Φ_u, Φ_s^+ are replaced by $\Phi'_u, \Phi_s'^+ = \Phi_s^+ \setminus \Phi'_u$, respectively. Since $(\mathbf{N}', \omega_{KS})$ does not lose the volume of $(M, \omega(t))$, by [3, Theorem 1.1] the metric completion of $(\mathbf{N}', \omega_{KS})$ is the limit $(M_\infty, \omega_\infty)$ of the KR flow (1.2) in the GH topology. Moreover M_∞ is homeomorphic to a $\mathbb{Q}$-Fano variety $\hat{M}_\infty$ and ω_{KS} can be extended to a singular KR soliton with respect to Y on $\hat{M}_\infty$ [54].

We need to prove that $\hat{M}_\infty$ is biholomorphic to $\overline{\mathbf{N}}'$. In fact, as in the proof of Theorem 3.2, we can consider two cases: ω_{KS} is KE or not. In the first case, the conclusion comes from the uniqueness result of Li–Xu–Wang [33, Theorem 1.3] for K-polystable limits of $\mathbb{C}^*$-degenerations. In the second case $Y \neq 0$, we can use the relatively modified K-stability for KR solitons with respect to Y to conclude that $\hat{M}_\infty$ is biholomorphic to $\overline{\mathbf{N}}'$. Moreover, $\hat{M}_\infty$ is the limit of the $\mathbb{C}^*$-degeneration of (M, J) induced by the soliton HVF Y. The proof is finished. ∎

As an application of Theorem 7.1, we prove the following.

Corollary 7.2. *There exists a KR soliton on a Fano horosymmetric manifold (M, J) if it is modified K-semistable with respect to a HVF $X \neq 0$. As a consequence, (M, J) is modified K-polystable with respect to X if it is modified K-semistable with respect to X.*

Proof. We notice that (M, J) is K-unstable by the modified K-semistability and so Case 1 in Theorem 7.1 is impossible. Since (M, J) admits a KR soliton in Case 2 and so it is modified K-polystable by [9, 24], we need to consider Case 3. In this case the KR soliton limit $(M_\infty, \omega_\infty)$ of the flow (1.2) is a $\mathbb{C}^*$-degeneration of M induced by a soliton HVF $Y \neq 0$ of $(M_\infty, \omega_\infty)$. By [20, 54], this degeneration corresponds to a minimizer of the H-invariant for 1-PS degenerations (also called $\mathbb{R}$-test-configurations) of (M, J). On the other hand, by the uniqueness result of Han–Li [25, Theorem 1.2] for minimizers of the H-invariant, the $\mathbb{C}^*$-degeneration induced by Y must be trivial, and so Y is conjugate to X, since (M, J) is modified K-semistable with respect to X. This fact also comes from the uniqueness of the minimizer of the H-invariant for **G**-equivariant 1-PS degenerations of (M, J) [57, Proposition 5.8, Corollary 5.10]. Thus M_∞ is biholomorphic to M and so (M, J) is in fact a Fano manifold which admits a smooth KR soliton ω_∞. Hence (M, J) is modified K-polystable [9, 24]. ∎

Remark 7.3. In the case of $X = 0$ in Corollary 7.2, modified K-semistability is the same as K-semistability. However, no proof of Corollary 7.2 is available for this case. We hope Corollary 7.2 is still true as for toric manifolds by Wang–Zhu's work [56] whenever $X = 0$. For another approach, see Remark 5.3 (1).

Theorem 1.5 follows from Theorem 7.1. Theorem 7.1 also implies Theorem 1.6.

Proof of Theorem 1.6. Without loss of generality [25, 55], we may assume that the initial metric ω_0 of (1.2) is **K**-invariant. By Theorem 7.1 and the assumption in Theorem 1.6, it suffices to consider Case 3 of Theorem 7.1. Then the set Φ_0 in (7.4) is empty or not. If it is not empty, we can use the argument in [37, proof of Lemma 4.4] (see also [65, Lemma 7.4]) for horosymmetric manifolds to show that the curvature along the flow must blow up. In fact, the flow will blow up if there are $\beta \in \Phi_+$ and a sequence $\{x_{t_i}\}$ such that (7.4) holds. Thus the conclusion of the theorem holds.

On the other hand, if Φ_0 is empty, the unipotent radical Φ'_u and the restricted positive root system Φ'^+_s of the limit horosymmetric space $\mathbf{N}'$ do not change. This means that $\mathbf{N}'$ is the same as the original $\mathbf{N}$. Thus $\mathbf{N}$ admits a KR soliton ω_{KS} with respect to the soliton HVF Y in (7.9) with full mass, determined by the solution ϕ of (7.8). As an application of Corollary 8.3, there exists a singular KR soliton with respect to Y with Kähler potential φ in $\mathcal{E}^1(M, -K_M)$ on the variety compactification M of $\mathbf{N}'$ (or $\mathbf{N}$). By uniqueness [7, 9, 10, 24], this singular metric should coincide with the GH limit ω_∞ of the KR flow as a singular KR soliton on M [54]. Moreover, we have

$$|\varphi|, |Y(\varphi)| \leq C.$$

Note that M is smooth. Hence, by the regularity of ω_{KS} [55, Lemma 7.2], ω_{KS} is in fact a global smooth KR soliton on M. In other words, (M, J) admits a KR soliton. This contradicts the assumption of the theorem. Therefore, Φ_0 cannot be empty and we have proved the theorem. ∎

8. Appendix: Singular KR solitons on horosymmetric varieties

In this appendix, we generalize the results on existence of KR solitons on horosymmetric manifolds [17, 19] to $\mathbb{Q}$-Fano horosymmetric varieties.

Let M be a $\mathbb{Q}$-Fano variety with klt-singularities and X a HVF which can be lifted to one on $\mathbb{C}P^N$ when $M \subset \mathbb{C}P^N$. We call a current in $2\pi c_1(M)$ a *singular KR soliton* with respect to X on M if it is a smooth metric on $\mathrm{Reg}(M)$ which satisfies the KR soliton equation (1.2) and whose weak Kähler potential φ belongs to the space $\mathcal{E}^1(M, -K_M)$ introduced in [7]. It has been shown in [36, Theorem 4.2] that a $\mathbf{K} \times \mathbf{K}$-invariant φ is in $\mathcal{E}^1(M, -K_M)$ for $\mathbb{Q}$-Fano **G**-compactification varieties if and only if its Legendre function u is in $\mathcal{E}^1_{\mathbf{K}\times\mathbf{K}}(2P)$, where

$$\mathcal{E}^1_{\mathbf{K}\times\mathbf{K}}(2P) = \Big\{ u \;\Big|\; u \geq 0 \text{ is convex, Weyl-invariant on } 2P, \\ u(O) = \inf_{2P_+} u = 0 \ (O \in 2P_+) \text{ and } \int_{2P_+} u\pi \, dy < \infty \Big\}. \tag{8.1}$$

The result of [36, Theorem 4.2] can be generalized to $\mathbb{Q}$-Fano horosymmetric varieties $\overline{\mathbf{N}}$ of a homogeneous space $\mathbf{N}$ when the integral condition in (8.1) is replaced by

$$\int_{2(P_+ + \rho_u)} u\pi_0(\cdot)\, dy < \infty,$$

where π_0 is defined by (2.28) and P_+ is replaced by the quotient space of the moment polytope P associated to $\frac{1}{l}c_1(\overline{\mathbf{N}}, K_{\overline{\mathbf{N}}}^{-l})$ by the restricted Weyl group.

As in the case of **G**-manifolds [17,38], we introduce a barycenter associated to a HVF X induced by an element in $\mathfrak{t}'$,

$$\mathrm{bar}_X(2P_+) = \frac{\int_{2(P_+ + \rho_u)} y e^{\theta_X(y)} \pi_0(y)\, dy}{\int_{2(P_+ + \rho_u)} e^{\theta_X(y)} \pi_0(y)\, dy}, \tag{8.2}$$

where $\theta_X(y)$ is a bounded potential of X associated to an admissible Kähler metric induced by the Fubini–Study metric of $\mathbb{C}P^N$ and $e^{\theta_X(y)}\pi_0\, dy$ is just the weighted MA measure $e^{\theta_X(y)}\omega_\varphi^n$ under the Legendre transformation. Moreover, we may normalize $\theta_X(y)$ so that

$$\int_{2(P_+ + \rho_u)} e^{\theta_X(y)} \pi_0(y)\, dy = \int_{2(P_+ + \rho_u)} \pi_0(y)\, dy = V_0.$$

The following is a version of [36, Theorem 1.2] for the existence of singular KR solitons on $\mathbb{Q}$-Fano horosymmetric varieties.

Theorem 8.1. *Let $\overline{\mathbf{N}}$ be a $\mathbb{Q}$-Fano horosymmetric variety with the associated fine moment polytope P. Let*

$$\rho = \frac{1}{2} \sum_{\alpha' \in \Phi_s^+ \cup \Phi_u} \alpha.$$

Then M admits a singular KR soliton with respect to X if

$$\mathrm{bar}_X(2P_+) \in 2\rho + \Xi, \tag{8.3}$$

where $\Xi = \{y \in \mathfrak{a}^* \mid (\alpha, y) > 0, \ \forall \alpha \in \Phi_s^+\}$ is the relative interior of the cone generated by Φ_s^+.

[36, Theorem 1.2] was proved by the variational method for the Ding energy, as done for toric varieties by Berman–Berndtsson [6]. At present, we can also use this method for the modified Ding energy [9, 24, 49] on $\mathbb{Q}$-Fano horosymmetric varieties. Here we shall assume that P in Theorem 8.1 is *fine* as in [36, Theorem 1.2] for **G**-compactification varieties in order to verify that the Ricci potential of the Guillemin metric is bounded above. But we believe this assumption can be removed as for general $\mathbb{Q}$-Fano **G**-compactification varieties in [36, Theorem 8.1] by using a recent result of Han–Li [24] to verify the uniform modified K-stability via (8.3). Actually, based on [17, 24] there is recent progress by Li–Li–Wang [35] who established a version of Theorem 8.1 for general $\mathbb{Q}$-Fano spherical varieties.

The inverse of Theorem 8.1 is also true for singular KE metrics on Fano **G**-compactification varieties [17, 36]. The result can be generalized to KR solitons even just defined on a horosymmetric space with full mass by the following lemma.

Lemma 8.2. *Let $\overline{\mathbf{N}}$ be a $\mathbb{Q}$-Fano horosymmetric variety. Suppose that there is a KR soliton ω_{KS} on $\mathbf{N}$ with respect to a HVF X induced by an element in $\mathfrak{t}'$, which is a form (2.17) determined by a solution ψ of (2.30) with full mass on $\overline{\mathbf{N}}$.*[6] *Then the associated moment polytope P of $\overline{\mathbf{N}}$ satisfies the barycenter condition (8.3) associated to X.*

Proof. The proof is almost the same as that of [16, Proposition 5.3] for KE metrics on **G**-manifolds. We can check (8.3) directly as follows. By (2.30), we have

$$\pi_0(\nabla\psi + 2\rho_u)\det(\psi)e^{\theta_X} = e^{-(\psi+j_0)},$$

where $j_0 = -\log J_0$. Let $\psi' = \psi - \langle 2\rho_u, x\rangle$ and $j_0' = j_0 - \langle 2\rho_u, x\rangle$. Then the above equation becomes

$$\pi_0(\nabla\psi')\det(\psi')e^{\theta_X} = e^{-(\psi'+j_0')} = e^{-w},$$

where $w = \psi' + j_0'$. Note that

$$\int_{\mathfrak{a}'_+} \pi_0(\nabla\psi)\det(\psi)\,dx = V_0. \tag{8.4}$$

Then by the full mass condition on ω_{KS}, we get

$$\int_{\mathfrak{a}'_+} \frac{\partial\psi'}{\partial\xi}e^{-w}\,dx = V_0\langle \mathrm{bar}_X(2P_+), \xi\rangle. \tag{8.5}$$

On the other hand, since

$$\frac{\partial j_0}{\partial\xi} < -2\langle\rho_0, \xi\rangle, \quad \text{where} \quad \rho_0 = \frac{1}{2}\sum_{\alpha'\in\Phi_s^+}\alpha',$$

we have

$$\int_{\mathfrak{a}'_+} \frac{\partial j_0'}{\partial\xi}e^{-w} < -2V_0\langle\rho, \xi\rangle. \tag{8.6}$$

[6] This is equivalent to $\mathrm{Image}(\nabla\psi) = P$ by [36, Lemma 4.5] since ψ is smooth on $\mathbf{N}$.

Thus combining (8.5) and (8.6), we obtain

$$V_0(\langle \mathrm{bar}(2P_+), \xi\rangle - 2\langle \rho, \xi\rangle) > \int_{\mathfrak{a}'_+} \frac{\partial \psi'}{\partial \xi} e^{-w}\, dx + \int_{\mathfrak{a}'_+} \frac{\partial j'_0}{\partial \xi} e^{-w}\, dx. \tag{8.7}$$

However, by the full mass condition on ω_{KS}, it is easy to see that for any $\xi \in \mathfrak{a}'$,

$$\int_{\mathfrak{a}'_+} \frac{\partial w}{\partial \xi} e^{-w}\, dx = -\int_{\mathfrak{a}'_+} \frac{\partial}{\partial \xi} e^{-w}\, dx = 0,$$

that is,

$$\int_{\mathfrak{a}'_+} \frac{\partial \psi'}{\partial \xi} e^{-w}\, dx + \int_{\mathfrak{a}'_+} \frac{\partial j'_0}{\partial \xi} e^{-w}\, dx = 0.$$

Hence, we derive

$$\langle (\mathrm{bar}(2P_+) - 2\rho), \xi\rangle > 0, \quad \forall \xi \in \mathfrak{a}'. \tag{8.8}$$

Clearly, (8.8) is equivalent to (8.3). The lemma is proved. ∎

By Theorem 8.1 and Lemma 8.2, we obtain the following.

Corollary 8.3. *Let $\overline{\mathbf{N}}$ be a $\mathbb{Q}$-Fano horosymmetric variety with the associated fine moment polytope. Suppose that there is a KR soliton ω_{KS} on $\mathbf{N}$ with respect to a HVF X induced by an element in $\mathfrak{t}'$, which is a form (2.17) determined by a solution ψ of (2.30) with full mass on $\overline{\mathbf{N}}$. Then there exists a singular KR soliton with respect to X on $\overline{\mathbf{N}}$ with Kähler potential in $\mathcal{E}^1(\overline{\mathbf{N}}, -K_{\overline{\mathbf{N}}})$.*

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