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Short note

A remark on the number of automorphisms of finite abelian groups

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Abstract. Let Ab_0 be the class of finite abelian groups and consider the function $f\colon \mathrm{Ab}_0 \to (0,\infty)$ given by $f(G) = |\mathrm{Aut}(G)|/|G|$, where $\mathrm{Aut}(G)$ is the automorphism group of a finite abelian group G. In this short note, we prove that the image of f is a dense set in $[0,\infty)$.

1 Introduction

A well-known question in group theory (see e.g. [5, Problem 12.77]) asks whether it is true that $|G|$ divides $|\mathrm{Aut}(G)|$ for every nonabelian finite p-group G. This was answered in negative in [3], where for each prime p, there was constructed a family $(G_n)_{n\in\mathbb{N}}$ of finite p-groups such that $|\mathrm{Aut}(G_n)|/|G_n|$ tends to zero as n tends to infinity. By considering the function

$$f(G) = \frac{|\mathrm{Aut}(G)|}{|G|}$$

for all finite groups G, the above result means that zero is an accumulation point of the image of f. Note that a similar result holds for the function

$$f'(G) = \frac{|\mathrm{Aut}(G)|}{\varphi(|G|)},$$

where φ denotes the Euler totient function (see e.g. [1, 2]). These constitute the starting point for our work.

Our main result shows that all nonnegative real numbers are accumulation points of the image of f, even if we restrict this function only to abelian groups.

Theorem 1.1. *The set*

$$\mathrm{Im}(f) = \{f(G) \mid G \in \mathrm{Ab}_0\}$$

is dense in $[0,\infty)$.

The proof of Theorem 1.1 follows the same steps as the proofs of [4, Theorem 1.1]. It is based on the next lemma which is a consequence of the proposition outlined on [6, p. 863].

Lemma 1.2. *Let* $(x_n)_{n\geq 1}$ *be a sequence of positive real numbers such that* $\lim_{n\to\infty} x_n = 0$ *and* $\sum_{n=1}^{\infty} x_n$ *is divergent. Then the set containing the sums of all finite subsequences of* $(x_n)_{n\geq 1}$ *is dense in* $[0,\infty)$.

It also uses the fact that the function f is multiplicative, that is,

$$f(G_1 \times G_2) = f(G_1)f(G_2)$$

for any finite groups G_1, G_2 of coprime orders.

Finally, we formulate a natural open problem related to the above theorem.

Open problem. Is it true that, for every $a \in [0,\infty) \cap \mathbb{Q}$, there is a finite (abelian) group G such that $f(G) = a$?

2 Proofs of the main results

First of all, we prove an auxiliary result.

Lemma 2.1. *The set* $\mathrm{Im}(f) \cap [0,1]$ *is dense in* $[0,1]$.

Proof. Let I be a finite subset of $\mathbb{N}$ and let p_i be the ith prime number. Since f is multiplicative, we have

$$f\Big(\prod_{i\in I} C_{p_i}\Big) = \prod_{i\in I} f(C_{p_i}) = \prod_{i\in I} \frac{p_i - 1}{p_i}$$

and so

$$A = \Big\{\prod_{i\in I} \frac{p_i-1}{p_i} \;\Big|\; I \subset \mathbb{N},\ |I| < \infty,\ p_i = i\text{th prime number}\Big\} \subseteq \mathrm{Im}(f) \cap [0,1].$$

Thus it suffices to prove that A is dense in $[0,1]$.

Consider the sequence $(x_i)_{i\geq 1} \subset (0,\infty)$, where $x_i = \ln(\frac{p_i}{p_i-1})$ for all $i \geq 1$. Clearly, $\lim_{i\to\infty} x_i = 0$. We have

$$\lim_{i\to\infty} \frac{x_i}{\frac{1}{p_i}} = 1.$$

Therefore, since the series $\sum_{i\geq 1} \frac{1}{p_i}$ is divergent, we deduce that the series $\sum_{i\geq 1} x_i$ is also divergent. So all hypotheses of Lemma 1.2 are satisfied, implying that

$$\overline{\Big\{\sum_{i\in I} x_i \;\Big|\; I \subset \mathbb{N}^*,\ |I| < \infty\Big\}} = [0,\infty).$$

This means

$$\overline{\Big\{\ln\Big(\prod_{i\in I} \frac{p_i}{p_i-1}\Big) \;\Big|\; I \subset \mathbb{N}^*,\ |I| < \infty,\ p_i = i\text{th prime number}\Big\}} = [0,\infty)$$

or equivalently

$$\overline{\Big\{\prod_{i\in I} \frac{p_i}{p_i-1} \;\Big|\; I \subset \mathbb{N}^*,\ |I| < \infty,\ p_i = i\text{th prime number}\Big\}} = [1,\infty).$$

Then

$$\overline{\left\{\prod_{i\in I}\frac{p_i-1}{p_i}\;\middle|\; I\subset\mathbb{N}^*,\ |I|<\infty,\ p_i=i\text{th prime number}\right\}}=[0,1]$$

and consequently

$$\overline{A}=[0,1],$$

as desired. ∎

Note that the conclusion of Lemma 2.1 remains valid if we restrict f to the class Ab_0' of finite abelian groups of odd order, that is, $\mathrm{Im}(f|_{\mathrm{Ab}_0'})\cap[0,1]$ is also dense in $[0,1]$.

We are now able to prove our main result.

Proof of Theorem 1.1. We have to prove that, for every $a\in[0,\infty)$ and every $\varepsilon>0$, there is $G\in\mathrm{Ab}_0$ such that $f(G)\in(a-\varepsilon,a+\varepsilon)$. If $a\in[0,1]$, this follows from Lemma 2.1. Assume now that $a\in(1,\infty)$. Since $\mathrm{Aut}(C_2^n)\cong\mathrm{GL}_n(2)$ has order $\prod_{k=0}^{n-1}(2^n-2^k)$, we have

$$\lim_{n\to\infty}f(C_2^n)=\lim_{n\to\infty}\frac{1}{2^n}\prod_{k=0}^{n-1}(2^n-2^k)=\infty$$

and so we can choose a finite elementary abelian 2-group G_1 such that $f(G_1)=b>a$. Then $\frac{a}{b}\in(0,1)$. Let $\varepsilon_1=\frac{\varepsilon}{b}$. By the above remark, there is a finite abelian group of odd order G_2 with $f(G_2)\in(\frac{a}{b}-\varepsilon_1,\frac{a}{b}+\varepsilon_1)$. It follows that $G=G_1\times G_2\in\mathrm{Ab}_0$ and

$$f(G)=f(G_1)f(G_2)=bf(G_2)\in(a-\varepsilon,a+\varepsilon).$$

This completes the proof. ∎

References

[1] J. N. Bray and R. A. Wilson, On the orders of automorphism groups of finite groups. *Bull. London Math. Soc.* **37** (2005), no. 3, 381–385 Zbl 1072.20029 MR 2131391

[2] J. N. Bray and R. A. Wilson, On the orders of automorphism groups of finite groups. II. *J. Group Theory* **9** (2006), no. 4, 537–545 Zbl 1103.20017 MR 2243245

[3] J. González-Sánchez and A. Jaikin-Zapirain, Finite p-groups with small automorphism group. *Forum Math. Sigma* **3** (2015), article no. e7, 11 Zbl 1319.20019 MR 3376735

[4] M.-S. Lazorec and M. Tărnăuceanu, A density result on the sum of element orders of a finite group. *Arch. Math. (Basel)* **114** (2020), no. 6, 601–607 Zbl 1481.20087 MR 4102339

[5] V. D. Mazurov and E. I. Khukhro (eds.), *The Kourovka notebook*. 17th edn., Russian Academy of Sciences Siberian Division, Institute of Mathematics, Novosibirsk, 2010 Zbl 1211.20001 MR 3235009

[6] Z. Nitecki, Cantorvals and subsum sets of null sequences. *Amer. Math. Monthly* **122** (2015), no. 9, 862–870 Zbl 1355.40001 MR 3418208

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