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A note on traces of operators on $L^\infty(\lambda)$

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Abstract. Let $(\Omega, \Sigma, \lambda)$ be a finite measure space and $\tau(L^\infty(\lambda), L^1(\lambda))$ be a natural Mackey topology on $L^\infty(\lambda)$. Let $T : L^\infty(\lambda) \to L^\infty(\lambda)$ be a Bochner representable operator, that is, there exists $g \in L^1(\lambda, L^\infty(\lambda))$ so that $T(u) = \int_\Omega u(\omega)g(\omega)\, d\lambda(\omega)$ for $u \in L^\infty(\lambda)$. It is shown that T is a nuclear operator between the locally convex space $(L^\infty(\lambda), \tau(L^\infty(\lambda), L^1(\lambda)))$ and the Banach space $L^\infty(\lambda)$ and T has a well-defined trace: $\operatorname{tr} T = \int_\Omega g(\omega)(\omega)\, d\lambda(\omega)$.

1. Introduction and preliminaries

Formulas for traces of kernel operators on Banach function spaces (in particular, $C(\Omega)$, $L^p(\lambda)$-spaces) have been the object of much study (see [2, 3, 7, 9, 10, 13]). In particular, Grothendieck [9] showed that if Ω is a compact Hausdorff space with a positive Borel measure λ on Ω and $k(\cdot,\cdot) \in C(\Omega \times \Omega)$, then the kernel operator $T : C(\Omega) \to C(\Omega)$ defined by

$$T(u) := \int_\Omega u(\omega)k(\cdot,\omega)\, d\lambda(\omega) \quad \text{for } u \in C(\Omega),$$

is nuclear and has a well-defined trace $\operatorname{tr} T = \int_\Omega k(\omega,\omega)\, d\lambda(\omega)$ (see [13, Theorem, p. 274]).

Grothendieck [8, Chap. I, p. 165] proved that the notion of "trace" can be defined for nuclear operators on Banach spaces with the approximation property (see [13, Lemma, pp. 210–211]).

From now on, let $(X, \|\cdot\|_X)$ be a real Banach space and X^* stand for its Banach dual. Let $\mathcal{L}(X, X)$ denote the Banach space of all bounded linear operators $T : X \to X$, equipped with the operator norm.

Assume that $T : X \to X$ is a nuclear operator, that is, there exist a bounded sequence (F_n) in X^*, a bounded sequence (x_n) in X and a sequence $(\alpha_n) \in \ell^1$ so

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that

$$T = \sum_{n=1}^{\infty} \alpha_n F_n \otimes x_n \quad \text{in } \mathcal{L}(X, X), \tag{1.1}$$

where $(\alpha_n F_n \otimes x_n)(x) = \alpha_n F_n(x) x_n$ for $x \in X$.

If X has the approximation property, then the trace of T is given by

$$\operatorname{tr} T = \sum_{n=1}^{\infty} \alpha_n F_n(x_n)$$

and it does not depend on the special choice of the nuclear representation (1.1) of T (see [7, Chap. 5, Theorem 1.2], [13, Lemma, pp. 210–211]).

Assume that $(\Omega, \Sigma, \lambda)$ is a complete finite measure space. Then the space

$$L^\infty(\lambda) := \Big\{ u \in L^0(\lambda) : \|u\|_\infty := \operatorname*{ess\,sup}_{\omega \in \Omega} |u(\omega)| < \infty \Big\},$$

equipped with the norm $\|\cdot\|_\infty$ is a Dedekind complete Banach lattice. It is known that $L^\infty(\lambda)$ has the approximation property (see [5, p. 245], [14]).

Let $L^1(\lambda, X)$ stand for the Banach space of λ-equivalence classes of all Bochner integrable functions $g : \Omega \to X$, equipped with the norm $\|g\|_1 := \int_\Omega \|g(\omega)\|_X \, d\lambda(\omega)$.

Definition 1.1. A bounded linear operator $T : L^\infty(\lambda) \to X$ is said to be *Bochner representable*, if there exists a unique $g \in L^1(\lambda, X)$ such that

$$T(u) = \int_\Omega u(\omega) g(\omega) \, d\lambda(\omega) \quad \text{for } u \in L^\infty(\lambda).$$

The aim of this paper is to find a formula for the trace of a Bochner representable operator $T : L^\infty(\lambda) \to L^\infty(\lambda)$.

2. Traces of Bochner representable operators on $L^\infty(\lambda)$

Recall that a linear functional F on $L^\infty(\lambda)$ is said to be *σ-order continuous* if $F(u_n) \to 0$ whenever $u_n(\omega) \to 0$ μ-a.e. and $\sup_n \|u_n\|_\infty < \infty$ (see [11, Chap. 6]).

Let $L^\infty(\lambda)^*_c$ stand for the linear subspace of $L^\infty(\lambda)^*$ consisting of all σ-order continuous functionals. Then we have (see [11, Theorem 6.6.1])

$$L^\infty(\lambda)^*_c = \big\{ F_v : v \in L^1(\lambda) \big\},$$

where $F_v(u) := \int_\Omega u(\omega) v(\omega) \, d\lambda(\omega)$ for $u \in L^\infty(\lambda)$ and $\|F_v\| = \|v\|_1$.

Let $\tau(L^\infty(\lambda), L^1(\lambda))$ denote the Mackey topology on $L^\infty(\lambda)$, with respect to the dual pair $\langle L^\infty(\lambda), L^\infty(\lambda)^*_c\rangle$ with the duality

$$\langle u, F_v\rangle := F_v(u) = \int_\Omega u(\omega)v(\omega)\, d\lambda(\omega) \quad \text{for } u \in L^\infty(\lambda),\ v \in L^1(\lambda).$$

This means that $\tau(L^\infty(\lambda), L^1(\lambda))$ is the topology of uniform convergence on the family of all convex, circled $\sigma(L^\infty(\lambda)^*_c, L^\infty(\lambda))$-compact sets in $L^\infty(\lambda)^*_c$ (see [1, p. 143]). It follows that every convex, circled $\sigma(L^\infty(\lambda)^*_c, L^\infty(\lambda))$-compact set in $L^\infty(\lambda)^*_c$ is $\tau(L^\infty(\lambda), L^1(\lambda))$-equicontinuous. It is known that $\tau(L^\infty(\lambda), L^1(\lambda))$ is a locally convex-solid Hausdorff topology with the Lebesgue property (see [6, Corollary 83H]).

Nuclear operators $T : L^\infty(\lambda) \to X$ between the Banach spaces $L^\infty(\lambda)$ and X have been studied by Swartz [16] and Nowak [12].

Grothendieck [8,9] introduced the concept of nuclear operators for locally convex spaces (see [15, Chap. 3, § 7], [17, § 47]).

Definition 2.1. A linear operator $T : L^\infty(\lambda) \to X$ is said to be a *nuclear operator* between the locally convex space $(L^\infty(\lambda), \tau(L^\infty(\lambda), L^1(\lambda)))$ and the Banach space X if there exist a $\tau(L^\infty(\lambda), L^1(\lambda))$-equicontinuous sequence (F_n) in $L^\infty(\lambda)^*_c$, a bounded sequence (x_n) in X and a sequence $(\alpha_n) \in \ell^1$ so that

$$T(u) = \sum_{n=1}^\infty \alpha_n F_n(u) x_n \quad \text{for } u \in L^\infty(\lambda),$$

that is,

$$T(u) = \sum_{n=1}^\infty \alpha_n F_n \otimes x_n \quad \text{for } \mathcal{L}(L^\infty(\lambda), X).$$

Now we can state our main result.

Theorem 2.1. *Assume that $T : L^\infty(\lambda) \to L^\infty(\lambda)$ is a Bochner representable operator, that is, there exists a unique $g \in L^1(\lambda, L^\infty(\lambda))$ so that*

$$T(u) = \int_\Omega u(\omega)g(\omega)\, d\lambda(\omega) \quad \textit{for all } u \in L^\infty(\lambda).$$

Then T is a nuclear operator between the locally convex space $(L^\infty(\lambda), \tau(L^\infty(\lambda), L^1(\lambda)))$ and the Banach space $L^\infty(\lambda)$ and has a well-defined trace

$$\operatorname{tr} T = \int_\Omega g(\omega)(\omega)\, d\lambda(\omega).$$

Proof. Let $L^1(\lambda)\,\widehat{\otimes}_\pi\, L^\infty(\lambda)$ denote the projective tensor product of the Banach spaces $L^1(\lambda)$ and $L^\infty(\lambda)$, equipped with the completed norm π (see [5, p. 227], [14, p. 17]). Note that for $z \in L^1(\lambda)\,\widehat{\otimes}_\pi\, L^\infty(\lambda)$, we have

$$\pi(z) = \inf\left\{\sum_{n=1}^{\infty} \alpha_n \|v_n\|_1 \|w_n\|_\infty\right\},$$

where the infimum is taken over all sequences (v_n) in $L^1(\lambda)$ and (w_n) in $L^\infty(\lambda)$ with $\lim \|v_n\|_1 = 0 = \lim \|w_n\|_\infty$ and $(\alpha_n) \in \ell^1$ such that $z = \sum_{n=1}^{\infty} \alpha_n v_n \otimes w_n$ in the norm π (see [14, Proposition 2.8]).

Note that $L^1(\lambda)\,\widehat{\otimes}_\pi\, L^\infty(\lambda)$ is isometrically isomorphic to the Banach space $L^1(\lambda, L^\infty(\lambda))$ by the isometry J, defined by

$$J(v \otimes w) := v(\cdot)w \quad \text{for } v \in L^1(\lambda),\ w \in L^\infty(\lambda)$$

(see [5, Example 10, p. 228], [14, Example 2.19, p. 29]). Then there exist sequences (v_n) in $L^1(\lambda)$ and (w_n) in $L^\infty(\lambda)$ with $\lim \|v_n\|_1 = 0 = \lim \|w_n\|_\infty$ and $(\alpha_n) \in \ell^1$ so that

$$J^{-1}(g) = \sum_{n=1}^{\infty} \alpha_n v_n \otimes w_n \quad \text{in } L^1(\lambda)\,\widehat{\otimes}_\pi\, L^\infty(\lambda).$$

Thus it follows that

$$g = J\left(\sum_{n=1}^{\infty} \alpha_n v_n \otimes w_n\right) = \sum_{n=1}^{\infty} \alpha_n v_n(\cdot) w_n \quad \text{in } L^1(\lambda, L^\infty(\lambda)),$$

and hence

$$g(\omega) = \sum_{n=1}^{\infty} \alpha_n v_n(\omega) w_n \quad \text{for } \omega \in \Omega.$$

Then for every $u \in L^\infty(\lambda)$, we get

$$T(u) = \int_\Omega u(\omega)\left(\sum_{n=1}^{\infty} \alpha_n v_n(\omega) w_n\right) d\lambda(\omega) = \sum_{n=1}^{\infty} \alpha_n \left(\int_\Omega u(\omega) v_n(\omega)\, d\lambda(\omega)\right) w_n.$$

Since $\lim \|v_n\|_1 = 0$, the set $\{v_n : n \in \mathbb{N}\}$ is uniformly integrable and in view of the Dunford–Pettis theorem (see [4, Theorem, p. 93]), we obtain that $\{v_n : n \in \mathbb{N}\}$ is a relatively $\sigma(L^1(\lambda), L^\infty(\lambda))$-compact subset of $L^1(\lambda)$. Then by the Krein–Smulian theorem (see [1, Theorem 10.15]), the convex, circled hull of the set $\{v_n : n \in \mathbb{N}\}$ is relatively $\sigma(L^1(\lambda), L^\infty(\lambda))$-compact in $L^1(\lambda)$, and hence the convex, circled hull of the set $\{F_{v_n} : n \in \mathbb{N}\}$ is relatively $\sigma(L^\infty(\lambda)^*_c, L^\infty(\lambda))$-compact in $L^\infty(\lambda)^*_c$. Hence the set $\{F_{v_n} : n \in \mathbb{N}\}$ is $\tau(L^\infty(\lambda), L^1(\lambda))$-equicontinuous in $L^\infty(\lambda)^*_c$.

This means that T is a nuclear operator between the locally convex space $(L^\infty(\lambda), \tau(L^\infty(\lambda), L^1(\lambda)))$ and the Banach space $L^\infty(\lambda)$ (see Definition 2.1). It follows that T is a nuclear operator between the Banach spaces $L^\infty(\lambda)$ and $L^\infty(\lambda)$ and we have

$$T = \sum_{n=1}^{\infty} \alpha_n F_{v_n} \otimes w_n \quad \text{in } \mathcal{L}(L^\infty(\lambda), L^\infty(\lambda)).$$

Hence, since $L^\infty(\lambda)$ has the approximation property, we get

$$\operatorname{tr} T = \sum_{n=1}^{\infty} \alpha_n F_{v_n}(w_n) = \sum_{n=1}^{\infty} \alpha_n \int_\Omega v_n(\omega) w_n(\omega)\, d\lambda(\omega). \tag{2.1}$$

For $n \in \mathbb{N}$, let $g_n := \sum_{i=1}^{n} \alpha_i v_i(\cdot) w_i$, that is, $g_n(\omega) = \sum_{i=1}^{n} \alpha_i v_i(\omega) w_i$ for $\omega \in \Omega$.

Then we have

$$\begin{aligned}
&\left| \int_\Omega g(\omega)(\omega)\, d\lambda(\omega) - \sum_{i=1}^{n} \alpha_i \int_\Omega v_i(\omega) w_i(\omega)\, d\lambda(\omega) \right| \\
&\quad = \left| \int_\Omega (g(\omega)(\omega) - g_n(\omega)(\omega))\, d\lambda(\omega) \right| \\
&\quad \le \int_\Omega |g(\omega)(\omega) - g_n(\omega)(\omega)|\, d\lambda(\omega) \\
&\quad \le \int_\Omega \|g(\omega) - g_n(\omega)\|_\infty\, d\lambda(\omega) \\
&\quad = \|g - g_n\|_1 \xrightarrow[n]{} 0.
\end{aligned}$$

Thus

$$\int_\Omega g(\omega)(\omega)\, d\lambda(\omega) = \sum_{n=1}^{\infty} \alpha_n \int_\Omega v_n(\omega) w_n(\omega)\, d\lambda(\omega),$$

and in view of (2.1), we get

$$\operatorname{tr} T = \int_\Omega g(\omega)(\omega)\, d\lambda(\omega).$$

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