Port. Math. (Online first)
DOI 10.4171/PM/2142

L_p-dual Brunn–Minkowski inequality for intersection bodies

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Abstract. In 2003, associated with the radial Minkowski additions of star bodies, Zhao and Leng established the dual Brunn–Minkowski inequality for intersection bodies. In this paper, associated with the L_p-radial Minkowski combinations of star bodies, we firstly prove the L_p-dual Brunn–Minkowski inequality for intersection bodies. Further, associated with the L_p-Minkowski combinations of convex bodies, we give the L_p-Brunn–Minkowski inequality for star dualities of intersection bodies.

1. Introduction and main results

The setting for this paper is the n-dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in $\mathbb{R}^n$, for the set of convex bodies containing the origin in their interiors in $\mathbb{R}^n$, we write $\mathcal{K}_o^n$. Let $\mathcal{S}_o^n$ denote the set of star bodies (with respect to origin) in $\mathbb{R}^n$. Let B denote the n-dimensional Euclidean unit ball centered at the origin, and the surface of B is written S^{n-1}. We use $V(K)$ to denote the n-dimensional volume of a body K.

The famous Brunn–Minkowski inequality for the volume is an important inequality in the theory of convex bodies. One form of it states the following: *if $K, L \in \mathcal{K}^n$, then*

$$V(K+L)^{\frac{1}{n}} \geq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}}, \tag{1.1}$$

with equality if and only if K and L are homothetic. Here, $K+L$ denotes the Minkowski addition of K and L.

In 1962, Firey (see [5]) introduced the L_p-Minkowski combinations of convex bodies (also called the Firey p-combinations) and established the following L_p-Brunn–Minkowski inequality. *If $K, L \in \mathcal{K}_o^n$ and $1 \leq p \leq +\infty$ (for $p = 1$, it can be assumed that $K, L \in \mathcal{K}^n$), then*

$$V(K +_p L)^{\frac{p}{n}} \geq V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}}. \tag{1.2}$$

Mathematics Subject Classification 2020: 52A20 (primary); 52A40 (secondary).
Keywords: intersection body, L_p-radial Minkowski combination, L_p-dual Brunn–Minkowski inequality, L_p-Brunn–Minkowski inequality, star duality.

Equality holds in (1.2) *for* $p = 1$ *if and only if* K *and* L *are homothetic, for* $1 < p < +\infty$ *if and only if* K *and* L *are dilates, for* $p = +\infty$ *if and only if* $K \subseteq L$ *or* $L \subseteq K$. Here, $K +_p L$ denotes the L_p-Minkowski addition of K and L.

The dual form of the L_p-Brunn–Minkowski inequality is the following L_p-dual Brunn–Minkowski inequality (see [6]). *If* $K, L \in \mathcal{S}_o^n$ *and real* $p \neq 0$*, then for* $0 < p < n$*,*

$$V(K \tilde{+}_p L)^{\frac{p}{n}} \leq V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}}; \tag{1.3}$$

for $-\infty \leq p < 0$ *or* $n < p \leq +\infty$*, then*

$$V(K \tilde{+}_p L)^{\frac{p}{n}} \geq V(K)^{\frac{p}{n}} + V(L)^{\frac{p}{n}}. \tag{1.4}$$

For $p \neq \pm\infty$*, equality hold in* (1.3) *and* (1.4) *if and only if* K *and* L *are dilates; for* $p = \pm\infty$*, equality holds in* (1.4) *if and only if* $K \subseteq L$ *or* $L \subseteq K$. Here, $K \tilde{+}_p L$ denotes the L_p-radial Minkowski addition of K and L.

In particular, the case $p = 1$ of inequality (1.3) shows the dual Brunn–Minkowski inequality as follows: *If* $K, L \in \mathcal{S}_o^n$*, then*

$$V(K \tilde{+} L)^{\frac{1}{n}} \leq V(K)^{\frac{1}{n}} + V(L)^{\frac{1}{n}},$$

with equality if and only if K *and* L *are dilates.* Here, $K \tilde{+} L = K \tilde{+}_1 L$ denotes the radial Minkowski addition of K and L.

The researches of the Brunn–Minkowski inequality and its dual versions in various forms have made a lot of achievements. For extensive and beautiful surveys on it we refer, e.g., to [1–4, 6–9, 12, 13, 15–17, 19–22, 25].

Associated with the radial Minkowski additions of star bodies, Zhao and Leng (see [23]) established the dual Brunn–Minkowski inequality for intersection bodies as follows.

Theorem 1.A. *If* $K, L \in \mathcal{S}_o^n$*, then*

$$V(I(K \tilde{+} L))^{\frac{1}{n(n-1)}} \leq V(IK)^{\frac{1}{n(n-1)}} + V(IL)^{\frac{1}{n(n-1)}}, \tag{1.5}$$

with equality if and only if K *and* L *are dilates. Here* IM *denotes the intersection body of* $M \in \mathcal{S}_o^n$.

In this paper, associated with the L_p-radial Minkowski combinations of star bodies, we firstly give the L_p-dual Brunn–Minkowski inequality for intersection bodies as follows.

Theorem 1.1. *If* $K, L \in \mathcal{S}_o^n$*,* $\lambda \in [0, 1]$ *and* p *is any real, then for* $0 < p < n-1$*,*

$$V(I(\lambda \odot K \tilde{+}_p (1-\lambda) \odot L))^{\frac{p}{n(n-1)}} \leq \lambda V(IK)^{\frac{p}{n(n-1)}} + (1-\lambda) V(IL)^{\frac{p}{n(n-1)}}; \tag{1.6}$$

for $-\infty \le p < 0$ *or* $n(n-1) < p \le +\infty$,

$$V(I(\lambda \odot K \,\tilde{+}_p\, (1-\lambda) \odot L))^{\frac{p}{n(n-1)}} \ge \lambda V(IK)^{\frac{p}{n(n-1)}} + (1-\lambda)V(IL)^{\frac{p}{n(n-1)}}; \tag{1.7}$$

for $p = 0$,

$$V(I(\lambda \odot K \,\tilde{+}_0\, (1-\lambda) \odot L)) \le V(IK)^{\lambda} V(IL)^{1-\lambda}. \tag{1.8}$$

When $\lambda \in (0,1)$, *equality holds in every above inequality for* $p \ne \pm\infty$ *if and only if* K *and* L *are dilates, for* $p = \pm\infty$ *if and only if* $K \subseteq L$ *or* $L \subseteq K$. *When* $\lambda = 0$ *or* $\lambda = 1$, *above inequalities all become equalities.*

Let $p = 1$ and $\lambda = \frac{1}{2}$ in Theorem 1.1, then inequality (1.6) yields inequality (1.5).

Next, respective to the L_p-Minkowski combinations of convex bodies, we prove the following L_p-Brunn–Minkowski inequality for star dualities of intersection bodies.

Theorem 1.2. *If* $K, L \in \mathcal{K}_o^n$, $1 \le p \le +\infty$ *(for* $p = 1$, *it can be assumed that* $K, L \in \mathcal{K}^n$*) and* $\lambda \in [0,1]$, *then*

$$\begin{aligned} &V(I^\circ(\lambda \cdot K +_p (1-\lambda) \cdot L))^{-\frac{p}{n(n-1)}} \\ &\quad \ge \lambda V(I^\circ K)^{-\frac{p}{n(n-1)}} + (1-\lambda)V(I^\circ L)^{-\frac{p}{n(n-1)}}. \end{aligned} \tag{1.9}$$

When $\lambda \in (0,1)$, *equality holds in* (1.9) *for* $p = 1$ *if and only if* K *and* L *are homothetic, for* $1 < p < +\infty$ *if and only if* K *and* L *are dilates, for* $p = +\infty$ *if and only if* $K \subseteq L$ *or* $L \subseteq K$. *When* $\lambda = 0$ *or* $\lambda = 1$, (1.9) *becomes an equality. Here,* $I^\circ M = (IM)^\circ$ *denotes the star duality of the intersection body* IM.

Obviously, the case $p = 1$ and $\lambda = \frac{1}{2}$ of inequality (1.9) implies a dual form of inequality (1.5) as follows: *If* $K, L \in \mathcal{K}^n$, *then*

$$V(I^\circ(K+L))^{-\frac{1}{n(n-1)}} \ge V(I^\circ K)^{-\frac{1}{n(n-1)}} + V(I^\circ L)^{-\frac{1}{n(n-1)}},$$

with equality if and only if K *and* L *are homothetic.*

2. Background material

2.1. Support functions and L_p-Minkowski combinations

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K,\cdot) : \mathbb{R}^n \to (-\infty, +\infty)$, is defined by (see [7, 16])

$$h(K,x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

For $1 \leq p \leq +\infty$, the L_p-Minkowski combinations (or called the Firey L_p-combinations) of convex bodies were introduced by Firey (see [5, 12]). For $K, L \in \mathcal{K}_o^n$, $1 \leq p < +\infty$ (for $p = 1$, it can be assumed that $K, L \in \mathcal{K}^n$) and $\lambda, \mu \geq 0$ (not both zero), the L_p-Minkowski combination, $\lambda \cdot K +_p \mu \cdot L \in \mathcal{K}_o^n$, of K and L is defined by

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot) = [\lambda h(K, \cdot)^p + \mu h(L, \cdot)^p]^{\frac{1}{p}}, \tag{2.1}$$

where $\lambda \cdot K = \lambda^{\frac{1}{p}} K$. If $\lambda = \mu = 1$, then $K +_p L$ is called the L_p-Minkowski addition of K and L. Obviously, if $p = 1$, then $\lambda \cdot K +_1 \mu \cdot L = \lambda K + \mu L$ is the Minkowski combination of K and L. For $p = +\infty$, according to the fact for $a, b \geq 0$,

$$\lim_{p \to +\infty} [\lambda a^p + \mu b^p]^{\frac{1}{p}} = \max\{a, b\}, \tag{2.2}$$

we define for $K, L \in \mathcal{K}_o^n$ (see [16]),

$$\lambda \cdot K +_{+\infty} \mu \cdot L = \mathrm{conv}(K \cup L). \tag{2.3}$$

From (2.1) and the Jensen's inequality, we easily know that *if* $K, L \in \mathcal{K}_o^n$, $1 < p < +\infty$ *and* $\lambda + \mu = 1$ *(*$\lambda, \mu \geq 0$*), then*

$$\lambda K + \mu L \subseteq \lambda \cdot K +_p \mu \cdot L. \tag{2.4}$$

Equality holds in (2.4) *if and only if* $K = L$.

Here, we deal with the equality condition of (2.4). Indeed, if $\lambda K + \mu L = \lambda \cdot K +_p \mu \cdot L$, i.e., for any $u \in S^{n-1}$,

$$[\lambda h(K, u) + \mu h(L, u)]^p = \lambda h(K, u)^p + \mu h(L, u)^p.$$

This implies $h(K, u) = h(L, u)$ for any $u \in S^{n-1}$, i.e., $K = L$. Obviously, if $K = L$, then equality holds in (2.4).

2.2. Radial functions and L_p-radial Minkowski combinations

If K is a compact star-shaped set (about the origin) in $\mathbb{R}^n$, its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \backslash \{0\} \to [0, +\infty)$, is defined by (see [7])

$$\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \backslash \{0\}.$$

If ρ_K is positive and continuous, K will be called a star body (about the origin).

From the radial function, we have the following volume formula of a body K:

$$V(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^n du. \tag{2.5}$$

For the radial function, we see that if $K \in \mathcal{S}_o^n$ and ζ is a subspace of $\mathbb{R}^n$, then for any $u \in S^{n-1} \cap \zeta$ (see [7]),

$$\rho(K \cap \zeta, u) = \rho(K, u). \tag{2.6}$$

For $-\infty \le p \le +\infty$, the L_p-radial Minkowski combinations of star bodies are defined as follows: For $K, L \in \mathcal{S}_o^n$, $\lambda, \mu \ge 0$ (not both zero), $-\infty < p < +\infty$ and $p \ne 0$, the L_p-radial Minkowski combination, $\lambda \odot K \tilde{+}_p \mu \odot L \in \mathcal{S}_o^n$, of K and L is defined by (see [16])

$$\rho(\lambda \odot K \tilde{+}_p \mu \odot L, \cdot) = [\lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p]^{\frac{1}{p}}. \tag{2.7}$$

If $\lambda = \mu = 1$, then $K \tilde{+}_p L$ is called the L_p-radial Minkowski addition of K and L. In particular, $K \tilde{+}_1 L = K \tilde{+} L$ is the radial Minkowski addition of K and L. For $p = \pm\infty$, from (2.2) and

$$\lim_{p \to -\infty} [\lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p]^{\frac{1}{p}} = \min\{\rho_K, \rho_L\}, \tag{2.8}$$

we define for $K, L \in \mathcal{S}_o^n$,

$$\lambda \odot K \tilde{+}_{+\infty} \mu \odot L = K \cup L, \tag{2.9}$$

$$\lambda \odot K \tilde{+}_{-\infty} \mu \odot L = K \cap L. \tag{2.10}$$

For $p = 0$, $\lambda \in [0, 1]$, define $\lambda \odot K \tilde{+}_0 (1 - \lambda) \odot L$ which is called the log-radial Minkowski combination of K and L by (see [18])

$$\begin{aligned} \rho(\lambda \odot K \tilde{+}_0 (1 - \lambda) \odot L, \cdot) &= \lim_{p \to 0} \rho(\lambda \odot K \tilde{+}_p (1 - \lambda) \odot L, \cdot) \\ &= \lim_{p \to 0} [\lambda \rho(K, \cdot)^p + (1 - \lambda) \rho(L, \cdot)^p]^{\frac{1}{p}} \\ &= \rho(K, \cdot)^{\lambda} \rho(L, \cdot)^{1-\lambda}. \end{aligned} \tag{2.11}$$

For the log-radial Minkowski combination, Wang and Liu (see [18]) established the following dual log-Brunn–Minkowski inequality: *If $K, L \in \mathcal{S}_o^n$ and $\lambda \in [0, 1]$, then*

$$V(\lambda \odot K \tilde{+}_0 (1 - \lambda) \odot L) \le V(K)^{\lambda} V(L)^{1-\lambda}, \tag{2.12}$$

with equality for $\lambda \in (0, 1)$ if and only if K and L are dilates. For $\lambda = 0$ or $\lambda = 1$, (2.12) *becomes an equality.*

2.3. Star dualities

In 1999, Moszyńska (see [14]) introduced the notion of star duality. For $K \in \mathcal{S}_o^n$, the star duality, K°, of K is given by

$$\rho(K^\circ, u) = \frac{1}{\rho(K, u)}, \tag{2.13}$$

for all $u \in \mathcal{S}^{n-1}$. From (2.13), we easily see that for $\lambda > 0$,

$$(\lambda K)^\circ = \frac{1}{\lambda} K^\circ. \tag{2.14}$$

2.4. Intersection bodies

Intersection bodies were first explicitly defined and named by Lutwak (see [11]). For each $K \in \mathcal{S}_o^n$, the intersection body, IK, of K is an origin-symmetric star body whose radial function in the direction $u \in S^{n-1}$ is equal to the $(n-1)$-dimensional volume of the section of K by the $(n-1)$-dimensional subspace $u^\perp$ orthogonal to u. That is, for any $u \in S^{n-1}$,

$$\rho(IK, u) = V_{n-1}(K \cap u^\perp), \tag{2.15}$$

where V_{n-1} denotes $(n-1)$-dimensional volume.

From (2.15), we know that the intersection body has the following property: If $K \in \mathcal{S}_o^n$, then for $\lambda > 0$,

$$I(\lambda K) = \lambda^{n-1} IK. \tag{2.16}$$

The intersection body is a very important object of study in the Brunn–Minkowski theory. A number of important results regarding intersection bodies come together in books [7, 16].

From (2.13) and (2.15), for the star duality $I^\circ K$ of intersection body IK, we have that for any $u \in S^{n-1}$,

$$\rho(I^\circ K, u)^{-1} = \rho(IK, u) = V_{n-1}(K \cap u^\perp). \tag{2.17}$$

Hence, (2.14), (2.16) and (2.17) show that for $\lambda > 0$,

$$I^\circ(\lambda K) = \frac{1}{\lambda^{n-1}} I^\circ K. \tag{2.18}$$

For the works of the star dualities of intersection bodies, also see [10, 24].

3. L_p-dual Brunn–Minkowski inequality for intersection bodies

Theorem 1.1 shows the L_p-dual Brunn–Minkowski inequality for intersection bodies. Here, we will prove Theorem 1.1.

Lemma 3.1. *If $K, L \in \mathcal{S}_o^n$, p is any real and $\lambda \in [0,1]$, then for any $u \in S^{n-1}$,*

$$(\lambda \odot K \tilde{+}_p (1-\lambda) \odot L) \cap u^\perp = \lambda \odot (K \cap u^\perp) \tilde{+}_p (1-\lambda) \odot (L \cap u^\perp). \tag{3.1}$$

Here $u^\perp$ denotes the $(n-1)$-dimensional subspace orthogonal to u.

Proof. For $p \neq 0$, according to (2.6) and (2.7), we have for any $v \in S^{n-1} \cap u^\perp$,

$$\begin{aligned}\rho((\lambda \odot K \tilde{+}_p (1-\lambda) \odot L) \cap u^\perp, v)^p &= \rho(\lambda \odot K \tilde{+}_p (1-\lambda) \odot L, v)^p \\ &= \lambda \rho(K, v)^p + (1-\lambda)\rho(L, v)^p \\ &= \lambda \rho(K \cap u^\perp, v)^p + (1-\lambda)\rho(L \cap u^\perp, v)^p \\ &= \rho(\lambda \odot (K \cap u^\perp) \tilde{+}_p (1-\lambda) \odot (L \cap u^\perp), v)^p.\end{aligned}$$

This gives the case $p \neq 0$ of (3.1).

For $p = 0$, by (2.6) and (2.11) we have that for any $v \in S^{n-1} \cap u^\perp$,

$$\begin{aligned}\rho((\lambda \odot K \tilde{+}_0 (1-\lambda) \odot L) \cap u^\perp, v) &= \rho(\lambda \odot K \tilde{+}_0 (1-\lambda) \odot L, v) \\ &= \rho(K, v)^\lambda \rho(L, v)^{1-\lambda} = \rho(K \cap u^\perp, v)^\lambda \rho(L \cap u^\perp, v)^{1-\lambda} \\ &= \rho(\lambda \odot (K \cap u^\perp) \tilde{+}_0 (1-\lambda) \odot (L \cap u^\perp), v).\end{aligned}$$

This provides the case $p = 0$ of (3.1). ■

Proof of Theorem 1.1. (1) For $0 < p < n-1$, applying inequality (1.3) to the $(n-1)$-dimensional case and combining with (3.1), we have that for $\lambda \in [0,1]$ and any $u \in S^{n-1}$,

$$\begin{aligned}V_{n-1}((\lambda \odot K \tilde{+}_p (1-\lambda) \odot L) \cap u^\perp)^{\frac{p}{n-1}} &= V_{n-1}(\lambda \odot (K \cap u^\perp) \tilde{+}_p (1-\lambda) \odot (L \cap u^\perp))^{\frac{p}{n-1}} \\ &\leq \lambda V_{n-1}(K \cap u^\perp)^{\frac{p}{n-1}} + (1-\lambda) V_{n-1}(L \cap u^\perp)^{\frac{p}{n-1}}.\end{aligned} \tag{3.2}$$

According to the equality condition of inequality (1.3), we see that equality holds in (3.2) for $\lambda \in (0,1)$ if and only if $K \cap u^\perp$ and $L \cap u^\perp$ are dilates for any $u \in S^{n-1}$, i.e., K and L are dilates (see [7, Theorem 7.1.1]).

Notice that $0 < p < n-1$ implies $\frac{n(n-1)}{p} > 1$. From this, by (2.5), (2.15), (3.2) and the Minkowski integral inequality we infer that

$$\begin{aligned}&V(I(\lambda \odot K \tilde{+}_p (1-\lambda) \odot L))^{\frac{p}{n(n-1)}} \\ &\quad = \left[\frac{1}{n}\int_{S^{n-1}} \rho(I(\lambda \odot K \tilde{+}_p (1-\lambda) \odot L), u)^n du\right]^{\frac{p}{n(n-1)}}\end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{1}{n}\int_{S^{n-1}} V_{n-1}((\lambda\odot K \,\widetilde{+}_p\, (1-\lambda)\odot L)\cap u^{\perp})^n du\right]^{\frac{p}{n(n-1)}}\\
&= \left[\frac{1}{n}\int_{S^{n-1}} [V_{n-1}((\lambda\odot K \,\widetilde{+}_p\, (1-\lambda)\odot L)\cap u^{\perp})^{\frac{p}{n-1}}]^{\frac{n(n-1)}{p}} du\right]^{\frac{p}{n(n-1)}}\\
&\le \Bigg[\frac{1}{n}\int_{S^{n-1}} [\lambda V_{n-1}(K\cap u^{\perp})^{\frac{p}{n-1}}\\
&\quad + (1-\lambda)V_{n-1}(L\cap u^{\perp})^{\frac{p}{n-1}}]^{\frac{n(n-1)}{p}} du\Bigg]^{\frac{p}{n(n-1)}}\\
&\le \lambda\left[\frac{1}{n}\int_{S^{n-1}} V_{n-1}(K\cap u^{\perp})^n du\right]^{\frac{p}{n(n-1)}}\\
&\quad + (1-\lambda)\left[\frac{1}{n}\int_{S^{n-1}} V_{n-1}(L\cap u^{\perp})^n du\right]^{\frac{p}{n(n-1)}}\\
&= \lambda\left[\frac{1}{n}\int_{S^{n-1}} \rho(IK,u)^n du\right]^{\frac{p}{n(n-1)}}\\
&\quad + (1-\lambda)\left[\frac{1}{n}\int_{S^{n-1}} \rho(IL,u)^n du\right]^{\frac{p}{n(n-1)}}\\
&= \lambda V(IK)^{\frac{p}{n(n-1)}} + (1-\lambda)V(IL)^{\frac{p}{n(n-1)}}. \qquad (3.3)
\end{aligned}
$$

From the equality conditions of inequality (3.2) and the Minkowski integral inequality, we know that equality holds in inequality (3.3) if and only if K and L are dilates. This gives inequality (1.6) and its equality condition.

(2) For $-\infty < p < 0$ or $n(n-1) < p < +\infty$, similar to the proof of inequality (1.6), applying inequality (1.4) to the $(n-1)$-dimensional case, we may obtain the reverse form of inequality (3.2). This together with the Minkowski integral inequality and notice $\frac{n(n-1)}{p} < 0$ or $0 < \frac{n(n-1)}{p} < 1$, we get inequality (1.7) and its equality condition.

(3) For $p = +\infty$, inequality (1.7) takes the following form:

$$
\begin{aligned}
&\lim_{p\to+\infty} V(I(\lambda\odot K \,\widetilde{+}_p\, (1-\lambda)\odot L))^{\frac{1}{n(n-1)}}\\
&\qquad \ge \lim_{p\to+\infty} [\lambda V(IK)^{\frac{p}{n(n-1)}} + (1-\lambda)V(IL)^{\frac{p}{n(n-1)}}]^{\frac{1}{p}}.
\end{aligned}
$$

This, together with (2.2) and (2.9), becomes that

$$
V(I(K\cup L))^{\frac{1}{n(n-1)}} \ge \max\{V(IK)^{\frac{1}{n(n-1)}}, V(IL)^{\frac{1}{n(n-1)}}\},
$$

i.e.,

$$
V(I(K\cup L)) \ge \max\{V(IK), V(IL)\}. \qquad (3.4)
$$

Because $K \cup L \supseteq K, L$, thus for $u \in S^{n-1}$,

$$\begin{aligned}\rho(I(K \cup L), u) &= V_{n-1}((K \cup L) \cap u^{\perp}) \\ &\geq \max\{V_{n-1}(K \cap u^{\perp}), V_{n-1}(L \cap u^{\perp})\} \\ &= \max\{\rho(IK, u), \rho(IL, u)\}.\end{aligned}$$

This means that (3.4) is true. Hence the case $p = +\infty$ of inequality (1.7) holds.

Equality holds in (3.4) if and only if $K \subseteq L$ or $L \subseteq K$. Indeed, if $K \subseteq L$ or $L \subseteq K$, then equality holds in (3.4). Conversely, for instance, assuming $\max\{V(IK), V(IL)\} = V(IK)$, if $V(I(K \cup L)) = V(IK)$, and notice that $K \cup L \supseteq K$, then $K \cup L = K$. This implies $L \subseteq K$.

For $p = -\infty$, inequality (1.7) becomes as follows:

$$\begin{aligned}&\lim_{p \to -\infty} V(I(\lambda \odot K \,\widetilde{+}_p\, (1-\lambda) \odot L))^{\frac{1}{n(n-1)}} \\ &\quad \leq \lim_{p \to -\infty} [\lambda V(IK)^{\frac{p}{n(n-1)}} + (1-\lambda) V(IL)^{\frac{p}{n(n-1)}}]^{\frac{1}{p}}.\end{aligned}$$

This, together with (2.8) and (2.10), gives that

$$V(I(K \cap L))^{\frac{1}{n(n-1)}} \leq \min\{V(IK)^{\frac{1}{n(n-1)}}, V(IL)^{\frac{1}{n(n-1)}}\},$$

i.e.,

$$V(I(K \cap L)) \leq \min\{V(IK), V(IL)\}. \tag{3.5}$$

Similar to the proof of (3.4), we can obtain inequality (3.5) and its equality condition, i.e., the case $p = -\infty$ of inequality (1.7) and its equality condition are true.

(4) For $p = 0$, applying inequality (2.12) to the $(n-1)$-dimensional case and combining with (3.1), we know that for $\lambda \in [0, 1]$ and any $u \in S^{n-1}$,

$$\begin{aligned}&V_{n-1}((\lambda \odot K \,\widetilde{+}_0\, (1-\lambda) \odot L) \cap u^{\perp}) \\ &\quad = V_{n-1}(\lambda \odot (K \cap u^{\perp}) \,\widetilde{+}_0\, (1-\lambda) \odot (L \cap u^{\perp})) \\ &\quad \leq V_{n-1}(K \cap u^{\perp})^{\lambda} V_{n-1}(L \cap u^{\perp})^{1-\lambda}.\end{aligned} \tag{3.6}$$

According to the equality condition of inequality (1.7), we see that equality holds in (3.6) for $\lambda \in (0, 1)$ if and only if $K \cap u^{\perp}$ and $L \cap u^{\perp}$ are dilates for any $u \in S^{n-1}$, i.e., K and L are dilates.

From (2.5), (2.15), (3.6) and the Hölder integral inequality, we deduce that for $\lambda \in (0, 1)$,

$$\begin{aligned}&V(I(\lambda \odot K \,\widetilde{+}_0\, (1-\lambda) \odot L)) \\ &\quad = \frac{1}{n} \int_{S^{n-1}} \rho(I(\lambda \odot K \,\widetilde{+}_0\, (1-\lambda) \odot L), u)^n du\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n}\int_{S^{n-1}} V_{n-1}((\lambda \odot K \,\widetilde{+}_0\, (1-\lambda)\odot L)\cap u^{\perp})^n du\\
&\le \frac{1}{n}\int_{S^{n-1}} [V_{n-1}(K\cap u^{\perp})^{\lambda} V_{n-1}(L\cap u^{\perp})^{1-\lambda}]^n du\\
&= \frac{1}{n}\int_{S^{n-1}} \rho(IK,u)^{n\lambda}\rho(IL,u)^{n(1-\lambda)} du\\
&\le \left[\frac{1}{n}\int_{S^{n-1}} (\rho(IK,u)^{n\lambda})^{\frac{1}{\lambda}} du\right]^{\lambda}\left[\frac{1}{n}\int_{S^{n-1}} (\rho(IL,u)^{n(1-\lambda)})^{\frac{1}{1-\lambda}} du\right]^{1-\lambda}\\
&= \left[\frac{1}{n}\int_{S^{n-1}} \rho(IK,u)^{n} du\right]^{\lambda}\left[\frac{1}{n}\int_{S^{n-1}} \rho(IL,u)^{n} du\right]^{1-\lambda}\\
&= V(IK)^{\lambda}V(IL)^{1-\lambda}.
\end{aligned}
$$

This yields inequality (1.8). And the equality conditions of (3.6) and the Hölder integral inequality give that equality holds in (1.8) for $\lambda \in (0,1)$ if and only if K and L are dilates.

To sum up, we complete the proof of Theorem 1.1. ■

4. L_p-Brunn–Minkowski inequality for star dualities of intersection bodies

Theorem 1.2 deals with the L_p-Brunn–Minkowski inequality for star dualities of intersection bodies. Now, we give its proof.

Lemma 4.1. *If $K, L \in \mathcal{K}^n$ and $\lambda \in [0,1]$, then for any $u \in S^{n-1}$,*

$$
\lambda(K\cap u^{\perp}) + (1-\lambda)(L\cap u^{\perp}) \subseteq (\lambda K + (1-\lambda)L)\cap u^{\perp}. \tag{4.1}
$$

When $\lambda \in (0,1)$, equality holds in (4.1) if K and L are homothetic.

Proof. For $\lambda \in (0,1)$ and any $u \in S^{n-1}$,

$$
\begin{aligned}
&\forall x = x_1 + x_2 \in \lambda(K\cap u^{\perp}) + (1-\lambda)(L\cap u^{\perp})\\
&\quad \Leftrightarrow x_1 \in \lambda(K\cap u^{\perp}) \text{ and } x_2 \in (1-\lambda)(L\cap u^{\perp})\\
&\quad \Leftrightarrow \lambda^{-1}x_1 \in K\cap u^{\perp} \text{ and } (1-\lambda)^{-1}x_2 \in L\cap u^{\perp}\\
&\quad \Leftrightarrow \lambda^{-1}x_1 \in K \text{ and } \lambda^{-1}x_1 \in u^{\perp},\ (1-\lambda)^{-1}x_2 \in L \text{ and } (1-\lambda)^{-1}x_2 \in u^{\perp}\\
&\quad \Leftrightarrow x_1 \in \lambda K \text{ and } x_1 \in \lambda u^{\perp},\ x_2 \in (1-\lambda)L \text{ and } x_2 \in (1-\lambda)u^{\perp}\\
&\quad \Leftrightarrow x_1 \in \lambda K \text{ and } x_1 \in u^{\perp},\ x_2 \in (1-\lambda)L \text{ and } x_2 \in u^{\perp}\\
&\quad \Rightarrow x = x_1 + x_2 \in \lambda K + (1-\lambda)L \text{ and } x = x_1 + x_2 \in u^{\perp}\\
&\quad \Leftrightarrow x \in [\lambda K + (1-\lambda)L]\cap u^{\perp}.
\end{aligned}
$$

This gives (4.1). We easily verify that equality holds in (4.1) for $\lambda \in (0,1)$ if K and L are homothetic. ■

Proof of Theorem 1.2. (1) If $p = 1$, from (4.1) and the $(n-1)$-dimensional case of inequality (1.1), we obtain that for $K, L \in \mathcal{K}^n$ and any $u \in S^{n-1}$,

$$\begin{aligned} &V_{n-1}([\lambda K + (1-\lambda)L] \cap u^{\perp})^{\frac{1}{n-1}} \\ &\quad \geq V_{n-1}(\lambda(K \cap u^{\perp}) + (1-\lambda)(L \cap u^{\perp}))^{\frac{1}{n-1}} \\ &\quad \geq \lambda V_{n-1}(K \cap u^{\perp})^{\frac{1}{n-1}} + (1-\lambda) V_{n-1}(L \cap u^{\perp})^{\frac{1}{n-1}}. \end{aligned} \tag{4.2}$$

Now we give the equality condition of (4.2). If equality holds in (4.2), i.e.,

$$\begin{aligned} &V_{n-1}([\lambda K + (1-\lambda)L] \cap u^{\perp})^{\frac{1}{n-1}} \\ &\quad = \lambda V_{n-1}(K \cap u^{\perp})^{\frac{1}{n-1}} + (1-\lambda) V_{n-1}(L \cap u^{\perp})^{\frac{1}{n-1}}, \end{aligned}$$

this, and (4.2) imply that

$$\begin{aligned} &V_{n-1}(\lambda(K \cap u^{\perp}) + (1-\lambda)(L \cap u^{\perp}))^{\frac{1}{n-1}} \\ &\quad = \lambda V_{n-1}(K \cap u^{\perp})^{\frac{1}{n-1}} + (1-\lambda) V_{n-1}(L \cap u^{\perp})^{\frac{1}{n-1}}. \end{aligned}$$

This, together with the equality condition of inequality (1.1), means that when $\lambda \in (0,1)$, $K \cap u^{\perp}$ and $L \cap u^{\perp}$ are homothetic for any $u \in S^{n-1}$, i.e., K and L are homothetic. Conversely, if K and L are homothetic, then equality holds in (4.2). Thus, equality holds in (4.2) for $\lambda \in (0,1)$ if and only if K and L are homothetic.

Hence, by (2.5), (4.2), (2.17) and the Minkowski integral inequality, we deduce that

$$\begin{aligned} &V(I^{\circ}(\lambda K + (1-\lambda)L))^{-\frac{1}{n(n-1)}} \\ &\quad = \left[\frac{1}{n}\int_{S^{n-1}} \rho(I^{\circ}(\lambda K + (1-\lambda)L), u)^n du\right]^{-\frac{1}{n(n-1)}} \\ &\quad = \left[\frac{1}{n}\int_{S^{n-1}} [\rho(I^{\circ}(\lambda K + (1-\lambda)L), u)^{-1}]^{-n} du\right]^{-\frac{1}{n(n-1)}} \\ &\quad = \left[\frac{1}{n}\int_{S^{n-1}} V_{n-1}((\lambda K + (1-\lambda)L) \cap u^{\perp})^{-n} du\right]^{-\frac{1}{n(n-1)}} \\ &\quad = \left[\frac{1}{n}\int_{S^{n-1}} [V_{n-1}((\lambda K + (1-\lambda)L) \cap u^{\perp})^{\frac{1}{n-1}}]^{-n(n-1)} du\right]^{-\frac{1}{n(n-1)}} \\ &\quad \geq \left[\frac{1}{n}\int_{S^{n-1}} [\lambda V_{n-1}(K \cap u^{\perp})^{\frac{1}{n-1}} \right. \\ &\qquad \left. + (1-\lambda) V_{n-1}(L \cap u^{\perp})^{\frac{1}{n-1}}]^{-n(n-1)} du\right]^{-\frac{1}{n(n-1)}} \end{aligned}$$

$$\geq \lambda\left[\frac{1}{n}\int_{S^{n-1}} V_{n-1}(K\cap u^{\perp})^{-n}du\right]^{-\frac{1}{n(n-1)}}$$
$$+(1-\lambda)\left[\frac{1}{n}\int_{S^{n-1}} V_{n-1}(L\cap u^{\perp})^{-n}du\right]^{-\frac{1}{n(n-1)}}$$
$$=\lambda\left[\frac{1}{n}\int_{S^{n-1}}\rho(I^{\circ}K,u)^{n}du\right]^{-\frac{1}{n(n-1)}}$$
$$+(1-\lambda)\left[\frac{1}{n}\int_{S^{n-1}}\rho(I^{\circ}L,u)^{n}du\right]^{-\frac{1}{n(n-1)}}$$
$$=\lambda V(I^{\circ}K)^{-\frac{1}{n(n-1)}}+(1-\lambda)V(I^{\circ}L)^{-\frac{1}{n(n-1)}}. \tag{4.3}$$

According to the equality conditions of inequality (4.2) and the Minkowski integral inequality, we know that equality holds in (4.3) for $\lambda\in(0,1)$ if and only if K and L are homothetic. From this, we obtain the case $p=1$ of inequality (1.9) and its equality condition.

(2) If $1<p<+\infty$, for $K,L\in\mathcal{K}_o^n$, let $\alpha=V(I^{\circ}K)^{-\frac{1}{n(n-1)}}$, $\beta=V(I^{\circ}L)^{-\frac{1}{n(n-1)}}$, $\overline{K}=\frac{1}{\alpha}K$, $\overline{L}=\frac{1}{\beta}L\in\mathcal{K}_o^n$. Then by (2.18) we get that

$$V(I^{\circ}\overline{K})^{-\frac{1}{n(n-1)}}=V\left(I^{\circ}\left(\frac{1}{\alpha}K\right)\right)^{-\frac{1}{n(n-1)}}=\frac{1}{\alpha}V(I^{\circ}K)^{-\frac{1}{n(n-1)}}=1,$$

i.e., $V(I^{\circ}\overline{K})=1$. Similarly, $V(I^{\circ}\overline{L})=1$. Since for $\lambda\in[0,1]$, $\overline{\lambda}=\frac{\lambda\alpha^{p}}{\lambda\alpha^{p}+(1-\lambda)\beta^{p}}\in[0,1]$, thus for any $u\in S^{n-1}$,

$$\begin{aligned}
&h(\overline{\lambda}\cdot\overline{K}+_p(1-\overline{\lambda})\cdot\overline{L},u)^{p}\\
&\quad=\overline{\lambda}h(\overline{K},u)^{p}+(1-\overline{\lambda})h(\overline{L},u)^{p}\\
&\quad=\frac{\lambda\alpha^{p}}{\lambda\alpha^{p}+(1-\lambda)\beta^{p}}h(\overline{K},u)^{p}+\frac{(1-\lambda)\beta^{p}}{\lambda\alpha^{p}+(1-\lambda)\beta^{p}}h(\overline{L},u)^{p}\\
&\quad=\frac{\lambda h(K,u)^{p}+(1-\lambda)h(L,u)^{p}}{\lambda\alpha^{p}+(1-\lambda)\beta^{p}}=\frac{h(\lambda\cdot K+_p(1-\lambda)\cdot L,u)^{p}}{\lambda\alpha^{p}+(1-\lambda)\beta^{p}}.
\end{aligned}$$

This and (2.4) yield that

$$\begin{aligned}
\lambda\cdot K+_p(1-\lambda)\cdot L&=[\lambda\alpha^{p}+(1-\lambda)\beta^{p}]^{\frac{1}{p}}[\overline{\lambda}\cdot\overline{K}+_p(1-\overline{\lambda})\cdot\overline{L}]\\
&\supseteq[\lambda\alpha^{p}+(1-\lambda)\beta^{p}]^{\frac{1}{p}}[\overline{\lambda}\overline{K}+(1-\overline{\lambda})\overline{L}].
\end{aligned}\tag{4.4}$$

The equality condition of (2.4) implies that equality holds in (4.4) if and only if $\overline{K}=\overline{L}$, i.e., $K=\frac{\alpha}{\beta}L$ which means that K and L are dilates.

From this, (4.4), (2.17) and (2.18) we deduce that

$$I^\circ(\lambda\cdot K+_p(1-\lambda)\cdot L)\subseteq I^\circ([\lambda\alpha^p+(1-\lambda)\beta^p]^{\frac{1}{p}}[\bar\lambda\overline{K}+(1-\bar\lambda)\overline{L}])$$
$$=[\lambda\alpha^p+(1-\lambda)\beta^p]^{-\frac{n-1}{p}}I^\circ(\bar\lambda\overline{K}+(1-\bar\lambda)\overline{L}). \quad (4.5)$$

Equality holds in (4.5) if and only if K and L are dilates.

Therefore, from (4.5), (2.18) and (4.3), and notice that $V(I^\circ\overline{K})=V(I^\circ\overline{L})=1$, we infer that

$$\begin{aligned}
&V(I^\circ(\lambda\cdot K+_p(1-\lambda)\cdot L))^{-\frac{1}{n(n-1)}}\\
&\geq V([\lambda\alpha^p+(1-\lambda)\beta^p]^{-\frac{n-1}{p}}I^\circ(\bar\lambda\overline{K}+(1-\bar\lambda)\overline{L}))^{-\frac{1}{n(n-1)}}\\
&=[(\lambda\alpha^p+(1-\lambda)\beta^p)^{-\frac{n(n-1)}{p}}]^{-\frac{1}{n(n-1)}}V(I^\circ(\bar\lambda\overline{K}+(1-\bar\lambda)\overline{L}))^{-\frac{1}{n(n-1)}}\\
&\geq[\lambda\alpha^p+(1-\lambda)\beta^p]^{\frac{1}{p}}[\bar\lambda V(I^\circ\overline{K})^{-\frac{1}{n(n-1)}}+(1-\bar\lambda)V(I^\circ\overline{L})^{-\frac{1}{n(n-1)}}]\\
&=[\lambda\alpha^p+(1-\lambda)\beta^p]^{\frac{1}{p}}(\bar\lambda+1-\bar\lambda)=[\lambda\alpha^p+(1-\lambda)\beta^p]^{\frac{1}{p}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
V(I^\circ(\lambda\cdot K+_p(1-\lambda)\cdot L))^{-\frac{p}{n(n-1)}}&\geq\lambda\alpha^p+(1-\lambda)\beta^p\\
&=\lambda V(I^\circ K)^{-\frac{p}{n(n-1)}}+(1-\lambda)V(I^\circ L)^{-\frac{p}{n(n-1)}}.
\end{aligned}$$

This is the case $1<p<+\infty$ of inequality (1.9). The equality condition of (4.5) gives that the equality holds in the case $1<p<+\infty$ of inequality (1.9) if and only if K and L are dilates.

(2) For $p=+\infty$, by (2.2) and (2.3), inequality (1.9) becomes the following form:

$$V(I^\circ(\mathrm{conv}(K\cup L)))^{-\frac{1}{n(n-1)}}\geq\max\{V(I^\circ K)^{-\frac{1}{n(n-1)}},V(I^\circ L)^{-\frac{1}{n(n-1)}}\},$$

or equivalently,

$$V(I^\circ(\mathrm{conv}(K\cup L)))\leq\min\{V(I^\circ K),V(I^\circ L)\}. \quad (4.6)$$

Since $K,L\subseteq\mathrm{conv}(K\cup L)$, thus $IK,IL\subseteq I(\mathrm{conv}(K\cup L))$. This together with (2.13) implies $I^\circ K,I^\circ L\supseteq I^\circ(\mathrm{conv}(K\cup L))$. Therefore, (4.6) is true.

Similar to the derivation of the equality condition of (3.4), we easily see that equality holds in (4.6) if and only if $K\subseteq L$ or $L\subseteq K$. So the case of $p=+\infty$ of inequality (1.9) is proven.

In summary, we complete the proof of Theorem 1.2. ∎

Acknowledgements. The author likes to sincerely thank the reviewer for very valuable and helpful comments and suggestions which made the paper more accurate and readable.

Funding. Research is supported in part by the Natural Science Foundation of China (Grant No.11371224).

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Received 22 July 2024.

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