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Complexity classes of Polishable subgroups

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Abstract. In this paper we further develop the theory of canonical approximations of Polishable subgroups of Polish groups, building on previous work of Solecki and Farah–Solecki. In particular, we obtain a complete characterization of the Borel complexity class of a Polishable subgroup in terms of its canonical approximation. As an application we provide a complete list of all the possible Borel complexity classes of Polishable subgroups of Polish groups, or equivalently of the ranges of continuous group homomorphisms between Polish groups. We also provide a complete list of all the possible Borel complexity classes of the ranges of: continuous group homomorphisms between non-Archimedean Polish groups; continuous linear maps between separable Fréchet spaces; and continuous linear maps between separable Banach spaces.

Keywords: Polish group, Polishable subgroup, Borel complexity class, Solecki subgroup, non-Archimedean Polish group, Fréchet space, Banach space.

1. Introduction

The goal of this paper is to exactly pin down the possible Borel complexity classes of Polishable subgroups of Polish groups. Most of the equivalence relations studied in the context of Borel complexity theory (and mathematics in general) arise as orbit equivalence relations associated with continuous actions of Polish groups on Polish spaces. Many of these actions can be seen as the (left) *translation action* associated with a continuous group homomorphism $\varphi : H \to G$ between Polish groups. In such a case, the image $\varphi(H)$ inside of G is Borel, and the *potential complexity class* (in the sense of Louveau [11]) of the orbit equivalence relation associated with the translation action of H on G is essentially the same as the Borel complexity class of $\varphi(H)$ inside of G; see Theorem 3.3 for a precise statement. It is thus an interesting problem to determine what are the possible values for the Borel complexity class of such a subgroup.

The study of Polishable subgroups of Polish groups, which are precisely the ranges of continuous homomorphisms between Polish groups, has been undertaken by several

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authors over a number of years. The problem of determining their complexity was considered as early as the 1970s, when Saint Raymond proved that there exist Polishable subgroups of $\mathbb{R}^{\mathbb{N}}$ that are arbitrarily high in the Borel hierarchy [16]. A construction of arbitrarily complex non-Archimedean Polishable subgroups of $\mathbb{Z}_2^{\mathbb{N}}$ was presented by Hjorth, Kechris, and Louveau [8]. Hjorth [7] constructed arbitrarily complex Polishable subgroups of any uncountable abelian Polish groups. Farah and Solecki [5], building on previous work of Saint Raymond [16] in the context of separable Fréchet spaces, related the least multiplicative Borel class containing a given Polishable subgroup to the length of the canonical approximation of that Polishable subgroup as in [17, 18].

In this paper, we refine the analysis from [5] by considering not only the multiplicative classes in the Borel hierarchy, but also the additive and difference classes. By relating the Borel complexity class of a Polishable subgroup to its canonical approximation, we completely characterize the possible Borel complexity classes of Polishable subgroups of Polish groups. As a proper open subgroup clearly has complexity class $\mathbf{\Delta}^0_1$, we restrict the analysis to Polishable subgroups that are not open.

Theorem 1.1. *If H is a Polishable subgroup of a Polish group G and H is not open, then the Borel complexity class of H is one of the following: $\mathbf{\Pi}^0_{1+\lambda}$, $\mathbf{\Sigma}^0_{1+\lambda+1}$, $D(\mathbf{\Pi}^0_{1+\lambda+n+1})$, $\mathbf{\Pi}^0_{1+\lambda+n+2}$ for $\lambda < \omega_1$ either zero or a limit ordinal, and $n < \omega$. Furthermore, each of these classes is the Borel complexity class of a Polishable subgroup of $\mathbb{Z}^{\mathbb{N}}$.*

Theorem 4.1 from [8, Section 4] shows that the complexity class $D(\mathbf{\Pi}^0_{1+\lambda+1})$, where λ is either zero or a countable limit ordinal, cannot arise in the context of Theorem 1.1 if one demands H to be non-Archimedean. In this case, we have the following characterization.

Theorem 1.2. *If H is a* non-Archimedean *Polishable subgroup of a Polish group G and H is not open, then the Borel complexity class of H in G is one of the following: $\mathbf{\Pi}^0_{1+\lambda}$, $\mathbf{\Sigma}^0_{1+\lambda+1}$, $D(\mathbf{\Pi}^0_{1+\lambda+n+2})$, $\mathbf{\Pi}^0_{1+\lambda+n+2}$ for $\lambda < \omega_1$ either zero or a limit ordinal, and $n < \omega$. Furthermore, each of these classes is the complexity class of a non-Archimedean Polishable subgroup of $\mathbb{Z}^{\mathbb{N}}$.*

The existence assertions in Theorems 1.1 and 1.2 are proved by providing a unified approach to the constructions in [16] and [8, Section 5] of arbitrarily complex Polishable subgroups, together with a careful analysis of their canonical approximations in the sense of Solecki [17, 18].

Theorem 1.1 entails in particular a negative answer to Question 6.3(1) from [3]. Let X be a separable Banach space with a Schauder basis $(x_n)_{n\in\mathbb{N}}$. Then the collection $\mathrm{coef}(X, (x_n))$ of $(\lambda_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that $\sum_{n\in\mathbb{N}} \lambda_n x_n$ converges in X is a Polishable subgroup of $\mathbb{R}^{\mathbb{N}}$. Question 6.3(1) in [3] asks whether there is an example of such a Polishable subgroup that is $\mathbf{\Delta}^0_3$ and not $D(\mathbf{\Sigma}^0_2)$. By Theorem 1.1, a $\mathbf{\Delta}^0_3$ Polishable subgroup of a Polish group must be $D(\mathbf{\Sigma}^0_2)$.

We also apply the techniques of this paper to provide a complete characterization of the Borel complexity classes of the ranges of continuous homomorphisms between separable Fréchet spaces and between separable Banach spaces.

Theorem 1.3. *The complexity classes in Theorem 1.1 form a complete list of all the Borel complexity classes of the ranges of non-surjective continuous linear maps between separable Fréchet spaces.*

Theorem 1.4. *The following is a complete list of all the Borel complexity classes of the ranges of non-surjective continuous linear maps between separable Banach spaces:* $\mathbf{\Pi}^0_1$, $\mathbf{\Sigma}^0_{1+\lambda+1}$, $D(\mathbf{\Pi}^0_{1+\lambda+n+1})$, $\mathbf{\Pi}^0_{1+\lambda+n+2}$ *for* $\lambda < \omega_1$ *either zero or a limit ordinal, and* $n < \omega$.

Continuous linear maps with arbitrarily complex range, with a fixed separable Banach space or separable Fréchet space as target, were constructed in [4, 12].

This paper is organized as follows. In Sections 2 and 3 we recall some definitions and known results concerning Polishable subgroups and their Borel complexity class. In Section 4 we recall the definition of the canonical approximation of a Polishable subgroup, whose elements we call Solecki subgroups, as they were originally described by Solecki [17]. In Section 5, building on the work of Farah and Solecki [5], we characterize the Solecki subgroups in terms of their Borel complexity class. This is then applied in Section 6 to obtain the characterization of complexity classes of Polishable subgroups as in Theorem 1.1. Section 7 shows that the length of the canonical approximation, called the Polishable rank in [5], coincides with a notion of rank originally considered by Saint Raymond [16] in the context of separable Fréchet spaces. The existence assertions in Theorems 1.1 and 1.2 are proved in Section 8. Finally, Sections 9 and 10 contain the proofs of Theorems 1.3 and 1.4, respectively.

Notation. In this paper, we use $\mathbb{N}$ to denote the set of positive integers *excluding zero*. As usual, we let ω be the first infinite ordinal, which can also be seen as the set of positive integers *including zero*.

2. Polishable subgroups

A *Polish space* is a second countable topological space whose topology is induced by a complete metric. A *Polish group* is a group in the category of Polish spaces, namely a Polish space that is endowed with a continuous group operation such that the function that maps each element to its inverse is also continuous (in fact, the latter requirement holds automatically; see [9, remark after Corollary 9.15]). A subgroup H of a Polish group G is *Polishable* if it is Borel and there exists a Polish group topology on H whose open sets are Borel in G. Notice that if such a topology on H exists, it is unique by [9, Theorem 9.10]. In the following, we will regard H as a Polish group with respect to its unique Polish group topology, which is in general finer than the subspace topology induced from G. Equivalently, H is a Polishable subgroup of G if and only if there exists a Polish group $\tilde{H}$ and a continuous group homomorphism $\varphi : \tilde{H} \to G$ with image equal to H. Noticing that one can assume without loss of generality that φ is an injection, the equivalence of the two definitions follows from [9, Theorem 9.10] and the fact that if $f : X \to Y$ is an injective Borel function between standard Borel spaces, then $f(A)$

is a Borel subset of Y and $f|_A$ is a Borel isomorphism between A and $f(A)$ [9, Theorem 15.1]. If G is a Polish group and H is a Polishable subgroup of G, then G is a *Polish H-space* with respect to the left translation action of H on G [1, Section 2.2]. We will denote by E_H^G the corresponding orbit equivalence relation. Recall that a Polish group G is *non-Archimedean* if it admits a basis of neighborhoods of the identity consisting of open subgroups; see [6, Theorem 2.4.1] for equivalent characterizations.

Lemma 2.1. *Suppose that G is a Polish group. Let $(G_n)_{n\in\omega}$ be a sequence of Polishable subgroups of G. Then $G_\omega := \bigcap_{n\in\omega} G_n$ is a Polishable subgroup of G. If G_n is non-Archimedean for every $n \in \omega$, then G_ω is non-Archimedean as well. If $A \subseteq G_\omega$ is such that A is dense in G_n for every $n \in \omega$, then A is dense in G_ω.*

Proof. The group G_ω is the image of the Polish group

$$Z := \left\{(x_n)_{n\in\omega} \in \prod_{n\in\omega} G_n : \forall n \in \omega,\ x_n = x_{n+1}\right\} \subseteq \prod_{n\in\omega} G_n$$

under the continuous injective group homomorphism $Z \to G$, $(x_n)_{n\in\omega} \mapsto x_0$. This shows that G_ω is Polishable. If each G_n is non-Archimedean, then Z is non-Archimedean, and hence G_ω is non-Archimedean as well. By the above, the sets of the form $W \cap G_\omega$, where W is a neighborhood of the identity in G_n for some $n \in \omega$, form a basis of neighborhoods of the identity in G_ω. Thus, if A is dense in G_n for every $n \in \omega$, then A is dense in G_ω. ■

3. Potential complexity

A *complexity class* Γ is a function $X \mapsto \Gamma(X)$ that assigns to each Polish space X a collection $\Gamma(X)$ of Borel subsets such that if X, Y are Polish spaces and $f : X \to Y$ is a continuous function, then $f^{-1}(A) \in \Gamma(X)$ for every $A \in \Gamma(Y)$. For a complexity class Γ, we let $D(\Gamma)$ be the complexity class consisting of *differences* between sets in Γ; see [9, Section 22.E] where it is denoted by $D_2(\Gamma)$. We let $\check{\Gamma}$ be the *dual* complexity class of Γ, such that $\check{\Gamma}(X)$ comprises the *complements* of the elements of $\Gamma(X)$. We say that Γ is *self-dual* if $\Gamma = \check{\Gamma}$. If Γ is a complexity class that is not self-dual, then we say that Γ is the complexity class of $A \subseteq X$ if $A \in \Gamma(X)$ and $A \notin \check{\Gamma}(X)$. We will be mainly interested in the complexity classes $\mathbf{\Sigma}_\alpha^0$, $\mathbf{\Pi}_\alpha^0$, $\mathbf{\Delta}_\alpha^0$, and $D(\mathbf{\Pi}_\alpha^0)$ for $\alpha \in \omega_1$; see [9, Section 11.B].

If X is a standard Borel space and E is an equivalence relation on X, then E has *potential complexity* Γ if there exists a Polish topology τ on X that induces the Borel structure of X such that $E \in \Gamma(\tau \times \tau)$ [11]. This is equivalent to the assertion that there exists a Borel equivalence relation F on a Polish space Y such that $F \in \Gamma(Y \times Y)$ and E is Borel reducible to F; see [6, Lemma 12.5.4]. The following result is essentially proved in [8, Section 5].

Proposition 3.1 (Hjorth–Kechris–Louveau). *Suppose that G is a Polish group, and X is a Polish G-space. For $x \in X$, denote by $[x]$ the corresponding G-orbit. Let Γ be a complexity class, and assume that the orbit equivalence relation E_G^X is potentially Γ. Suppose that α is a countable ordinal.*

(1) *If* Γ *is* $\mathbf{\Pi}^0_\alpha$ *for* $\alpha \geq 2$, $\mathbf{\Sigma}^0_\alpha$ *for* $\alpha \geq 3$, *or* $D(\mathbf{\Pi}^0_\alpha)$ *for* $\alpha \geq 2$, *then* $\{x \in X : [x] \in \Gamma\}$ *is comeager in* X.

(2) *If* Γ *is* $\check{D}(\mathbf{\Pi}^0_\alpha)$ *for* $\alpha \geq 3$, *then* $\{x \in X : [x]$ *is either* $\mathbf{\Pi}^0_\alpha$ *or* $\mathbf{\Sigma}^0_\alpha\}$ *is comeager in* X.

Proof. Fix a countable open basis $\{U_i : i \in \omega\}$ of G. Below we adopt the Vaught transform notation as in [6, Section 3.2]. By [9, Theorem 8.38], there exists a dense G_δ set $W \subseteq X$ such that $E^X_G \cap (W \times W) \in \Gamma(W \times W)$. Notice that W^{*G} is also a dense G_δ subset of X. Fix $x \in W \cap W^{*G}$. Thus, $[x] \cap W \in \Gamma(W)$.

If $\Gamma = \mathbf{\Sigma}^0_\alpha$ for $\alpha \geq 3$, then $[x] \cap W = A \cap W$ for some $A \in \mathbf{\Sigma}^0_\alpha(X)$. Then $[x] = ([x] \cap W)^{\Delta G} = (A \cap W)^{\Delta G}$ is $\mathbf{\Sigma}^0_\alpha$ in X.

If $\Gamma = \mathbf{\Pi}^0_\alpha$ for $\alpha \geq 2$, then $[x] \cap W = B \cap W$ for some $B \in \mathbf{\Pi}^0_\alpha(X)$. Then $[x] = (W \cap [x])^{*G} = (B \cap W)^{*G}$ is $\mathbf{\Pi}^0_\alpha$ in X.

If $\Gamma = D(\mathbf{\Pi}^0_\alpha)$ for $\alpha \geq 2$, then $W \cap [x] = A \cap B \cap W$ where $A \in \mathbf{\Sigma}^0_\alpha(X)$ and $B \in \mathbf{\Pi}^0_\alpha(X)$. Thus, $[x] = A^{\Delta G} \cap (B \cap W)^{*G} \in D(\mathbf{\Pi}^0_\alpha)$.

If $\Gamma = \check{D}(\mathbf{\Pi}^0_\alpha)$ for $\alpha \geq 3$, then $W \cap [x] = (A \cap W) \cup (B \cap W)$ where $A \in \mathbf{\Sigma}^0_\alpha(X)$ and $B \in \mathbf{\Pi}^0_\alpha(X)$. Thus, $[x] = (A \cap W)^{\Delta G}$ or $[x] = (B \cap W)^{*G}$. Hence, either $[x]$ is $\mathbf{\Sigma}^0_\alpha$ or $[x]$ is $\mathbf{\Pi}^0_\alpha$. ∎

A similar proof gives the following.

Lemma 3.2. *Suppose that* G *is a Polish group, and* H *is a Polishable subgroup of* G. *Let* α *be a countable ordinal. If* H *is* $\check{D}(\mathbf{\Pi}^0_\alpha)$, *then* H *is either* $\mathbf{\Pi}^0_\alpha$ *or* $\mathbf{\Sigma}^0_\alpha$.

Proof. Adopt the notation of the Vaught transform with respect to the left translation action of H on G. We have $H = A \cup B$ where A is $\mathbf{\Sigma}^0_\alpha$ and B is $\mathbf{\Pi}^0_\alpha$. If $x \in H$, then either $x \in A^{\Delta H}$ or $x \in B^{*H}$. Since $A^{\Delta H}$ and B^{*H} are H-invariant, either $H \subseteq A^{\Delta H}$ or $H \subseteq B^{*H}$. Since $A^{\Delta H}$ and B^{*H} are contained in H, either $H = A^{\Delta H}$ or $H = B^{*H}$. This concludes the proof. ∎

Applying Proposition 3.1 to the left translation action associated with a Polishable subgroup of a Polish group, we obtain items (1) and (3) of the following result. The proof of (2) is postponed to Section 6.

Theorem 3.3. *Suppose that* G *is a Polish group, and* H *is a Polishable subgroup of* G. *Denote by* E^G_H *the corresponding coset equivalence relation.*

(1) E^G_H *is potentially* $\mathbf{\Pi}^0_2$ *if and only if* H *is closed* G.

(2) E^G_H *is potentially* $\mathbf{\Sigma}^0_2$ *if and only if* H *is* $D(\mathbf{\Pi}^0_2)$ *in* G.

(3) *Let* Γ *be one of the following complexity classes:* $\mathbf{\Sigma}^0_\alpha$ *for* $\alpha \neq 2$, $\mathbf{\Pi}^0_\alpha$, *and* $D(\mathbf{\Pi}^0_\alpha)$. *Then* E^G_H *is potentially* Γ *in* G *if and only if* H *is* Γ *in* G.

Proof. (1) Suppose that E^G_H is potentially $\mathbf{\Pi}^0_2$. By [6, Lemma 12.5.3], E^G_H is smooth. Thus, H is closed by [18, p. 574].

(2) The forward implication is a particular instance of Proposition 3.1, while the converse follows from Lemma 6.5 below.

(3) Only the forward implication requires proof. If $\Gamma = \mathbf{\Sigma}^0_1$ then E^G_H has countably many classes by [6, Lemma 12.5.2]. Thus, H has countable index in G, and hence it is nonmeger. Therefore, H is open by [6, Theorem 2.3.2]. If Γ is $\mathbf{\Pi}^0_1$ or $\mathbf{\Pi}^0_2$ or $D(\mathbf{\Pi}^0_1)$, then $H \in \mathbf{\Pi}^0_1(G) \subseteq \Gamma(G)$ by part (1). If Γ is $\mathbf{\Pi}^0_\alpha$ or $\mathbf{\Sigma}^0_\alpha$ for $\alpha \geq 3$, or $D(\mathbf{\Pi}^0_\alpha)$ for $\alpha \geq 2$, the conclusion follows from Proposition 3.1. ■

We now recall a result concerning the possible complexity classes of Polishable subgroups from [5, Corollary 3.4].

Proposition 3.4 (Farah–Solecki). *Suppose that G is a Polish group, and H is a Polishable subgroup of G. If $\lambda < \omega_1$ is either zero or a limit ordinal, and H is $\mathbf{\Pi}^0_{1+\lambda+1}$ in G, then H is $\mathbf{\Pi}^0_{1+\lambda}$ in G.*

4. Solecki subgroups

Suppose that G is a Polish group, and H is a Polishable subgroup of G. Then H admits a canonical approximation by Polishable subgroups indexed by countable ordinals. As these were originally described by Solecki [17], we call them *Solecki subgroups* of G associated with H. They have also been considered in [5, 18].

The result of [17, Lemma 2.3] implies that G has a smallest $\mathbf{\Pi}^0_3$ subgroup containing H, which we denote by $s^H_1(G)$. It can be explicitly described as the subgroup of G defined by

$$\bigcap_V \bigcup_{z_0, z_1 \in H} (z_0 \overline{V}^G \cap \overline{V}^G z_1),$$

where V ranges over the open neighborhoods of the identity in H, and $\overline{V}^G$ is the closure of V inside of G. It is proved in [17, Lemma 2.3] that $s^H_1(G)$ satisfies the following properties—see also [19, Lemma 4.5] and [5, Section 3]:

- $s^H_1(G)$ is a Polishable subgroup of G;
- H is dense in $s^H_1(G)$;
- a neighborhood basis of $x \in s^H_1(G)$ consists of the sets of the form $\overline{Wx}^G \cap s^H_1(G)$ where W is an open neighborhood of the identity in H;
- if $A \subseteq G$ is $\mathbf{\Pi}^0_3$ and contains H, then $A \cap s^H_1(G)$ is comeager in the Polish group topology of $s^H_1(G)$.

Lemma 4.1. *Suppose that G is a Polish group, and H is a non-Archimedean Polishable subgroup of G. Then a neighborhood basis of the identity in $s^H_1(G)$ consists of the sets of the form $\overline{W}^G \cap s^H_1(G)$ where W is an open subgroup of H. In particular, $s^H_1(G)$ is non-Archimedean.*

Proof. Since H is non-Archimedean, the first assertion follows from the remarks above. If W is an open subgroup of H, then $\overline{W}^G \cap s^H_1(G)$ is a subgroup of $s^H_1(G)$ with nonempty interior, whence it is an open subgroup by Pettis' Theorem [15, Corollary 3.1]. Therefore, the second assertion follows from the first one. ■

A similar argument to the one in [17, proof of Lemma 2.3] gives the following.

Lemma 4.2. *Suppose that G is a Polish group, and that N is a Polishable subgroup of G. Let H be a Polishable subgroup of G such that*

(1) *$N \subseteq H$ and N is dense in the Polish group topology of H;*

(2) *for every open neighborhood V of the identity in N, $\overline{V}^G \cap H$ contains an open neighborhood of the identity in H.*

If $A \subseteq G$ is $\mathbf{\Pi}^0_3$ and contains N, then $A \cap H$ is comeager in H. In particular, $H \subseteq s^N_1(G)$. If H is furthermore $\mathbf{\Pi}^0_3$, then $H = s^N_1(G)$.

Proof. It suffices to consider the case when A is $\mathbf{\Sigma}^0_2$. In this case, there exist closed subsets L_k of G for $k \in \omega$ such that $A = \bigcup_{k\in\omega} L_k$. Suppose that $U \subseteq H$ is a nonempty open set. Since N is dense in H, $U \cap N$ is a nonempty open subset of N. By the Baire Category Theorem, there exists $k_0 \in \omega$ such that $L_{k_0} \cap U \cap N$ is not meager in N. Thus, there exist $x \in N$ and an open neighborhood V of the identity in N such that

$$Vx \subseteq L_{k_0} \cap U \cap N.$$

Since L_{k_0} is closed in G, we know that $\overline{Vx}^G \subseteq L_{k_0}$. By (2), there is an open neighborhood W of the identity in H such that $Wx \subseteq \overline{Vx}^G \subseteq L_{k_0}$. This shows that $x \in U$ is in the interior of $L_{k_0} \cap H \subseteq A \cap H$. Since this holds for every nonempty open subset of H, we see that $A \cap H$ contains a dense open subset of H, and hence it is comeager in H. This concludes the proof. ■

The sequence of *Solecki subgroups* $s^H_\alpha(G)$ for $\alpha < \omega_1$ of G associated with H is defined recursively by setting

- $s^H_0(G) = \overline{H}^G$;
- $s^H_{\alpha+1}(G) = s^H_1(s^H_\alpha(G))$ for $\alpha < \omega_1$;
- $s^H_\lambda(G) = \bigcap_{\beta<\lambda} s^H_\beta(G)$ for a limit ordinal $\lambda < \omega_1$.

Using Lemma 2.1 at the limit stage, one can prove by induction on $\alpha < \omega_1$ that $s^H_\alpha(G)$ is a Polishable subgroup of G, and H is dense in $s^H_\alpha(G)$. Furthermore, by Lemma 4.1, if H is non-Archimedean, then $s^H_\alpha(G)$ is non-Archimedean for every $1 \leq \alpha < \omega_1$. It is proved in [17, Theorem 2.1] that there exists $\alpha < \omega_1$ such that $s^H_\alpha(G) = H$. We call the least countable ordinal α such that $s^H_\alpha(G) = H$ the *Solecki rank* of H in G.

One can define the Polish groups $s^H_\alpha(G)$ solely in terms of H endowed with the subspace topology inherited from G. Indeed, $s^H_0(G)$ can be seen as the completion of H with respect to a suitable metric that induces the subspace topology inherited from G; see [17, Section 2.1]. Using Lemma 4.2 one can describe the Solecki subgroups of products, as follows.

Lemma 4.3. *Suppose that, for every $n \in \mathbb{N}$, G_n is a Polish group, and N_n is a Polishable subgroup. Define $G = \prod_{n\in\omega} G_n$ and $N = \prod_{n\in\omega} N_n$. Then*

$$s_\gamma^N(G) = \prod_{n\in\omega} s_\gamma^{N_n}(G_n) \quad \text{for every } \gamma < \omega_1.$$

Proof. It suffices to consider the case when $\gamma = 1$. In this case, set

$$H_n := s_1^{N_n}(G_n) \quad \text{for } n \in \omega, \quad \text{and} \quad H := \prod_{n\in\omega} H_n.$$

Then H is a $\mathbf{\Pi}_3^0$ Polishable subgroup of G, $N \subseteq H$, and N is dense in H. If V is an open neighborhood of the identity in N, then there exist $n \in \omega$ and open neighborhoods V_i of the identity in N_i for $i < n$ such that V contains

$$\prod_{i<n} V_i = \{x \in N : \forall i < n, x_i \in V_i\}.$$

For $i < n$, $\overline{V}_i^{G_i} \cap H_i$ contains an open neighborhood W_i of the identity in H_i. Therefore, $\overline{V}^G \cap H$ contains

$$\prod_{i<n} W_i = \{x \in H : \forall i < n, x_i \in W_i\},$$

which is an open neighborhood of the identity in H. The conclusion thus follows from Lemma 4.2. ∎

5. Complexity of Solecki subgroups

Suppose that G is a Polish group, and H is a Polishable subgroup of G. For a complexity class Γ, we define $\Gamma(G)|_H$ to be the collection of sets of the form $A \cap H$ for $A \in \Gamma(G)$. The following results are essentially established in [5]. In the statements and proofs, we adopt the Vaught transform notation in reference to the action of H on G by left translation; see [6, Section 3.2].

Lemma 5.1. *Suppose that G is a Polish group, H is a Polishable subgroup of G, and $\alpha, \beta < \omega_1$ are ordinals. Then*

$$\mathbf{\Sigma}_{1+\beta}^0(s_\alpha^H(G)) \subseteq \mathbf{\Sigma}_{1+\alpha+\beta}^0(G)|_{s_\alpha^H(G)},$$
$$\mathbf{\Pi}_{1+\beta}^0(s_\alpha^H(G)) \subseteq \mathbf{\Pi}_{1+\alpha+\beta}^0(G)|_{s_\alpha^H(G)}.$$

Proof. It is proved in [5, Theorem 3.1] by induction on α that

$$\mathbf{\Sigma}_1^0(s_\alpha^H(G)) \subseteq \mathbf{\Sigma}_{1+\alpha}^0(G)|_{s_\alpha^H(G)}.$$

By taking complements, we find that

$$\mathbf{\Pi}_1^0(s_\alpha^H(G)) \subseteq \mathbf{\Pi}_{1+\alpha}^0(G)|_{s_\alpha^H(G)}.$$

This is the case $\beta = 0$ of the statement above. The rest follows by induction on β. ∎

Lemma 5.2. *Suppose that G is a Polish group, H is a Polishable subgroup of G, $\alpha, \beta < \omega_1$, and $U \subseteq H$ is open in H. If $A \in \mathbf{\Sigma}^0_{1+\alpha+\beta}(G)$ and $B \in \mathbf{\Pi}^0_{1+\alpha+\beta}(G)$, then $A^{\Delta U} \cap s^H_\alpha(G) \in \mathbf{\Sigma}^0_{1+\beta}(s^H_\alpha(G))$ and $B^{*U} \cap s^H_\alpha(G) \in \mathbf{\Pi}^0_{1+\beta}(s^H_\alpha(G))$.*

Proof. When $\beta = 0$, the assertion about A is the content of Claim 3.3 in the proof of [5, Theorem 3.1]. The assertion about B follows by taking complements. This concludes the proof when $\beta = 0$. The case of an arbitrary β is established by induction on β using the properties of the Vaught transform; see [6, Proposition 3.2.5]. ■

Corollary 5.3. *Suppose that G is a Polish group, H is a Polishable subgroup of G, and $\alpha, \beta < \omega_1$. Let L be a Borel subgroup of G containing H. If $L \in \mathbf{\Sigma}^0_{1+\alpha+\beta}(G)$, then $L \cap s^H_\alpha(G) \in \mathbf{\Sigma}^0_{1+\beta}(s^H_\alpha(G))$. If $L \in \mathbf{\Pi}^0_{1+\alpha+\beta}(G)$, then $L \cap s^H_\alpha(G) \in \mathbf{\Pi}^0_{1+\beta}(s^H_\alpha(G))$. If $L \in D(\mathbf{\Pi}^0_{1+\alpha+\beta})(G)$, then $L \cap s^H_\alpha(G) \in D(\mathbf{\Pi}^0_{1+\beta})(s^H_\alpha(G))$.*

Proof. Observe that $L = L^{*H} = L^{\Delta H}$. Thus, the first two assertions follow immediately from Lemma 5.2. If $L = A \cap B$ where A is $\mathbf{\Sigma}^0_{1+\beta+1}$ in G and B is $\mathbf{\Pi}^0_{1+\beta+1}$ in G, then we have $L \cap s^H_\alpha(G) = A^{\Delta H} \cap B^{*H} \cap s^H_\alpha(G)$ where $A^{\Delta H} \cap s^H_\alpha(G) \in \mathbf{\Sigma}^0_{1+\beta}(s^H_\alpha(G))$ and $B^{*H} \cap s^H_\alpha(G) \in \mathbf{\Pi}^0_{1+\beta}(s^H_\alpha(G))$ by Lemma 5.2. It follows that $L \cap s^H_\alpha(G) \in D(\mathbf{\Pi}^0_{1+\beta})(s^H_\alpha(G))$. ■

Recall that, by Proposition 3.4, if α is either zero or a countable limit ordinal, and H is a $\mathbf{\Pi}^0_{1+\alpha+1}$ Polishable subgroup of a Polish group, then H is $\mathbf{\Pi}^0_{1+\alpha}$.

Theorem 5.4. *Suppose that G is a Polish group, H is a Polishable subgroup of H, and $\alpha < \omega_1$. Then $s^H_\alpha(G)$ is the smallest $\mathbf{\Pi}^0_{1+\alpha+1}$ subgroup of G containing H.*

Proof. It is established in [5, proof of Theorem 3.1] that $s_\alpha(G)$ is a $\mathbf{\Pi}^0_{1+\alpha+1}$ Polishable subgroup of G. We now prove the minimality assertion by induction on α. For $\alpha = 0$ this follows from the fact that $s^H_0(G) = \bar{H}^G$. Suppose that it holds for α. We now prove that it holds for $\alpha + 1$. Let L be a $\mathbf{\Pi}^0_{1+\alpha+2}$ subgroup of G containing H. Thus, $L \cap s^H_\alpha(G)$ is a $\mathbf{\Pi}^0_{1+\alpha+2}$ subgroup of $s^H_\alpha(G)$. Then by Corollary 5.3 we find that $L \cap s^H_\alpha(G) \in \mathbf{\Pi}^0_3(s_\alpha(H))$. As $s^H_{\alpha+1}(G) = s^H_1(s^H_\alpha(G))$ is the smallest $\mathbf{\Pi}^0_3(s^H_\alpha(G))$ subgroup of $s^H_\alpha(G)$, this implies that $s^H_{\alpha+1}(G) \subseteq L \cap s^H_\alpha(G) \subseteq L$.

Suppose that α is a limit ordinal and the conclusion holds for every $\beta < \alpha$. Fix an increasing sequence (α_n) in α such that $\alpha = \sup_n \alpha_n$. Suppose that L is a $\mathbf{\Pi}^0_{1+\alpha}$ subgroup of G containing H. Since L is $\mathbf{\Pi}^0_{1+\alpha}$ in G, we can write $L = \bigcap_{n\in\omega} A_n$ where, for every $n \in \omega$, $A_n \in \mathbf{\Pi}^0_{1+\alpha_n}(G)$. Then by Lemma 5.2,

$$A_n^{*H} \cap s^H_{\alpha_n}(G) \in \mathbf{\Pi}^0_1(s_{\alpha_n}(G)).$$

Since $H \subseteq L \subseteq A_n$ we have $H \subseteq A_n^{*H} \cap s^H_{\alpha_n}(G)$. Since H is dense in $s^H_{\alpha_n}(G)$, this implies that $s^H_{\alpha_n}(G) \subseteq A_n^{*H}$. Therefore,

$$s^H_\alpha(G) = \bigcap_{n\in\omega} s^H_{\alpha_n}(G) \subseteq \bigcap_{n\in\omega} A_n^{*H} = L^{*H} = L.$$

This shows that $s^H_\alpha(G) \subseteq L$, concluding the proof. ■

Lemma 5.5. *Suppose that G is a Polish group, H is a Polishable subgroup of G, and $\alpha < \omega_1$. Let L be a subgroup of $s_\alpha^H(G)$.*

(1) *If $L \in \mathbf{\Pi}_3^0(s_\alpha^H(G))$, then $L \in \mathbf{\Pi}_{1+\alpha+2}^0(G)$.*
(2) *If $L \in D(\mathbf{\Pi}_2^0)(s_\alpha^H(G))$, then $L \in D(\mathbf{\Pi}_{1+\alpha+1}^0)(G)$.*
(3) *If $L \in \mathbf{\Sigma}_2^0(s_\alpha^H(G))$ and α is either zero or limit, then $L \in \mathbf{\Sigma}_{1+\alpha+1}^0(G)$.*

Proof. By Lemma 5.1 we have

$$\mathbf{\Pi}_3^0(s_\alpha^H(G)) \subseteq \mathbf{\Pi}_{1+\alpha+2}^0(G)|_{s_\alpha^H(G)},$$
$$D(\mathbf{\Pi}_2^0)(s_\alpha^H(G)) \subseteq D(\mathbf{\Pi}_{1+\alpha+1}^0)(G)|_{s_\alpha^H(G)}.$$

Furthermore, $s_\alpha^H(G) \in \mathbf{\Pi}_{1+\alpha+1}^0(G)$ by Theorem 5.4. Therefore,

$$\mathbf{\Pi}_3^0(s_\alpha^H(G)) \subseteq \mathbf{\Pi}_{1+\alpha+2}^0(G)|_{s_\alpha^H(G)} \subseteq \mathbf{\Pi}_{1+\alpha+2}^0(G),$$
$$D(\mathbf{\Pi}_2^0)(s_\alpha^H(G)) \subseteq D(\mathbf{\Pi}_{1+\alpha+1}^0)(G)|_{s_\alpha^H(G)} \subseteq D(\mathbf{\Pi}_{1+\alpha+1}^0)(G).$$

This concludes the proof of (1) and (2).

When α is either zero or limit, by Theorem 5.4 and Proposition 3.4 we have

$$s_\alpha^H(G) \in \mathbf{\Pi}_{1+\alpha}^0(G) \subseteq \mathbf{\Sigma}_{1+\alpha+1}^0(G).$$

Therefore, in this case

$$\mathbf{\Sigma}_2^0(s_\alpha^H(G)) \subseteq \mathbf{\Sigma}_{1+\alpha+1}^0(G)|_{s_\alpha^H(G)} \subseteq \mathbf{\Sigma}_{1+\alpha+1}^0(G).$$

This concludes the proof of (3). ∎

Lemma 5.6. *Suppose that, for every $k \in \omega$, G_k is a Polish group and H_k is a subgroup of G_k. Define $G = \prod_{k\in\omega} G_k$ and $H = \prod_{k\in\omega} H_k$. Assume that, for every $k \in \omega$, H_k is $\mathbf{\Pi}_\alpha^0$ in G_k, and for every $\beta < \alpha$ there exist infinitely many $k \in \omega$ such that H_k is not $\mathbf{\Pi}_\beta^0$ in G_k. Then $\mathbf{\Pi}_\alpha^0$ is the complexity class of H in G.*

Proof. We have $H = \prod_{k\in\omega} H_k \subseteq \prod_{k\in\omega} G_k$. Clearly, H is $\mathbf{\Pi}_\alpha^0$. By [9, Theorem 22.10], for every $k \in \omega$ and $\beta < \alpha$ such that H_k is not $\mathbf{\Pi}_\beta^0$, H_k is $\mathbf{\Sigma}_\beta^0$-hard [9, Definition 22.9]. Therefore, H is $\mathbf{\Pi}_\alpha^0$-hard, and hence H is not $\mathbf{\Sigma}_\alpha^0$ by [9, Theorem 22.10] again. ∎

Lemma 5.7. *Suppose that, for every $k \in \omega$, G_k is a Polish group, H_k is a Polishable subgroup of G_k, and $\alpha < \omega_1$. Define $G = \prod_{k\in\omega} G_k$ and $H = \prod_{k\in\omega} H_k$. If H_k has Solecki rank α in G_k for every $k \in \omega$, then $\mathbf{\Pi}_{1+\alpha+1}^0$ is the complexity class of H in G if α is a successor ordinal, and $\mathbf{\Pi}_{1+\alpha}^0$ is the complexity class of H in G if α is either zero or a limit ordinal.*

Proof. Define $\lambda = 1 + \alpha + 1$ if α is a successor ordinal, and $\lambda = 1 + \alpha$ if α is either zero or a limit ordinal. By Theorem 5.4, for every $k \in \omega$, H_k is $\mathbf{\Pi}_\lambda^0$ but not $\mathbf{\Pi}_\beta^0$ for $\beta < \lambda$ in G_k. Therefore, by Lemma 5.6, $\mathbf{\Pi}_\lambda^0$ is the complexity of H in G. ∎

6. Complexity of Polishable subgroups

The goal of this section is to establish the following theorem, characterizing the possible values of the complexity class of a Polishable subgroup of a Polish group.

Theorem 6.1. *Suppose that G is a Polish group, and H is a Polishable subgroup of G that is not open. Let $\alpha = \lambda + n$ be the Solecki rank of H in G, where $\lambda < \omega_1$ is either zero or a limit ordinal and $n < \omega$.*

(1) *Suppose that $n = 0$. Then $\mathbf{\Pi}^0_{1+\lambda}$ is the complexity class of H in G.*
(2) *Suppose that $n \geq 1$.*
   (a) *If $H \in \mathbf{\Pi}^0_3(s^H_{\lambda+n-1}(G))$ and $H \notin D(\mathbf{\Pi}^0_2)(s^H_{\lambda+n-1}(G))$, then $\mathbf{\Pi}^0_{1+\lambda+n+1}$ is the complexity class of H in G.*
   (b) *If $n \geq 2$ and $H \in D(\mathbf{\Pi}^0_2)(s^H_{\lambda+n-1}(G))$, then $D(\mathbf{\Pi}^0_{1+\lambda+n})$ is the complexity class of H in G.*
   (c) *If $n = 1$, $H \in D(\mathbf{\Pi}^0_2)(s^H_{\lambda}(G))$, and $H \notin \mathbf{\Sigma}^0_2(s^H_{\lambda}(G))$, then $D(\mathbf{\Pi}^0_{1+\lambda+1})$ is the complexity class of H in G.*
   (d) *If $n = 1$ and $H \in \mathbf{\Sigma}^0_2(s^H_{\lambda}(G))$, then $\mathbf{\Sigma}^0_{1+\lambda+1}$ is the complexity class of H in G.*

Furthermore, if H is non-Archimedean then case (2c) *is excluded.*

Theorems 1.1 and 1.2 are immediate consequences of Theorem 6.1. We will obtain Theorem 6.1 as a consequence of a number of *complexity reduction* lemmas. We fix a Polish group G and a Polishable subgroup H of G. We adopt the Vaught transform notation, in reference to the left translation action of H on G.

Lemma 6.2. *If H is $\mathbf{\Delta}^0_3$, then H is $D(\mathbf{\Pi}^0_2)$.*

Proof. Since H is $\mathbf{\Pi}^0_3$ in G, we have $H = s^H_1(G)$. Thus, H has a countable basis $\{V_n : n \in \omega\}$ of neighborhoods of the identity such that $\overline{V}^G_n \cap H = V_n$ for every $n \in \omega$. Indeed, if $\{W_n : n \in \omega\}$ is a countable basis of open neighborhoods of the identity in H, then $\{\overline{W}^G_n \cap H : n \in \omega\}$ is a countable basis of neighborhoods of the identity in H. If $V_n = \overline{W}^G_n \cap H$, then $W_n \subseteq V_n \subseteq \overline{W}^G_n$ and hence $\overline{V}^G_n = \overline{W}^G_n$ and $V_n = \overline{V}^G_n \cap H$.

Let also $\{U_\ell : \ell \in \omega\}$ be a countable basis for the Polish group topology of H. Since H is $\mathbf{\Sigma}^0_3$, we can write $H = \bigcup_{k\in\omega} F_k$ where F_k is $\mathbf{\Pi}^0_2$ in G. Thus, $H = \bigcup_{k,\ell\in\omega} F^{*U_\ell}_k$ where, by Lemma 5.2, $F^{*U_\ell}_k$ is closed in H and $\mathbf{\Pi}^0_2$ in G. Hence, without loss of generality we can assume that F_k is closed in H for every $k \in \omega$. By the Baire Category Theorem, we can assume that $V_0 \subseteq F_0$. Fix a countable dense subset $\{z_m : m \in \omega\}$ of H. Since $H = s^H_1(G)$, we see that an element $x \in G$ is in H if and only if for every $k \in \omega$ there exist $m_0, m_1 \in \omega$ such that $xz_{m_0} \in \overline{V}^G_k$ and $z_{m_1}x \in \overline{V}^G_k$.

We claim that an element $x \in G$ is in H if and only if (1) there exists $m \in \omega$ such that $xz_m \in \overline{V}^G_0$, and (2) for all $m \in \omega$, if $xz_m \in \overline{V}^G_0$, then $xz_m \in F_0$. This will witness that H is $D(\mathbf{\Pi}^0_2)$ in G.

Indeed, since $\{z_m : m \in \omega\}$ is dense in H, if $x \in H$, then there exists $m \in \omega$ such that $xz_m \in V_0 \subseteq \overline{V}_0^G$. Furthermore, if $m \in \omega$ is such that $xz_m \in \overline{V}_0^G$, then $xz_m \subseteq \overline{V}_0^G \cap H = V_0 \subseteq F_0$. Conversely suppose that there exists $m_0 \in \omega$ such that $xz_{m_0} \in \overline{V}_0^G$, and for all $m \in \omega$, if $xz_m \in \overline{V}_0^G$ then $xz_m \in F_0$. Then $xz_{m_0} \in F_0 \subseteq H$ and hence $x \in H$. ■

Lemma 6.3. *If H is $\mathbf{\Pi}_4^0$ in G, then H has a countable basis of neighborhoods of the identity consisting of sets that are in $\mathbf{\Pi}_2^0(G)|_H$.*

Proof. Define $\tilde{H} = s_1^H(G)$. By Theorem 5.4, $H = s_2^H(G) = s_1^H(\tilde{H})$. Thus a neighborhood basis of the identity in H consists of sets of the form $\overline{W}^{\tilde{H}} \cap H$ where W is an open neighborhood of the identity in H. By Lemma 5.1,

$$\overline{W}^{\tilde{H}} \in \mathbf{\Pi}_1^0(s_1^H(G)) \subseteq \mathbf{\Pi}_2^0(G)|_{s_1^H(G)}.$$

Therefore,

$$\overline{W}^{\tilde{H}} \cap H \in \mathbf{\Pi}_2^0(G)|_H.$$

This concludes the proof. ■

Lemma 6.4. *Suppose that H is $\mathbf{\Sigma}_3^0$ in G, and define $\tilde{H} = s_1^H(G)$. Then the following hold:*

(1) $\tilde{H} = s_1^H(\tilde{H})$.

(2) *We can write H as the union of an increasing sequence $(F_k)_{k\in\omega}$ such that F_k is $\mathbf{\Pi}_2^0$ in G and closed in $\tilde{H}$ for every $k \in \omega$.*

(3) *H has a countable family of neighborhoods of the identity of the form V^{*W} where W, V are neighborhoods of the identity in H, W is open in H and $V \in \mathbf{\Pi}_2^0(G)$.*

Proof. (1) This is a consequence of Theorem 5.4.

(2) We can write $H = \bigcup_{k\in\omega} F_k$ where F_k is $\mathbf{\Pi}_2^0$ in G. Fix a countable basis $\{V_n : n \in \omega\}$ for the topology of H. Let also $\{z_m : n \in \omega\}$ be a countable dense subset of H. Then $H = \bigcup_{n,k\in\omega} F_k^{*V_n}$ where, by Lemma 5.2, $F_k^{*V_n}$ is closed in $\tilde{H}$ and $\mathbf{\Pi}_2^0$ in G. Thus, without loss of generality we can assume that F_k is closed in $\tilde{H}$ for every $k \in \omega$.

(3) Let $(F_k)_{k\in\omega}$ be as in (2). By the Baire Category Theorem, we can assume that F_0 is a neighborhood of the identity. Fix an open neighborhood V_0 of the identity in H contained in F_0. By Lemma 6.3, there exist symmetric neighborhoods U, V, W of the identity in H such that W is open in H and $U, V \in \mathbf{\Pi}_2^0(G)|_H$ and $WU \subseteq V \subseteq V^2 \subseteq V_0$. Since $U \subseteq V \subseteq V_0 \subseteq F_0$ and $F_0 \in \mathbf{\Pi}_2^0(G)$, we find that $U, V \in \mathbf{\Pi}_2^0(G)$. Furthermore, $U \subseteq V^{*W} \subseteq V^2 \subseteq V_0$. As this holds for every open neighborhood of the identity in H contained in F_0, this concludes the proof that H has a countable basis of neighborhoods of the identity consisting of sets of the form V^{*W} where $V \in \mathbf{\Pi}_2^0(G)$ and W is open in H. ■

Lemma 6.5. *If H is $\mathbf{\Sigma}_3^0$ in G, then the coset equivalence relation E_H^G is potentially $\mathbf{\Sigma}_2^0$, and H is $D(\mathbf{\Pi}_2^0)$ in G.*

Proof. By Lemma 6.4, we can fix a countable basis $\{V_m : m \in \omega\}$ of neighborhoods of the identity in H consisting of sets of the form $V_m = B_m^{*A_m}$ where $B_m \in \mathbf{\Pi}_2^0(G)$ and A_m is open in H. Fix also a countable dense subset $\{h_k : k \in \omega\}$ of H. Let $(U_n)_{n\in\omega}$ be an enumeration of the countable set $\{V_m h_k : m, k \in \omega\}$.

Let $H \times G$ be the product Polish group. By [1, Theorem 5.1.8] applied to the left translation action $a : H \curvearrowright G$, together with [1, Theorem 5.1.3], there exists a Polish topology t on G such that the left translation action $a : H \curvearrowright (G, t)$ is continuous, U_n is t-closed for every $n \in \omega$, t is finer than the Polish group topology of G, and it generates the same Borel structure as the Polish group topology of G.

Fix a metric d on G compatible with t. For a closed subset C of G and $x \in G$ we define

$$d(x, C) = \inf\{d(x, c) : c \in C\}.$$

Let $K(G, t)$ be the space of t-closed subsets of G. We regard $K(G, t)$ as endowed with the *Wijsman topology*, which is obtained by declaring a net (C_i) to converge to C if and only if, for every $x \in X$, $(d(C_i, x))$ converges to $d(C, x)$ in $\mathbb{R}$. This turns $K(G, t)$ into a Polish space [2, Theorem 4.3]. The Borel σ-algebra on $K(G, t)$ is the σ-algebra generated by sets of the form

$$\{C \in K(G, t) : C \cap W \neq \varnothing\},$$

where W is some t-open subset of G [9, Section 12.C]. The relation $C_0 \subseteq C_1$ for closed subsets C_0, C_1 of G is closed in $K(G, t)$, since $C_0 \subseteq C_1$ if and only if $d(C_1, x) \leq d(C_0, x)$ for every $x \in G$.

Define the Borel function $G \to K(G, t)^\omega$ by $x \mapsto (U_n x)_{n\in\omega}$. Notice that this function is indeed Borel: if $W \subseteq G$ is t-open, then the set

$$\{x \in G : U_n x \cap W \neq \varnothing\} = \bigcup_{u \in U_n} u^{-1} W$$

is t-open, and hence Borel, for every $n \in \omega$.

Note that, for $x, y \in G$, $x E_H^G y$ if and only if $U_\ell x \subseteq U_0 y$ for some $\ell \in \omega$. Indeed, if $x E_H^G y$ then $Hx = Hy$. Thus, $U_0 y x^{-1} \subseteq H$ is closed and nonmeager in the Polish topology of H, and hence there exists $\ell \in \omega$ such that $U_\ell \subseteq U_0 y x^{-1}$. Conversely, if there exists $\ell \in \omega$ such that $U_\ell x \subseteq U_0 y$ then $Hx \cap Hy \neq \varnothing$ and $x E_H^G y$.

This shows that E_H^G is potentially $\mathbf{\Sigma}_2^0$ and in particular potentially $D(\mathbf{\Pi}_2^0)$. It now follows from Proposition 3.1 that H is $D(\mathbf{\Pi}_2^0)$ in G. ∎

Lemma 6.6. *If λ is a limit ordinal and H is $\mathbf{\Sigma}_\lambda^0$ in G, then there exists $\mu < \lambda$ such that H is $\mathbf{\Pi}_\mu^0$ in G.*

Proof. Let α be the Solecki rank of H in G. If $\alpha < \lambda$ then H is $\mathbf{\Pi}_{1+\alpha+1}^0$ in G and hence the conclusion holds. Suppose that $\alpha \geq \lambda$. We know that $H = \bigcup_{k\in\omega} F_k$ where each F_k is $\mathbf{\Sigma}_{\mu_k}^0$ in G for some $\mu_k < \lambda$. As in the proof of Lemma 6.2, by Lemma 5.2 we can assume without loss of generality that each F_k is closed in H. By the Baire Category Theorem, we can assume that F_0 is nonmeager in H. Thus, $H = F_0^{\Delta H}$ is $\mathbf{\Sigma}_{\mu_0}^0$ and in particular $\mathbf{\Pi}_{\mu_0+1}^0$ in G. ∎

Lemma 6.7. *If H is $\mathbf{\Delta}^0_{1+\lambda+n+1}$ in G for some $1 \leq n < \omega$ and $\lambda < \omega_1$ either zero or limit, then H is $D(\mathbf{\Pi}^0_{1+\lambda+n})$ in G.*

Proof. Fix a countable open basis $\{V_n : n \in \omega\}$ for H. We have

$$H = s^H_{\lambda+n}(G) = s^H_1(s^H_{\lambda+n-1}(G)) \in \mathbf{\Pi}^0_3(s^H_{\lambda+n-1}(G)).$$

Furthermore, we can write $H = \bigcup_{k\in\omega} F_k$ where F_k is $\mathbf{\Pi}^0_{1+\lambda+n}$ in G for every $k \in \omega$. Thus,

$$H = \bigcup_{k,\ell\in\omega} F_k^{*V_\ell}$$

where $F_k^{*V_\ell} \in \mathbf{\Pi}^0_2(s^H_{\lambda+n-1}(G))$ by Lemma 5.2. Consequently, $H \in \mathbf{\Sigma}^0_3(s^H_{\lambda+n-1}(G))$. Hence, by Lemma 6.5, $H \in D(\mathbf{\Pi}^0_2)(s^H_{\lambda+n-1}(G))$. But $D(\mathbf{\Pi}^0_2)(s^H_{\lambda+n-1}(G))$ is contained in $D(\mathbf{\Pi}^0_{1+\lambda+n})(G)$ by Lemma 5.5, concluding the proof. ∎

Lemma 6.8. *If H is $\mathbf{\Sigma}^0_{1+\lambda+n+1}$ in G for some $1 \leq n < \omega$ and $\lambda < \omega_1$ either zero or limit, then H is $D(\mathbf{\Pi}^0_{1+\lambda+n})$ in G.*

Proof. Fix a countable open basis $\{V_n : n \in \omega\}$ for H. Let α be the Solecki rank of H in G. Since H is $\mathbf{\Pi}^0_{1+\lambda+n+2}$ we deduce that $\alpha \leq \lambda + n + 1$ by Theorem 5.4. If $\alpha \leq \lambda + n$ then H is $\mathbf{\Delta}^0_{1+\lambda+n+1}$ and hence H is $D(\mathbf{\Pi}^0_{1+\lambda+n})$ by Lemma 6.7. Suppose that $\alpha = \lambda + n + 1$. Thus, $H = s^H_1(s^H_{\lambda+n}(G))$. We can write $H = \bigcup_{k\in\omega} F_k$ where F_k is $\mathbf{\Pi}^0_{1+\lambda+n}$ in G for every $k \in \omega$. Hence, $H = \bigcup_{k,\ell\in\omega} F_k^{*V_\ell}$ where, by Lemma 5.2, $F_k^{*V_\ell}$ is $\mathbf{\Pi}^0_2$ in $s^H_{\lambda+n-1}(G)$. Thus, $H \in \mathbf{\Sigma}^0_3(s^H_{\lambda+n-1}(G))$. By Lemma 6.5, this implies that $H \in D(\mathbf{\Pi}^0_2)(s^H_{\lambda+n-1}(G))$. By Lemma 5.5, we know that $D(\mathbf{\Pi}^0_2)(s^H_{\lambda+n-1}(G)) \subseteq D(\mathbf{\Pi}^0_{1+\lambda+n})(G)$, concluding the proof. ∎

The following proposition is essentially proved in [8].

Proposition 6.9 (Hjorth–Kechris–Louveau). *Suppose that G is a Polish group, and H is a* non-Archimedean *Polishable subgroup of G. Suppose that $\lambda < \omega_1$ is either zero or limit. If H is $\mathbf{\Sigma}^0_{1+\lambda+2}$, then H is $\mathbf{\Sigma}^0_{1+\lambda+1}$.*

Proof. For λ a limit ordinal, this is a consequence of [8, Theorem 4.1] and Proposition 3.1. Suppose that $\lambda = 0$. If H is a non-Archimedean Polish group that is a Polishable $\mathbf{\Sigma}^0_3$ subgroup of G, then the proof of Lemma 6.4 shows that there exists an open subgroup U of H that is $\mathbf{\Pi}^0_2$ in G. Since a $\mathbf{\Pi}^0_2$ subgroup of a Polish group is closed, U is closed in G, and hence H is $\mathbf{\Sigma}^0_2$ in G. ∎

We have now all the ingredients to present a proof of Theorem 6.1.

Proof of Theorem 6.1. (1) We know that H is closed if and only if its Solecki rank is zero. Suppose now that λ is a limit ordinal and $n = 0$. By Theorem 5.4, H is $\mathbf{\Pi}^0_\lambda$ and not $\mathbf{\Pi}^0_\mu$ for $\mu < \lambda$. Thus, H is not $\mathbf{\Sigma}^0_\lambda$ by Lemma 6.6.

(2a) By Lemma 5.5, H is $\mathbf{\Pi}^0_{1+\lambda+n+1}$. It remains to prove that H is not $\mathbf{\Sigma}^0_{1+\lambda+n+1}$. Suppose that H is $\mathbf{\Sigma}^0_{1+\lambda+n+1}$. Then by Lemma 6.8, H is $D(\mathbf{\Pi}^0_{1+\lambda+n})$. Thus, by Corollary 5.3, $H \in D(\mathbf{\Pi}^0_2)(s^H_{\lambda+n-1}(G))$, contradicting the hypothesis.

(2b) By Lemma 5.5 we see that H is $D(\mathbf{\Pi}^0_{1+\lambda+n})$. It remains to prove that H is not $\check{D}(\mathbf{\Pi}^0_{1+\lambda+n})$. Suppose otherwise. Then by Lemma 3.2, H is either $\mathbf{\Pi}^0_{1+\lambda+n}$ or $\mathbf{\Sigma}^0_{1+\lambda+n}$. If H is $\mathbf{\Pi}^0_{1+\lambda+n}$ then by Theorem 5.4 and Proposition 3.4, $\lambda + n - 1$ is the Solecki rank of H in G, contradicting the hypothesis. If H is $\mathbf{\Sigma}^0_{1+\lambda+n}$, then $H \in \mathbf{\Sigma}^0_2(s^H_{\lambda+n-1}(G))$ by Corollary 5.3, contradicting the hypothesis.

(2c) By Lemma 5.5, H is $D(\mathbf{\Pi}^0_{1+\lambda+n})$. The same proof as for (2b) shows that H is not $\check{D}(\mathbf{\Pi}^0_{1+\lambda+n})$.

(2d) By Lemma 5.5, H is $\mathbf{\Sigma}^0_{1+\lambda+n}$. The same proof as for (2b) shows that H is not $\mathbf{\Pi}^0_{1+\lambda+n}$.

When H is non-Archimedean, case (2c) is excluded by Proposition 6.9. ■

7. The Saint Raymond rank

Saint Raymond [16, Définition 18] introduced a notion of rank (called *degree* therein) for Fréchetable subspaces of Fréchet spaces; see Section 9. In this section we consider a natural generalization of that notion to Polishable subgroups of Polish groups. Recall that for a complexity class Γ, and a Polishable subgroup H of a Polish group G, we denote by $\Gamma(G)|_H$ the collection of sets of the form $A \cap H$ for $A \in \Gamma(G)$.

Definition 7.1. Suppose that G is a Polish group, and $H \subseteq G$ is a Polishable subgroup. The *Saint Raymond rank* of H is the least countable ordinal α such that every open subset in the Polish group topology of H belongs to $\mathbf{\Sigma}^0_{1+\alpha}(G)|_H$.

Suppose that X, Y are Polish spaces, and α is a countable ordinal. As in [16, p. 216], one can define $\mathscr{B}_\alpha(X, Y)$ to be the set of Borel functions that have class α as in [10, Section 31], that is, are $\mathbf{\Sigma}^0_{1+\alpha}$-measurable; see [9, Definition 24.2]. By definition, the Saint Raymond rank of H is the least countable ordinal α such that the identity function of H belongs to $\mathscr{B}_\alpha(X, Y)$ where X is equal to H endowed with the subspace topology inherited from G, and Y is equal to H endowed with its Polish group topology. Adapting an argument of Tsankov [19], we now show that the Saint Raymond rank and the Solecki rank of a Polishable subgroup of G coincide.

Theorem 7.2. *Suppose that G is a Polish group, and $H \subseteq G$ is a Polishable subgroup. Then the Saint Raymond rank is equal to the Solecki rank.*

Proof. By Lemma 5.1, the Saint Raymond rank is less than or equal to the Solecki rank. We prove the reverse inequality as in [19, proof of Proposition 4.6]. If H has Saint Raymond rank α, then every open set in H belongs to $\mathbf{\Sigma}^0_{1+\alpha}(G)|_H$. Suppose that U is an open neighborhood of the identity in H, and let V be an open neighborhood of the identity in H such that $V^{-1}V \subseteq U$. Then there exists $A \in \mathbf{\Sigma}^0_{1+\alpha}(G)$ such that $A \cap H = V$. Thus, $1 \in A^{\Delta V}$, where $A^{\Delta V} \cap s^H_\alpha(G)$ is open in $s^H_\alpha(G)$ by Lemma 5.2. Furthermore, $A^{\Delta V} \cap H \subseteq V^{-1}V \subseteq U$. This shows that U contains a neighborhood of the identity with respect to the topology on H induced by $s^H_\alpha(G)$. This shows that the Polish topology on H is the subspace topology induced by $s^H_\alpha(G)$, whence H is closed in $s^H_\alpha(G)$. As H

is dense in $s_\alpha^H(G)$, we deduce that $H = s_\alpha^H(G)$. This shows that H has Solecki rank at most α. ∎

8. Polishable subgroups in each complexity class

The goal of this section is to prove the following theorem. Recall that a Polish group is *CLI* if it admits a compatible complete left-invariant metric, or equivalently its left uniformity is complete [13].

Theorem 8.1. *Let Γ be one of the possible complexity classes of Polishable subgroups from Theorem 1.1. Suppose that G is a nontrivial CLI Polish group. Then there exists a CLI Polishable subgroup of $G^{\mathbb{N}}$ whose complexity class is Γ.*

Remark 8.2. After replacing G with $G^{\mathbb{N}}$, we can assume that G is not discrete. We will assume that G is not discrete in the rest of this section.

Recall that a *pseudo-length function* on a group H is a function $L : H \to [0, +\infty)$ such that, for $h, h' \in H$,

- $L(1_H) = 0$;
- $L(h^{-1}) = L(h)$;
- $L(hh') \leq L(h) + L(h')$.

A *length function* is a pseudo-length function L such that $L(h) = 0 \Rightarrow h = 1_H$ for $h \in H$. A (pseudo-)length function L gives rise to a left-invariant (pseudo-)metric d defined by setting $d(h, h') = L(h^{-1}h')$, and every left-invariant metric arises in this fashion.

Suppose that G is a CLI Polish group, and let L_G be a length function on G that induces the Polish topology on G. We define the length functions L_1 and L_∞ on $G^{\mathbb{N}}$, with corresponding left-invariant metrics d_1 and d_∞, by setting

$$L_1((g_n)_{n\in\mathbb{N}}) := \sum_{n\in\mathbb{N}} L_G(g_n) \quad \text{and} \quad L_\infty((g_n)_{n\in\mathbb{N}}) := \sup_{n\in\mathbb{N}} L_G(g_n).$$

for a sequence $(g_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$. We say that $(g_n)_{n\in\mathbb{N}}$ is *L_G-summable* if

$$L_1((g_n)_{n\in\mathbb{N}}) < \infty,$$

and has *bounded (left) L_G-variation* if

$$\sum_{n\in\mathbb{N}} L_G(g_{n+1}^{-1} g_n) < \infty.$$

We let $\ell_1(G, L_G) \subseteq G^{\mathbb{N}}$ be the CLI Polishable subgroup of L_G-summable sequences, $\mathrm{bv}_0(G, L_G) \subseteq G^{\mathbb{N}}$ be the CLI Polishable subgroup of vanishing sequences of bounded L_G-variation, and $\mathrm{c}(G) \subseteq G^{\mathbb{N}}$ be the CLI Polishable subgroup of convergent sequences.

Fix, for each limit ordinal $\lambda < \omega_1$, an increasing cofinal sequence $(\lambda_i)_{i\in\mathbb{N}}$ in λ. If $\gamma = \delta + 1$ is a successor ordinal, define $\gamma_i = \delta$ for every $i \in \mathbb{N}$. By recursion on $\alpha < \omega_1$, define $I_0^0 = \{(\varnothing, \varnothing)\}$ where $\varnothing$ is the empty tuple, and I_0^α to be the set of tuples $(n_0, \dots, n_d; \beta_0, \dots, \beta_d)$ for $d \in \omega$, $n_0, \dots, n_d \in \mathbb{N}$, $0 = \beta_0 < \cdots < \beta_d = \alpha_{n_d}$, and $(n_0, \dots, n_{d-1}; \beta_0, \dots, \beta_{d-1}) \in I_0^{\alpha_{n_d}}$.

Similarly for a fixed $\gamma < \omega_1$ we define I_γ^α by recursion on $\alpha \geq \gamma$, by setting $I_\gamma^\gamma = \{(\varnothing, \varnothing)\}$, and I_γ^α to be the set of tuples $(n_0, \dots, n_d; \beta_0, \dots, \beta_d)$ for $d \in \omega$, $n_0, \dots, n_d \in \mathbb{N}$, $\gamma = \beta_0 < \cdots < \beta_d = \alpha_{n_d}$, and $(n_0, \dots, n_{d-1}; \beta_0, \dots, \beta_{d-1}) \in I_\gamma^{\alpha_{n_d}}$. Notice that, by definition, if $(n_0, \dots, n_d; \beta_0, \dots, \beta_d) \in I_\gamma^\alpha$ for some $\gamma \geq 1$, then $(m, n_0, \dots, n_d; \gamma_m, \beta_0, \dots, \beta_d) \in I_{\gamma_m}^\alpha$ for every $m \in \mathbb{N}$.

Thus, for example, for $1 \leq k < \omega$, $I_\gamma^{\gamma+k}$ is the set of tuples

$$(n_0, \dots, n_{k-1}; \gamma, \gamma + 1, \dots, \gamma + k - 1)$$

for $n_0, \dots, n_{k-1} \in \mathbb{N}$, and $I_\gamma^{\gamma+\omega}$ is the set of tuples

$$(n_0, \dots, n_d; \gamma, \gamma + 1, \dots, (\gamma + \omega)_{n_d})$$

for $d \in \omega$ such that $(\gamma + \omega)_{n_d} \geq \gamma$, and $n_0, \dots, n_d \in \mathbb{N}$.

We also define $I_\alpha^\alpha = \{(\varnothing; \varnothing)\}$. We denote by I^α the union of I_γ^α for $\gamma \leq \alpha$. If $\gamma \leq \alpha$, we denote by $I_{\leq\gamma}^\alpha$ the union of I_δ^α for $\delta \leq \gamma$, and by $I_{<\gamma}^\alpha$ the union of I_δ^α for $\delta < \gamma$. For $(n; \beta)$ and $(m; \tau)$ in I^α we define $(n; \beta) \leq (m; \tau)$ if and only if there exist $\gamma_0 \leq \gamma_1 \leq \alpha$ such that $(n; \beta) \in I_{\gamma_0}^\alpha$, $(m; \tau) \in I_{\gamma_1}^\alpha$, m is a tail of n, and τ is a tail of β, i.e., for some $\ell \leq d < \omega$, $(n; \beta) = (n_0, \dots, n_d; \beta_0, \dots, \beta_d)$, $(m; \tau) = (m_0, \dots, m_\ell; \tau_0, \dots, \tau_\ell)$, and for $0 \leq i \leq \ell$, $m_i = n_{i+d-\ell}$ and $\tau_i = \beta_{i+d-\ell}$. We regard I^α as an ordered set with this order relation. Observe that $I_{\leq\gamma}^\alpha$ and $I_{<\gamma}^\alpha$ are downward-closed. For a subset F of I^α, we denote by $F_\downarrow$ its downward closure. Notice also that if $F \subseteq I^\alpha$ is finite, and $(n; \beta) \in I_\gamma^\alpha$ for some $\gamma \geq 1$ is such that $(k, n; \gamma_k, \beta) \in F_\downarrow$ for infinitely many $k \in \mathbb{N}$, then $(n; \beta) \in F_\downarrow$.

Fix a countable ordinal α. We will define, by recursion on $\gamma < \alpha$, a decreasing sequence $(P_\gamma)_{\gamma<\alpha}$ of CLI Polishable subgroups of $G^{I_0^\alpha}$. Furthermore, for $x \in P_\gamma$, we will define the values $x(n; \beta) \in G$ for $(n; \beta) \in I_\gamma^\alpha$. If $\gamma \geq 1$ and $(n; \beta) \in I_\gamma^\alpha$, then we let $\boldsymbol{x}(n; \beta)$ be the convergent sequence $(x(k, n; \gamma_k, \beta))_{k\in\omega}$ in G with limit $x(n; \beta)$. If $(n; \beta) \in I_0^\alpha$, then we let $\boldsymbol{x}(n; \beta)$ be the sequence constantly equal to $x(n; \beta)$.

Define P_0 to be $G^{I_0^\alpha}$. This is a CLI Polish group with topology induced by the pseudo-length functions

$$L_0^{(n;\beta)}(x) = L_G(x(n; \beta))$$

for $(n; \beta) \in I_0^\alpha$. Suppose that $1 \leq \gamma \leq \alpha$, and that P_δ has been defined for all $\delta < \gamma$. Define $P_{<\gamma} = \bigcap_{\delta<\gamma} P_\delta$, and let $P_\gamma \subseteq P_{<\gamma}$ consist of those $x \in P_{<\gamma}$ such that, for every $(n; \beta) \in I_\gamma^\alpha$, the sequence $\boldsymbol{x}(n; \beta) := (x(i, n; \gamma_i, \beta))_{i\in\mathbb{N}}$ is convergent. For $x \in P_\gamma$ and $(n; \beta) \in I_\gamma^\alpha$, we define $x(n; \beta)$ to be the limit of $\boldsymbol{x}(n; \beta)$. Then the Polish topology on P_γ is induced by the restriction to P_γ of the continuous pseudo-length functions on P_δ for $\delta < \gamma$, together with the pseudo-length functions $L_\gamma^{(n;\beta)}(x) = L_\infty(\boldsymbol{x}(n; \beta))$ for $(n; \beta) \in I_\gamma^\alpha$. This concludes the recursive definition of the CLI Polishable subgroups P_γ

of $G^{I_0^\alpha}$ for $\gamma \leq \alpha$. Notice that in particular P_α consists of the elements $x \in P_{<\alpha}$ such that the sequence $\boldsymbol{x}(\alpha) := (x(n;\alpha_n))_{n\in\mathbb{N}}$ belongs to $\mathrm{c}(G)$. We also define S_α and D_α to be the subgroups of P_α consisting of the elements $x \in P_{<\alpha}$ such that the sequence $\boldsymbol{x}(\alpha)$ belongs to $\ell_1(G, L_G)$ and $\mathrm{bv}_0(G, L_G)$, respectively. Theorem 8.1 will be a consequence of the following.

Theorem 8.3. *Fix $\alpha = 1 + \lambda + n < \omega_1$ where λ is a limit ordinal or zero and $n < \omega$.*

(1) *If $n = 0$ and λ is limit, then $P_{<\lambda}$ has Solecki rank λ in $G^{I_0^\alpha}$, and complexity class $\boldsymbol{\Pi}^0_\lambda$.*
(2) *If $n = 0$, then $S_{1+\lambda}$, $D_{1+\lambda}$, and $P_{1+\lambda}$ have Solecki rank $\lambda + 1$ in $G^{I_0^\alpha}$, and complexity class $\boldsymbol{\Sigma}^0_{1+\lambda+1}$, $D(\boldsymbol{\Pi}^0_{1+\lambda+1})$, and $\boldsymbol{\Pi}^0_{1+\lambda+2}$ respectively.*
(3) *If $n \geq 1$, then $S_{1+\lambda+n}$, $D_{1+\lambda+n}$, and $P_{1+\lambda+n}$ have Solecki rank $\lambda + n + 1$ in $G^{I_0^\alpha}$, and complexity class $D(\boldsymbol{\Pi}^0_{1+\lambda+n+1})$, $D(\boldsymbol{\Pi}^0_{1+\lambda+n+1})$, and $\boldsymbol{\Pi}^0_{1+\lambda+n+2}$ respectively.*

We will obtain Theorem 8.3 as a consequence of a number of lemmas.

Lemma 8.4. *Suppose that $\gamma < \alpha$, F is a finite subset of $I^\alpha_{\leq\gamma}$, and $x \in P_\gamma$. Define $y \in G^{I_0^\alpha}$ by setting, for $(n;\beta) \in I_0^\alpha$,*

$$y(n;\beta) := \begin{cases} x(n;\beta) & \text{if } (n;\beta) \in F_\downarrow, \\ 1_G & \text{otherwise.} \end{cases} \tag{8.1}$$

Then $y \in S_\alpha$, and (8.1) holds for every $(n;\beta) \in I^\alpha$.

Proof. We prove by induction on $\sigma < \alpha$ that $y \in P_\sigma$, and that (8.1) holds for every $(n;\beta) \in I^\alpha_\sigma$. For $\sigma = 0$, this holds by definition. Suppose that the conclusion holds for every $\delta < \sigma$. Fix $(n;\beta) \in I^\alpha_\sigma$. If $(n;\beta) \in F_\downarrow$ then necessarily $\sigma \leq \gamma$, and for every $k \in \mathbb{N}$, $(k, n; \sigma_k, \beta) \in I^\alpha_{\sigma_k} \cap F_\downarrow$ and hence by the inductive hypothesis, we have $y(k, n; \sigma_k, \beta) = x(k, n; \sigma_k, \beta)$. Since $x \in P_\gamma \subseteq P_{\sigma_k}$, the sequence $\boldsymbol{y}(n;\beta) = \boldsymbol{x}(n;\beta)$ converges to $x(n;\beta)$. Thus, $y(n;\beta) = x(n;\beta)$. If $(n;\beta) \notin F_\downarrow$ then there exists k_0 such that, for all $k \geq k_0$, $(k, n; \sigma_k, \beta) \notin F_\downarrow$. Therefore, $\boldsymbol{y}(n;\beta)$ is eventually equal to 1_G, and thus $y(n;\beta) = 1_G$. This shows that $y \in P_\sigma$. This concludes the proof by induction.

By the above, we know that $y \in P_{<\alpha}$. For $k \in \mathbb{N}$ such that $\alpha_k > \gamma$ we have $y(k, \alpha_k) = 1_G$ and hence $y \in S_\alpha$. ∎

Lemma 8.5. *For every $\gamma < \alpha$, S_α is dense in P_γ.*

Proof. Suppose that $x \in P_\gamma$, and let V be a neighborhood of x in P_γ. Then there exist $\varepsilon > 0$ and a finite subset F of $I^\alpha_{\leq\gamma}$ such that

$$\bigcap_{(n;\beta)\in F} \{z \in P_\gamma : d_\infty(\boldsymbol{x}(n;\beta), \boldsymbol{z}(n;\beta)) < \varepsilon\} \subseteq V.$$

Define $z \in G^{I_0^\alpha}$ by setting, for $(n;\beta) \in I_0^\alpha$,

$$z(n;\beta) = \begin{cases} x(n;\beta) & \text{if } (n;\beta) \in F_\downarrow, \\ 1_G & \text{otherwise.} \end{cases} \tag{8.2}$$

Then by Lemma 8.4, we have $z \in S_\alpha$ and (8.2) holds for all $(n;\beta) \in I^\alpha$. In particular, $z \in V$. ■

Lemma 8.6. *For every $\gamma < \alpha$ and every open neighborhood V of the identity in S_α, $\overline{V}^{P_{<\gamma}} \cap P_\gamma$ contains an open neighborhood of the identity in P_γ.*

Proof. Let V be a neighborhood of the identity in S_α. Fix a finite subset F of I^α and $\varepsilon > 0$ such that

$$\{w \in S_\alpha : L_1(\boldsymbol{w}(\alpha)) < \varepsilon\} \cap \bigcap_{(n;\beta)\in F} \{w \in S_\alpha : L_\infty(\boldsymbol{w}(n;\beta)) < \varepsilon\} \subseteq V.$$

Define

$$N = \max_{\substack{\gamma<\delta\leq\alpha \\ F\cap I^\alpha_\delta \neq \varnothing}} \max\{n \in \mathbb{N} : \delta_n \leq \gamma\}$$

Define also the finite subset

$$B = (F \cap I^\alpha_{\leq\gamma}) \cup \{(k, n; \delta_k, \beta) : \gamma < \delta \leq \alpha, k \leq N, (n;\beta) \in F \cap I^\alpha_\delta\}$$

of $I^\alpha_{\leq\gamma}$. Consider the open neighborhood W of the identity in P_γ defined by

$$W = \left\{x \in P_\gamma : \sum_{n\leq N} L_G(x(n;\alpha_n)) < \varepsilon\right\} \cap \bigcap_{(n;\beta)\in B} \{x \in P_\gamma : L_\infty(\boldsymbol{x}(n;\beta)) < \varepsilon\}.$$

We claim that $W \subseteq \overline{V}^{P_{<\gamma}} \cap P_\gamma$. Suppose that $x \in W$. Let U be an open neighborhood of x in $P_{<\gamma}$. Then there exist a finite subset A of $I^\alpha_{<\gamma}$ containing $B \cap I^\alpha_{<\gamma}$ and $\varepsilon_1 > 0$ such that

$$\bigcap_{(n;\beta)\in A} \{z \in P_{<\gamma} : d_\infty(\boldsymbol{x}(n;\beta), \boldsymbol{z}(n;\beta)) < \varepsilon_1\} \subseteq U.$$

We need to prove that $U \cap V \neq \varnothing$.

We define $z \in G^{I^\alpha_0}$ by setting, for $(n;\beta) \in I^\alpha_0$,

$$z(n;\beta) := \begin{cases} x(n;\beta) & \text{if } (n;\beta) \in A_\downarrow, \\ 1_G & \text{otherwise.} \end{cases} \tag{8.3}$$

Then by Lemma 8.4, we have $z \in S_\alpha$ and (8.3) holds for all $(n;\beta) \in I^\alpha$. In particular, $z \in U$. We now show that $z \in V$, i.e., $L_1(\boldsymbol{z}(\alpha)) < \varepsilon$ and, for every $(n;\beta) \in F$, $L_\infty(\boldsymbol{z}(n;\beta)) < \varepsilon$. We have

$$L_1(\boldsymbol{z}(\alpha)) = \sum_{n\in\mathbb{N}} L_G(z(n;\alpha_n)) \leq \sum_{n\leq N} L_G(x(n;\alpha_n)) < \varepsilon.$$

If $(n;\beta) \in F \cap I^\alpha_{<\gamma}$, then $\boldsymbol{z}(n;\beta) = \boldsymbol{x}(n;\beta)$. As $(n;\beta) \in B$ and $x \in W$, this implies that $L_\infty(\boldsymbol{z}(n;\beta)) = L_\infty(\boldsymbol{x}(n;\beta)) < \varepsilon$. If $(n;\beta) \in F \cap I^\alpha_\gamma$, then

$$\begin{aligned} L_\infty(\boldsymbol{z}(n;\beta)) &= \sup_{k\in\mathbb{N}} L_G(z(k, n; \gamma_k, \beta)) \\ &\leq \sup_{k\in\mathbb{N}} L_G(x(k, n; \gamma_k, \beta)) = L_\infty(\boldsymbol{x}(n;\beta)) < \varepsilon \end{aligned}$$

since $(n;\beta) \in B$ and $x \in W$. If $(n;\beta) \in F \cap I_\delta^\alpha$ for some $\delta > \gamma$, then

$$\begin{aligned} L_\infty(\boldsymbol{z}(n;\beta)) &= \sup_{k \in \mathbb{N}} L_G(z(k,n;\delta_k,\beta)) \\ &\leq \max_{k \leq N} L_G(x(k,n;\delta_k,\beta)) \leq \max_{k \leq N} L_\infty(\boldsymbol{x}(k,n;\delta_k,\beta)) < \varepsilon \end{aligned}$$

since $(k,n;\delta_k,\beta) \in B$ for $k \leq N$, and $x \in W$. This shows that $z \in V$, concluding the proof. ∎

Proposition 8.7. *For $\gamma < \alpha$ we have*

$$s_\gamma^{S_\alpha}(G^{I_0^\alpha}) = s_\gamma^{D_\alpha}(G^{I_0^\alpha}) = s_\gamma^{P_\alpha}(G^{I_0^\alpha}) = s_\gamma^{P_{<\alpha}}(G^{I_0^\alpha}) = P_{<(1+\gamma)}.$$

Proof. Since $S_\alpha \subseteq D_\alpha \subseteq P_\alpha \subseteq P_{<(1+\gamma)}$, it suffices to prove that $s_\gamma^{S_\alpha}(G^{I_0^\alpha}) = P_{<(1+\gamma)}$. We use induction on $\gamma < \alpha$. For $\gamma = 0$, S_α is dense in $G^{I_0^\alpha} = P_0$ by Lemma 8.5, and hence $s_0^{S_\alpha}(G^{I_0^\alpha}) = P_0$. Suppose that the conclusion holds for every $\delta < \gamma$. If γ is limit, then

$$s_\gamma^{S_\alpha}(G^{I_0^\alpha}) = \bigcap_{\delta<\gamma} s_{\delta+1}^{S_\alpha}(G^{I_0^\alpha}) = \bigcap_{\delta<\gamma} P_{1+\delta} = P_{<\gamma} = P_{<(1+\gamma)}.$$

Suppose that $\gamma = \delta + 1$. Then, by the inductive hypothesis,

$$s_\gamma^{S_\alpha}(G^{I_0^\alpha}) = s_{\delta+1}^{S_\alpha}(G^{I_0^\alpha}) = s_1^{S_\alpha}(s_\delta^{S_\alpha}(G^{I_0^\alpha})) = s_1^{S_\alpha}(P_{<(1+\delta)}).$$

Thus, it remains to prove that

$$s_1^{S_\alpha}(P_{<(1+\delta)}) = P_{1+\delta}.$$

Notice that $P_{1+\delta}$ is a $\boldsymbol{\Pi}_3^0$ subgroup of $P_{<(1+\delta)}$. Thus, the conclusion follows from Lemma 4.2, in view of Lemmas 8.5 and 8.6. ∎

Lemma 8.8. *For every $\gamma \leq \alpha$, there is a continuous group homomorphism $\Phi : G^{I_{<\gamma}^\alpha} \to P_{<\gamma}$ such that*

$$\Phi(z)(k,n;\gamma_k,\beta) = z(k,n;\gamma_k,\beta) \quad \text{for every } z \in G^{I_{<\gamma}^\alpha},\ (n;\beta) \in I_\gamma^\alpha, \text{ and } k \in \mathbb{N}.$$

Proof. For $z \in G^{I_{<\gamma}^\alpha}$, define $\Phi(z) := x \in G^{I_0^\alpha}$ by setting, for $(m;\tau) \in I_0^\alpha$,

$$x(m;\tau) := \begin{cases} z(k,n;\gamma_k,\beta) & \text{if } (m,\tau) \leq (k,n;\gamma_k,\beta) \text{ for some } k \in \mathbb{N} \text{ and } (n;\beta) \in I_\gamma^\alpha, \\ 1_G & \text{otherwise.} \end{cases} \tag{8.4}$$

One can prove by induction on $\delta < \gamma$ that $x \in P_\delta$, and (8.4) holds for all $(m;\tau) \in I_\delta^\alpha$ Thus, $\Phi : G^{I_{<\gamma}^\alpha} \to P_{<\gamma}$ is a well-defined group homomorphism. Since $P_{<\gamma}$ is a Polishable subgroup of $G^{I_0^\alpha}$, its Borel structure is the one inherited from $G^{I_0^\alpha}$. This easily implies that Φ is Borel, and hence continuous. ∎

Lemma 8.9. (1) *$\boldsymbol{\Sigma}_2^0$ is the complexity class of $\ell_1(G,L_G)$ in $G^{\mathbb{N}}$.*

(2) *$D(\boldsymbol{\Pi}_2^0)$ is the complexity class of $\mathrm{bv}_0(G,L_G)$ in $G^{\mathbb{N}}$.*

(3) *$\boldsymbol{\Pi}_3^0$ is the complexity class of $\mathrm{c}(G)$ in $G^{\mathbb{N}}$.*

Proof. (1) It is clear that $\ell_1(G, L_G)$ is $\mathbf{\Sigma}^0_2$ in $G^{\mathbb{N}}$. Since $\ell_1(G, L_G)$ is a dense, proper subgroup of $G^{\mathbb{N}}$, it is not closed.

(2) We have $(g_n)_{n\in\mathbb{N}} \in \mathrm{bv}_0(G, L_G)$ if and only if

$$\sum_{n\in\mathbb{N}} L_G(g_{n+1}^{-1} g_n) < \infty$$

and for all $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ there exists $n \geq n_0$ such that $L_G(g_n) < \varepsilon$. This shows that $\mathrm{bv}_0(G, L_G)$ is $D(\mathbf{\Pi}^0_2)$ in $G^{\mathbb{N}}$. It remains to prove that it is not $\mathbf{\Sigma}^0_2$. Suppose for contradiction that

$$\mathrm{bv}_0(G, L_G) = \bigcup_{k\in\mathbb{N}} F_k,$$

where $F_k \subseteq G^{\mathbb{N}}$ is closed for every $k \in \mathbb{N}$. By the Baire Category Theorem, we can assume that F_0 contains a neighborhood of the identity. Thus, there exists $\varepsilon > 0$ such that

$$\left\{(g_n)_{n\in\mathbb{N}} \in \mathrm{bv}_0(G, L_G) : \sum_{n\in\mathbb{N}} L_G(g_{n+1}^{-1} g_n) < \varepsilon \text{ and } \sup_{n\in\mathbb{N}} L_G(g_n) < \varepsilon\right\} \subseteq F_0.$$

Since we are assuming that G is not discrete—see Remark 8.2—there exists $g \in G$ such that $0 < L_G(g) < \varepsilon$. For $N \in \mathbb{N}$, define $x^{(N)} \in \mathrm{bv}_0(G, L_G)$ by setting

$$x_k^{(N)} = \begin{cases} g & \text{if } k \leq N, \\ 1_G & \text{otherwise.} \end{cases}$$

Then $x^{(N)} \in F_0$ for every $N \in \mathbb{N}$. The sequence $(x^{(N)})_{N\in\mathbb{N}}$ converges in $G^{\mathbb{N}}$ to the sequence $x \in G^{\mathbb{N}}$ constantly equal to g. Since F_0 is closed in $G^{\mathbb{N}}$, we find that $x \in \mathrm{bv}_0(G, L_G)$, which contradicts $L_G(g) > 0$.

(3) By definition, $\mathrm{c}(G)$ is $\mathbf{\Pi}^0_3$ in $G^{\mathbb{N}}$. By Theorem 3.3, it suffices to prove that $\mathrm{c}(G)$ is not potentially $\mathbf{\Sigma}^0_2$. Let E_0 be the relation of tail equivalence in $2^{\mathbb{N}}$, and let $E_0^{\mathbb{N}}$ be the corresponding product equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}} = 2^{\mathbb{N}\times\mathbb{N}}$. Then $\mathbf{\Pi}^0_3$ is the potential complexity class of $E_0^{\mathbb{N}}$, for example by Lemma 5.7 and Theorem 3.3.

Thus, it suffices to define a Borel function $2^{\mathbb{N}\times\mathbb{N}} \to G^{\mathbb{N}}$ that is a Borel reduction from $E_0^{\mathbb{N}}$ to the coset relation of $\mathrm{c}(G)$ inside $G^{\mathbb{N}}$. We argue as in [6, Lemma 8.5.3]. Fix a bijection $\langle\cdot,\cdot\rangle : \mathbb{N}\times\mathbb{N} \to \omega$ such that if $n \leq n'$ and $m \leq m'$, then $\langle n, m\rangle \leq \langle n', m'\rangle$. Fix also a sequence $(g_n)_{n\in\mathbb{N}}$ in G such that $0 < L_G(g_n) < 2^{-(n+1)}$ for every $n \in \mathbb{N}$. Define $\Xi : 2^{\mathbb{N}\times\mathbb{N}} \to G^{\omega}$, $\varphi \mapsto a$, by setting

$$a_{\langle n,m\rangle} = \begin{cases} g_n, & \varphi(n, m) = 1, \\ 1_G, & \varphi(n, m) = 0. \end{cases}$$

Fix $\varphi, \psi \in 2^{\mathbb{N}\times\mathbb{N}}$. Define $\Xi(\varphi) = a$ and $\Xi(\psi) = b$.

Suppose that $\varphi E_0^{\mathbb{N}} \psi$. Thus for every $n \in \mathbb{N}$ there exists $M_n \in \mathbb{N}$ such that $\varphi(n, m) = \psi(n, m)$ for $m \geq M_n$. Fix $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon$. Define then

$$M = \max\{M_n : n < N\}.$$

We claim that for $k \geq \langle N, M\rangle$ we have $L_G(a_k^{-1}b_k) < \varepsilon$. Indeed, suppose $k \geq \langle N, M\rangle$. Then $k = \langle n, m\rangle$ for some $n, m \in \mathbb{N}$. If $n \geq N$ then

$$L_G(a_k^{-1}b_k) \leq L_G(a_k) + L_G(b_k) \leq 2L_G(g_n) < 2^{-n} \leq 2^{-N} < \varepsilon.$$

If $n < N$ then we must have $m \geq M \geq M_n$, and hence $\varphi(n, m) = \psi(n, m)$ and $a_k = b_k$. This shows that $a^{-1}b \in \mathrm{c}(G)$.

Conversely, suppose that $a^{-1}b \in \mathrm{c}(G)$. Fix $n_0 \in \mathbb{N}$. Then there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$, $L_G(a_k^{-1}b_k) < L_G(g_{n_0})$. Thus, for $k \geq k_0$, if $k = \langle n_0, m\rangle$ for some $m \in \mathbb{N}$, we must have $a_k = b_k$ and $\varphi(n_0, m) = \psi(n_0, m)$. Hence, if $m_0 \in \mathbb{N}$ is such that $\langle n_0, m_0\rangle \geq k_0$, we must have $\varphi(n_0, m) = \psi(n_0, m)$ for all $m \geq m_0$. As this holds for every $n_0 \in \mathbb{N}$, we have $\varphi E_0^{\mathbb{N}} \psi$, concluding the proof. ■

Corollary 8.10. *For every $\gamma < \alpha$, P_γ is a proper subgroup of $P_{<\gamma}$. The complexity class of S_α, D_α, and P_α inside $P_{<\alpha}$ is $\boldsymbol{\Sigma}_2^0$, $D(\boldsymbol{\Pi}_2^0)$, and $\boldsymbol{\Pi}_3^0$, respectively.*

Proof. Fix $\gamma < \alpha$ and $(n; \beta) \in I_\gamma^\alpha$. By Lemma 8.8 there exists $x \in P_{<\gamma}$ such that $\boldsymbol{x}(n; \beta)$ is not convergent. Such an x does not belong to P_γ, so P_γ is a proper subgroup of $P_{<\gamma}$.

We now prove the assertion about P_α; the other assertions are proved in a similar fashion. Recall that $I_\alpha^\alpha = \{(\varnothing; \varnothing)\}$. Define

$$H = \{x \in G^{I^\alpha} : (x(k; \alpha_k))_{k\in\mathbb{N}} \in \mathrm{c}(G)\}.$$

By Lemma 8.9, $\boldsymbol{\Pi}_3^0$ is the complexity class of H in G^{I^α} and of $\mathrm{c}(G)$ in $G^{\mathbb{N}}$.

By Lemma 8.8 there exists a continuous group homomorphism $\Phi : G^{I^\alpha} \to P_{<\alpha}$ such that $\Phi(x)(k; \alpha_k) = x(k; \alpha_k)$ for every $k \in \mathbb{N}$, and hence $\Phi^{-1}(P_\alpha) = H$. Similarly, the function $\Psi : P_{<\alpha} \to G^{\mathbb{N}}$, $x \mapsto \boldsymbol{x}(\alpha)$, is a continuous group homomorphism such that $\Psi^{-1}(\mathrm{c}(G)) = P_\alpha$. Thus, $\boldsymbol{\Pi}_3^0$ is the complexity class of P_α in $P_{<\alpha}$. ■

Proof of Theorem 8.3. By the first assertion in Corollary 8.10 and Proposition 8.7, we know that S_α, D_α, P_α all have Solecki rank $\alpha + 1$ in P_0, and $P_{<\alpha}$ has Solecki rank α in P_0 if α is limit. The conclusion now follows by applying Theorem 6.1 and the second assertion in Corollary 8.10. ■

Recall that a (*pseudo-*)*ultralength function* on a group H is a (pseudo-)length function L such that $L(hh') \leq \max\{L(h), L(h')\}$ for $h, h' \in H$. A Polish group G is non-Archimedean if and only if it admits a compatible ultralength function [6, Theorem 2.4.1]. Much as above, one can prove the following statement; see also [8, Section 5].

Theorem 8.11. *Let Γ be one of the possible complexity classes of non-Archimedean Polishable subgroups from Theorem 1.2. Suppose that G is a countable discrete group. Then there exists a non-Archimedean CLI Polishable subgroup of $G^{\mathbb{N}}$ whose complexity class is Γ.*

Define $H := G^{\mathbb{N}}$. This is a non-Archimedean CLI group. The topology on H is induced by the ultralength function

$$L_H((g_n)_{n\in\mathbb{N}}) = \exp(-\min\{n \in \mathbb{N} : g_n \neq 1_G\}).$$

Notice that the subgroup $\mathrm{c}(H)$ of $H^{\mathbb{N}}$ of convergent sequences is a non-Archimedean CLI Polishable subgroup of $H^{\mathbb{N}}$ of complexity class $\mathbf{\Pi}^0_3$, with topology induced by the ultralength function

$$L_\infty((h_n)_{n\in\mathbb{N}}) = \max\{L_H(h_n) : n \in \mathbb{N}\}.$$

The subgroup $\sigma(H)$ of $H^{\mathbb{N}}$ consisting of the sequences $(h_n)_{n\in\mathbb{N}}$ such that the sequence $(h_n(0))_{n\in\mathbb{N}}$ in G is eventually equal to 1_G is a non-Archimedean CLI Polishable subgroup of $H^{\mathbb{N}}$ of complexity class $\mathbf{\Sigma}^0_2$.

Fix $\alpha < \omega_1$. We will define by recursion on $\gamma \le \alpha$ a decreasing sequence $(F_\gamma)_{\gamma<\alpha}$ of non-Archimedean Polishable subgroups of $H^{I_0^\alpha}$. We will also recursively define, for $x \in F_\gamma$ and $(n;\beta) \in I_0^\beta$, the values $x(n;\beta) \in H$. We set $F_0 = H^{I_0^\alpha}$. If F_δ has been defined for every $\delta < \gamma$, define $F_{<\gamma} = \bigcap_{\delta<\gamma} F_\delta$. Let F_γ contain those $x \in F_{<\gamma}$ such that, for every $(n;\beta) \in I_\gamma^\alpha$, the sequence $\boldsymbol{x}(n;\beta) := (x(k,n;\gamma_k,\beta))_{k\in\omega}$ is convergent in H. For $x \in F_\gamma$ and $(n;\beta) \in I_\gamma^\alpha$, we define $x(n;\beta)$ to be the limit of $\boldsymbol{x}(n;\beta)$. Then the non-Archimedean Polish group topology on F_γ is induced by the restriction of the continuous pseudo-ultralength functions on F_δ for $\delta < \gamma$ together with the pseudo-ultralength function

$$L_\gamma^{(n;\beta)}(x) = L_\infty(\boldsymbol{x}(n;\beta))$$

for $(n;\beta) \in I_\gamma^\alpha$. This concludes the recursive definition of the non-Archimedean Polishable subgroups F_γ of $H^{I_0^\alpha}$ for $\gamma \le \alpha$. Notice that in particular F_α contains all elements $x \in F_{<\alpha}$ such that $\boldsymbol{x}(\alpha) := (x(n;\alpha_n))_{n\in\mathbb{N}}$ belongs to $c(H)$. Define Z_α to consist of those elements $x \in F_{<\alpha}$ such that $\boldsymbol{x}(\alpha)$ belongs to $\sigma(H)$. The same argument as above gives the following.

Theorem 8.12. *Adopt the notations above. Suppose that $\alpha = 1 + \lambda + n$ where $\lambda < \omega_1$ is either limit or zero and $n < \omega$.*

(1) *If $n = 0$ and λ is limit, then $F_{<\lambda}$ has Solecki rank λ in $H^{I_0^\lambda}$, and complexity class $\mathbf{\Pi}^0_\lambda$.*
(2) *If $n = 0$, then $Z_{1+\lambda}$ and $F_{1+\lambda}$ have Solecki rank $\lambda + 1$ in $H^{I_0^\alpha}$, and complexity class $\mathbf{\Sigma}^0_{1+\lambda+1}$ and $\mathbf{\Pi}^0_{1+\lambda+2}$, respectively.*
(3) *If $n \ge 1$, then $Z_{1+\lambda}$ and $F_{1+\lambda}$ have Solecki rank $\lambda + n + 1$ in $H^{I_0^\alpha}$, and complexity class $D(\mathbf{\Pi}^0_{1+\lambda+n+1})$ and $\mathbf{\Pi}^0_{1+\lambda+n+2}$, respectively.*

9. Fréchetable subspaces

In this section and the following one, we assume all the vector spaces to be over the reals. Similar considerations apply to complex vector spaces. Recall that a *Fréchet space* is a locally convex topological vector space whose topology is induced by a complete, translation-invariant metric. Thus, the additive group of a separable Fréchet space is a Polish group. In analogy with the notion of Polishable subgroup of a Polish group, we consider the notion of Fréchetable subspace of a separable Fréchet space.

Definition 9.1. Suppose that X is a separable Fréchet space, and Y is a subspace of X. Then we say that Y is *Fréchetable* if there exists a separable Fréchet space topology on Y whose open sets are Borel in X.

This notion was considered by Saint Raymond [16]: a subspace Y of X is Fréchetable if and only if *it has a separable model* according to [16, Définition 1]. Notice that a Fréchetable subspace of X is, in particular, a Polishable subgroup of the additive group of X. Thus, if it exists, the separable Fréchet space topology on Y as in Definition 9.1 is unique; see also [14, Corollary 4.38]. A subspace Y of a separable Fréchet space X is Fréchetable if and only if there exists a separable Fréchet space Z and a continuous linear map $\varphi : Z \to X$ with image equal to Y [16, Proposition 4]. If Y is a Fréchetable subspace of X, then the separable Fréchet space topology on Y is the *finest* locally convex topological vector space topology on Y that makes all the Borel linear functionals on Y continuous [16, Théorème 9]. Furthermore, a subspace Y of X is Fréchetable if and only if it is a Polishable subgroup of the additive group of X, and the Polish topology on Y has a basis of zero neighborhoods consisting of absolutely convex sets [14, Proposition 3.33, Corollary 3.36].

Lemma 9.2. *Suppose that X is a separable Fréchet space, and Y a Fréchetable subspace of X. The first Solecki subgroup $s_1^Y(X)$ of X relative to Y, where X is regarded as an additive Polish group and Y as a Polishable subgroup of X, is a Fréchetable subspace of X.*

Proof. By definition, if $x \in X$, then $x \in Y$ if and only if for every open neighborhood V of zero in Y there exists $z \in Y$ such that $x + z \in \overline{V}^G$. If $x \in Y$, $\lambda \in \mathbb{R}$ is nonzero, and V is an open neighborhood of zero in Y, then there exists $z \in Y$ such that $x + z \in \overline{\lambda^{-1}V}^G$, whence $\lambda x + \lambda z \in \overline{V}^G$. This shows that $\lambda x \in s_1^Y(X)$, so $s_1^Y(X)$ is a subspace of X.

We now show that $s_1^Y(X)$ is Fréchetable. Since Y is a separable Fréchet space, by the remarks above it has a basis $(V_n)_{n\in\omega}$ of zero neighborhoods consisting of absolutely convex sets. Then $(\overline{V}_n^G \cap s_1^Y(X))_{n\in\omega}$ is a basis of zero neighborhoods in $s_1^Y(X)$ consisting of absolutely convex sets. Thus, $s_1^Y(X)$ is a Fréchetable subspace of X by the remarks above again. ∎

As an immediate consequence of Lemma 9.2 and Theorem 5.4 by induction on $\alpha < \omega_1$ we have the following.

Theorem 9.3. *Suppose that X is a separable Fréchet space, Y is a Fréchetable subspace of X, and $\alpha < \omega_1$. Then the α-th Solecki subgroup $s_\alpha^Y(X)$ of X relative to Y, where X and Y are regarded as additive groups, is the smallest $\mathbf{\Pi}^0_{1+\alpha+1}$ Fréchetable subspace of X containing Y.*

A similar proof to that of Theorem 8.1 gives the following.

Theorem 9.4. *Let Γ be one of the possible complexity classes of Polishable subgroups from Theorem 1.1. Suppose that X is a nontrivial separable Fréchet space. Then there exists a Fréchetable subspace of $X^{\mathbb{N}}$ whose complexity class is Γ.*

10. Banachable subspaces

Let V be a separable Fréchet space. A subspace $X \subseteq V$ is *Banachable* if it is the image of a continuous linear map $T : Z \to V$ for some separable Banach space Z. Equivalently, X admits a Banach space topology that makes the inclusion map $X \to V$ continuous. The Solecki subgroups whose index is a successor ordinal associated with a Banachable subspace of a separable Fréchet space are also Banachable. Recall that a subset A of a topological vector space Z is *bounded* if for every zero neighborhood U in Z there exists $t > 0$ such that $A \subseteq tU$. A Fréchet space is a Banach space if and only if it has a bounded zero neighborhood.

Proposition 10.1. *Suppose that V is a separable Fréchet space, and $X \subseteq V$ is Banachable. Then $s^X_{\alpha+1}(V) \subseteq V$ is Banachable for every $\alpha < \omega_1$.*

Proof. It suffices to consider the case $\alpha = 0$. Let B be a bounded zero neighborhood in X. Define $C := \overline{B}^V \cap s^X_1(V)$. As B in bounded in X, $(2^{-n}B)_{n\in\omega}$ is a basis of zero neighborhoods in X. Thus, $(2^{-n}C)_{n\in\omega}$ is a basis of zero neighborhoods in $s^X_1(V)$. Hence, C is a bounded zero neighborhood in $s^X_1(V)$, so $s^X_1(V)$ is a Banach space. ∎

In this section, using the methods from Section 8 and Theorem 8.3 we will prove the following characterization of the possible complexity classes of Banachable subspaces.

Theorem 10.2. *The following is a complete list of all the possible complexity classes of proper Banachable subspaces of separable Fréchet spaces: $\mathbf{\Pi}^0_1$, $\mathbf{\Pi}^0_{1+\lambda+n+1}$, $D(\mathbf{\Pi}^0_{1+\lambda+n})$, and $\mathbf{\Sigma}^0_{1+\lambda+1}$ for $\lambda < \omega_1$ either zero or limit and $1 \leq n < \omega$. Furthermore, for every complexity class Γ in this list and any nontrivial separable Banach space Z, there exists a Banachable subspace of* $\mathrm{c}_0(\mathbb{N}, Z)$ *that has complexity class Γ.*

We begin by showing that a Banachable subspace of a separable Fréchet space cannot have complexity class $\mathbf{\Pi}^0_\lambda$ for a countable limit ordinal λ.

Proposition 10.3. *Suppose that V is a separable Fréchet space, and $X \subseteq V$ is a Fréchetable subspace. Suppose that $s^X_\alpha(V)$ is Banachable for some limit ordinal α. Then X has Solecki rank less than α.*

Proof. Without loss of generality, we can assume that $X = s^X_\alpha(V)$. Since X is Banachable, there exists $B \subseteq X$ such that $(2^{-n}B)_{n\in\omega}$ forms a basis of neighborhoods of zero in X. Since $X = \bigcap_{\beta<\alpha} s^X_\beta(V)$, there exists $\beta < \alpha$ and a neighborhood C of 0 in $s^X_\beta(V)$ such that $B = C \cap s^X_\alpha(V)$. Thus, X is endowed with the subspace topology inherited from $s^X_\beta(V)$. Hence, X is closed in $s^X_\beta(V)$. Since X is also dense in $s^X_\beta(V)$, we see that $X = s^X_\beta(V)$, so X has Solecki rank at most β. ∎

Corollary 10.4. *Suppose that V is a separable Fréchet space, and $X \subseteq V$ is a Banachable subspace. If X is $\mathbf{\Pi}^0_\lambda$ for some limit ordinal $\lambda < \omega_1$, then X is $\mathbf{\Pi}^0_\beta$ for some $\beta < \lambda$.*

Proof. By Theorem 6.1, $X = s^X_\lambda(V)$ is Banachable. Thus, by Proposition 10.3, X has Solecki rank β for some $\beta < \lambda$, and hence X is $\mathbf{\Pi}^0_{1+\beta+1}$ by Theorem 6.1 again. ∎

In order to conclude the proof of Theorem 10.2 it remains to prove that all the complexity classes from the statement of Theorem 10.2 can arise. Fix a countable ordinal α. We adopt the notation from Section 8. We regard I_γ^α as a set *fibered* over α, with respect to the map $I^\alpha \to \alpha$, $(n;\beta) \mapsto \gamma$, such that $(n;\beta) \in I_\gamma^\alpha$. For $\gamma < \alpha$ we define

$$J_\gamma^\alpha = \{(k,\sigma) \in \mathbb{N} \times (\alpha+1) : \sigma_k < \gamma < \sigma \leq \alpha\}.$$

We also regard J_γ^α as a set fibered over α with respect to the function $J_\gamma^\alpha \to \alpha$, $(k,\sigma) \mapsto \sigma$. We then define the *fibered product*

$$J_\gamma^\alpha * I^\alpha = \{((k,\sigma),(n;\beta)) : (k,\sigma) \in J_\gamma^\alpha,\ (n;\beta) \in I_\sigma^\alpha\}.$$

Remark 10.5. The projection map $J_\sigma^\alpha * I^\alpha \to I^\alpha$ is finite-to-one. Indeed, suppose that $((k,\sigma),(n;\beta)) \in J_\gamma^\alpha * I^\alpha$. Then $\gamma < \sigma$, and hence $\{k \in \mathbb{N} : \sigma_k < \gamma\}$ is finite.

Fix a nontrivial separable Banach space Z. We denote the norm of $z \in Z$ by $|z|$. We consider the Banach spaces

$$\ell_1(Z) = \Big\{(x_n) \in Z^{\mathbb{N}} : \sum_{n\in\mathbb{N}} |x_n| < \infty\Big\},$$
$$\mathrm{bv}_0(Z) = \Big\{(x_n) \in Z^{\mathbb{N}} : \sum_{n\in\mathbb{N}} |z_n - z_{n+1}| < \infty \text{ and } (z_n)_{n\in\mathbb{N}} \text{ is vanishing}\Big\}.$$

Set $X_0 := \mathrm{c}_0(I_0^\alpha, Z)$. We now define, by recursion on $\gamma \leq \alpha$, Banachable subspaces X_γ and Fréchetable subspaces $X_{<\gamma}$ of X_0 such that $X_\gamma \subseteq X_{<\gamma} \subseteq X_\delta$ for $\delta < \gamma \leq \alpha$. Furthermore, for $x \in X_\gamma$, we define the values $x(n;\beta) \in Z$ for $(n;\beta) \in I_\gamma^\alpha$, such that the linear functional $x \mapsto x(n;\beta)$ on X_γ is continuous. If $\gamma \geq 1$ and $(n;\beta) \in I_\gamma^\alpha$, then we let $\boldsymbol{x}(n;\beta)$ be the convergent sequence $(kx(k,n;\gamma_k,\beta))_{k\in\omega}$ with limit $x(n;\beta)$. If $(n;\beta) \in I_0^\alpha$, then we let $\boldsymbol{x}(n;\beta)$ be the sequence constantly equal to $x(n;\beta)$. Suppose that $1 \leq \gamma \leq \alpha$, and that X_δ has been defined for $\delta < \gamma$ in such a way that X_δ is a separable Banach space with norm $\|\cdot\|_{X_\delta}$.

Define $X_{<\gamma}$ to be the intersection of X_δ for $\delta < \gamma$. Define the continuous linear map

$$T_\gamma^0 : X_{<\gamma} \to (Z^{\mathbb{N}})^{I_{\leq\gamma}^\alpha} \quad \text{by} \quad T_\gamma^0(x) = (\boldsymbol{x}(n;\beta))_{(n;\beta)\in I_{\leq\gamma}^\alpha}.$$

Define also the continuous linear map

$$T_\gamma^1 : X_{<\gamma} \to Z^{J_\gamma^\alpha * I^\alpha} \quad \text{by} \quad T_\gamma^1(x) = (kx(k,n;\sigma_k,\beta))_{((k,\sigma),(n;\beta))\in J_\gamma^\alpha * I^\alpha}.$$

Define $X_\gamma \subseteq X_{<\gamma}$ to be the intersection of the preimage of

$$\mathrm{c}_0(I_{\leq\gamma}^\alpha, \mathrm{c}(\mathbb{N}, Z)) \subseteq (Z^{\mathbb{N}})^{I_\gamma^\alpha}$$

under T_γ^0 and the preimage of

$$\mathrm{c}_0(J_\gamma^\alpha * I^\alpha, Z) \subseteq Z^{J_\gamma^\alpha * I^\alpha}$$

under T_γ^1. It follows from Lemma 10.7 below that X_γ is a separable Banach space with respect to the norm

$$\|x\|_{X_\gamma} = \max\{\|T^0_\gamma(x)\|_{c_0(I^\alpha_{\leq\gamma}, c(\mathbb{N},Z))}, \|T^1_\gamma(x)\|_{c_0(J^\alpha_\gamma * I^\alpha, Z)}\}$$

for $x \in X_\gamma$. Observe that in particular

$$X_\alpha = \{x \in X_{<\alpha} : \boldsymbol{x}(\alpha) \in c(\mathbb{N}, Z)\}$$

and

$$\|x\|_{X_\alpha} = \max\left\{\sup_{\gamma<\alpha} \|x\|_{X_\gamma}, \|\boldsymbol{x}(\alpha)\|_{c(\mathbb{N},Z)}\right\}$$

for $x \in X_\alpha$, where $\boldsymbol{x}(\alpha) := (kx(k;\alpha_k))_{k\in\mathbb{N}}$. Define also $S_\alpha \subseteq D_\alpha \subseteq X_\alpha$ by setting

$$S_\alpha = \left\{x \in X_{<\alpha} : \sup_{\gamma<\alpha} \|x\|_{X_\gamma} < \infty \text{ and } \boldsymbol{x}(\alpha) \in \ell_1(Z)\right\},$$

$$D_\alpha = \left\{x \in X_{<\alpha} : \sup_{\gamma<\alpha} \|x\|_{X_\gamma} < \infty \text{ and } \boldsymbol{x}(\alpha) \in \mathrm{bv}_0(Z)\right\},$$

where

$$\boldsymbol{x}(\alpha) = (x(k;\alpha_k))_{k\in\mathbb{N}}.$$

Then S_α is a separable Banach space with respect to the norm

$$\|x\|_{S_\alpha} = \max\{\|x\|_{X_\alpha}, \|\boldsymbol{x}(\alpha)\|_{\ell_1(Z)}\}$$

and D_α is a separable Banach space with respect to the norm

$$\|x\|_{D_\alpha} = \max\{\|x\|_{X_\alpha}, \|\boldsymbol{x}(\alpha)\|_{\mathrm{bv}_0(Z)}\}.$$

The existence statement in Theorem 10.2 will be a consequence of the following result.

Theorem 10.6. *Fix $\alpha = 1 + \lambda + n < \omega_1$ where λ is a limit ordinal or zero and $n < \omega$.*

(1) *If $n = 0$ and λ is limit, then $X_{<\lambda}$ has Solecki rank λ in X_0, and complexity class $\boldsymbol{\Pi}^0_\lambda$.*
(2) *If $n = 0$, then $S_{1+\lambda}$, $D_{1+\lambda}$, and $X_{1+\lambda}$ have Solecki rank $\lambda + 1$ in X_0, and complexity class $\boldsymbol{\Sigma}^0_{1+\lambda+1}$, $D(\boldsymbol{\Pi}^0_{1+\lambda+1})$, and $\boldsymbol{\Pi}^0_{1+\lambda+2}$ respectively.*
(3) *If $n \geq 1$, then $S_{1+\lambda+n}$, $D_{1+\lambda+n}$, and $X_{1+\lambda+n}$ have Solecki rank $\lambda + n + 1$ in X_0, and complexity class $D(\boldsymbol{\Pi}^0_{1+\lambda+n+1})$, $D(\boldsymbol{\Pi}^0_{1+\lambda+n+1})$, and $\boldsymbol{\Pi}^0_{1+\lambda+n+2}$ respectively.*

The rest of this section is devoted to the proof of Theorem 10.6.

Lemma 10.7. *We have*

$$\|x\|_{X_\delta} \leq \|x\|_{X_\gamma} \quad \textit{for } \delta < \gamma \leq \alpha \textit{ and } x \in X_\gamma.$$

Proof. It suffices to prove that

$$\|T^1_\delta(x)\|_{c_0(J^\alpha_\delta * I^\alpha, Z)} \leq \|x\|_{X_\gamma}.$$

Suppose that $((k,\sigma),(n;\beta)) \in J^\alpha_\delta * I^\alpha$. Then $\sigma_k < \delta < \sigma$ and $(n;\beta) \in I^\alpha_\sigma$. Suppose initially that $\sigma \leq \gamma$. Then $(n;\beta) \in I^{\alpha}_{\leq\gamma}$ and hence

$$|kx(k,n;\sigma_k,\beta)| \leq \|\boldsymbol{x}(n;\beta)\|_{\infty} \leq \|x\|_{X_\gamma}.$$

Suppose now that $\gamma < \sigma$. Then $((k,\sigma),(n;\beta)) \in J^{\alpha}_{\gamma} * I^{\alpha}$ and hence

$$|kx(k,n;\sigma_k,\beta)| \leq \|T^1_\gamma(x)\|_{c_0(J^{\alpha}_{\gamma} * I^{\alpha},\mathbb{R})} \leq \|x\|_{X_\gamma}.$$

This concludes the proof. ∎

Lemma 10.8. *Fix $\gamma < \alpha$ and $x \in X_\gamma$. Let $F \subseteq I^{\alpha}_{\leq\gamma}$ be a finite set. Define $z \in Z^{I^{\alpha}_0}$ by setting, for $(n;\beta) \in I^{\alpha}_0$,*

$$z(n;\beta) = \begin{cases} x(n;\beta) & \text{if } (n;\beta) \in F_{\downarrow}, \\ 0 & \text{otherwise.} \end{cases} \tag{10.1}$$

Then $z \in S_\alpha$ and (10.1) holds for every $(n;\beta) \in I^{\alpha}$.

Proof. We prove by induction on $\sigma \leq \alpha$ that $z \in X_\sigma$ and (10.1) holds for every $(n;\beta) \in I^{\alpha}_{\sigma}$. Define

$$\tilde{F} = \{(m;\gamma) \in I^{\alpha} : \exists (n;\beta) \in F,\ (n;\beta) \leq (m;\gamma)\}.$$

Case $\sigma = 0$: (10.1) holds for $(n;\beta) \in I^{\alpha}_0$ by definition of z. As $x \in X_\gamma$, for every $\varepsilon > 0$ there exists a finite subset E of $I^{\alpha}_{\leq\gamma}$ such that $\|\boldsymbol{x}(n;\beta)\|_\infty < \varepsilon$ for $(n;\beta) \in I^{\alpha}_{\leq\gamma} \setminus E$. Thus, if $(n;\beta) \in I^{\alpha}_0 \setminus E$, then

$$\|\boldsymbol{z}(n;\beta)\|_\infty = |z(n;\beta)| \leq |x(n;\beta)| < \varepsilon.$$

This shows that $z \in X_0$.

Case $1 \leq \sigma \leq \gamma$: Fix $(n;\beta) \in I^{\alpha}_{\sigma}$. If $(n;\beta) \in F_{\downarrow}$ then $(k,n;\sigma_k,\beta) \in F_{\downarrow}$ for every $k \in \mathbb{N}$. Thus, by the inductive hypothesis,

$$kz(k,n;\sigma_k,\beta) = kx(k,n;\sigma_k,\beta) \quad \text{for every } k \in \mathbb{N},$$

and hence

$$\boldsymbol{z}(n;\beta) = \boldsymbol{x}(n;\beta).$$

Since by assumption $\boldsymbol{x}(n;\beta)$ is a convergent sequence with limit $x(n;\beta)$, it follows that $\boldsymbol{z}(n;\beta)$ is a convergent sequence with limit $z(n;\beta) = x(n;\beta)$. If $(n;\beta) \notin F_{\downarrow}$ then there exists $N \in \mathbb{N}$ such that for $k > N$, $(k,n;\sigma_k,\beta) \notin F_{\downarrow}$. By the inductive hypothesis, we know that $kz(k,n;\sigma_k,\beta) = 0$ for $k > N$. Thus, the sequence $\boldsymbol{z}(n;\beta)$ is eventually zero with limit $z(n;\beta) = 0$.

We now show that $z \in X_\sigma$. Fix $\varepsilon > 0$. Since $x \in X_\gamma$, there exist a finite subset $E \subseteq I^{\alpha}_{\leq\gamma}$ such that $\|\boldsymbol{x}(n;\beta)\|_\infty < \varepsilon$ for $(n;\beta) \in I^{\alpha}_{\leq\gamma} \setminus E$, and a finite subset $E' \subseteq J^{\alpha}_{\gamma} * I^{\alpha}$ such that $|kx(k,n;\tau_k,\beta)| < \varepsilon$ for $((k,\tau),(n;\beta)) \in (J^{\alpha}_{\gamma} * I^{\alpha}) \setminus E'$. Define

$$E'' = E' \cup \{((k,\tau),(n;\beta)) \in J^{\alpha}_{\sigma} * I^{\alpha} : (n;\beta) \in E\}$$

If $(n;\beta) \in I^{\alpha}_{\leq\sigma} \setminus E$ then

$$\|\boldsymbol{z}(n;\beta)\|_{\infty} \leq \|\boldsymbol{x}(n;\beta)\|_{\infty} \leq \varepsilon.$$

If $((k,\tau),(n;\beta)) \in (J^{\alpha}_{\sigma} * I^{\alpha}) \setminus E''$ then $\tau_k < \sigma < \tau$ and $(n;\beta) \in I^{\alpha}_{\tau}$. If $\tau \leq \gamma$ then $(n;\beta) \in I^{\alpha}_{\leq\gamma} \setminus E$ and hence

$$|kz(k,n;\tau_k,\beta)| \leq \|\boldsymbol{x}(n;\beta)\|_{\infty} \leq \varepsilon.$$

If $\gamma < \tau$ then $((k,\tau),(n;\beta)) \in (J^{\alpha}_{\gamma} * I^{\alpha}) \setminus E'$ and hence

$$|kz(k,n;\tau_k,\beta)| \leq |kx(k,n;\tau_k,\beta)| \leq \varepsilon.$$

Case $\sigma > \gamma$: Fix $(n;\beta) \in I^{\alpha}_{\sigma}$. Then by the inductive assumption,

$$z(k,n;\sigma_k,\beta) = 0$$

for $k > N$. Thus, the sequence $\boldsymbol{z}(n;\beta)$ is eventually zero, and $z(n;\beta) = 0$.

We now show that $z \in X_{\sigma}$. Fix $\varepsilon > 0$. Since $x \in X_{\gamma}$, there exist a finite set $E \subseteq I^{\alpha}_{\leq\gamma}$ such that

$$\|\boldsymbol{x}(n;\beta)\|_{\infty} < \varepsilon \quad \text{for } (n;\beta) \in I^{\alpha}_{\leq\gamma} \setminus E,$$

and a finite set $E' \subseteq J^{\alpha}_{\gamma} * I^{\alpha}$ such that

$$|kx(k,n;\tau_k,\beta)| < \varepsilon \quad \text{for } ((k,\tau),(n;\beta)) \in (J^{\alpha}_{\gamma} * I^{\alpha}) \setminus E'.$$

Define $\tilde{E} = E \cup \tilde{F}$ and

$$\tilde{E}' = \tilde{F} \cup E' \cup \{((k,\tau),(n;\beta)) \in J^{\alpha}_{\sigma} * I^{\alpha} : (n;\beta) \in \tilde{E}\},$$

which is finite by Remark 10.5. Fix $(n;\beta) \in I^{\alpha}_{\leq\sigma} \setminus \tilde{E}$. Fix $\delta \leq \sigma$ such that $(n;\beta) \in I^{\alpha}_{\delta}$. If $\delta \leq \gamma$, then $(n;\beta) \in I^{\alpha}_{\leq\gamma} \setminus E$ and hence

$$\|\boldsymbol{z}(n;\beta)\|_{\infty} \leq \|\boldsymbol{x}(n;\beta)\|_{\infty} \leq \varepsilon.$$

Suppose that $\delta > \gamma$, and fix $k \in \mathbb{N}$. If $z(k,n;\delta_k,\beta) \neq 0$ then $(k,n;\delta_k,\beta) \in F$, so $(n;\beta) \in \tilde{F} \subseteq \tilde{E}$, contradicting the hypothesis. Thus, $\boldsymbol{z}(n;\beta)$ is constantly equal to zero. Fix $((k,\tau),(n;\beta)) \in (J^{\alpha}_{\sigma} * I^{\alpha}) \setminus \tilde{E}'$. Thus, $\tau_k < \sigma < \tau$ and $(n;\beta) \in I^{\alpha}_{\tau}$. If $\tau_k < \gamma$, then $((k,\tau),(n;\beta)) \in (J^{\alpha}_{\gamma} * I^{\alpha}) \setminus E'$ and hence

$$|kz(k,n;\tau_k,\beta)| \leq |kx(k,n;\tau_k,\beta)| \leq \varepsilon.$$

Suppose that $\tau_k = \gamma$. If $z(k,n;\tau_k,\beta)$ is nonzero, then $(k,n;\tau_k,\beta) \in F$ and hence $(n;\beta) \in \tilde{F} \subseteq \tilde{E}$, contradicting the assumption that $((k,\tau),(n;\beta)) \notin \tilde{E}'$. If $\tau_k > \gamma$ then $(k,n;\tau_k,\beta) \notin F_{\downarrow}$ and hence $z(k,n;\tau_k,\beta) = 0$. This concludes the inductive proof.

Finally, to see that $z \in S_{\alpha}$ observe that if $N = \max\{k \in \mathbb{N} : \alpha_k \leq \gamma\}$ then

$$\sum_{k\in\mathbb{N}} |z(k;\alpha_k)| \leq \sum_{k\leq N} |x(k;\alpha_k)| < \infty. \qquad \blacksquare$$

The next lemma is similar to the previous one, with the difference that the finite set F is supposed to be a subset of $I^\alpha_{<\gamma}$ instead of $I^\alpha_{\leq\gamma}$.

Lemma 10.9. *Fix $\gamma < \alpha$ and $x \in X_\gamma$. Let $F \subseteq I^\alpha_{<\gamma}$ be a finite set. Define $z \in Z^{I^\alpha_0}$ by setting, for $(n; \beta) \in I^\alpha_0$,*

$$z(n;\beta) = \begin{cases} x(n;\beta) & \text{if } (n;\beta) \in F_\downarrow, \\ 0 & \text{otherwise.} \end{cases} \tag{10.2}$$

Then $z \in S_\alpha$, $\|z\|_{X_\alpha} \leq \|x\|_{X_\gamma}$, and (10.2) holds for every $(n;\beta) \in I^\alpha$. Furthermore,

$$\|z\|_{S_\alpha} \leq \max\Big\{\sum_{k\leq N} |x(k;\alpha_k)|, \|x\|_{X_\gamma}\Big\},$$

where $N = \max\{k \in \mathbb{N} : \alpha_k < \gamma\}$.

Proof. It follows from Lemma 10.8 that $z \in S_\alpha$ and (10.2) holds for every $(n;\beta) \in I^\alpha_\sigma$. We now prove by induction on $\sigma \leq \alpha$ that $\|z\|_{X_\sigma} \leq \|x\|_{X_\gamma}$. Suppose that the conclusion holds for every $\delta < \sigma$.

Case $\sigma = 0$: If $(n;\beta) \in I^\alpha_0$, then $|z(n;\beta)| \leq |x(n;\beta)|$. This shows that $\|z\|_{X_0} \leq \|x\|_{X_0} \leq \|x\|_{X_\gamma}$.

Case $1 \leq \sigma \leq \gamma$: For $(n;\beta) \in I^\alpha_{\leq\sigma}$, we have

$$\|\boldsymbol{z}(n;\beta)\|_\infty \leq \|\boldsymbol{x}(n;\beta)\|_\infty \leq \|x\|_{X_\gamma}.$$

Fix $((k,\tau),(n;\beta)) \in J^\alpha_\sigma * I^\alpha$. Thus, $\tau_k < \sigma < \tau$ and $(n;\beta) \in I^\alpha_\tau$. If $\tau \leq \gamma$, then $(n;\beta) \in I^\alpha_{\leq\gamma}$ and hence

$$|kz(k,n;\tau_k,\beta)| \leq |kx(k,n;\tau_k,\beta)| \leq \|\boldsymbol{x}(n;\beta)\|_\infty \leq \|x\|_{X_\gamma}.$$

If $\gamma < \tau$, then $((k,\tau),(n;\beta)) \in J^\alpha_\gamma * I^\alpha$ and hence

$$|kz(k,n;\tau_k,\beta)| \leq |kx(k,n;\tau_k,\beta)| \leq \|x\|_{X_\gamma}.$$

Case $\sigma > \gamma$: Suppose that $(n;\beta) \in I^\alpha_\delta$ for some $\delta \leq \sigma$. If $\delta \leq \gamma$, then

$$\|\boldsymbol{z}(n;\beta)\|_\infty \leq \|\boldsymbol{x}(n;\beta)\|_\infty \leq \|x\|_{X_\gamma}.$$

If $\gamma < \delta$ then for every $k \in \mathbb{N}$, either $\delta_k < \gamma$, in which case $((k,\delta),(n;\beta)) \in J^\alpha_\gamma * I^\alpha$ and hence

$$|kz(k,n;\delta_k,\beta)| \leq |kx(k,n;\delta_k,\beta)| \leq \|x\|_{X_\gamma},$$

or $\gamma \leq \delta_k$, in which case $(k,n;\delta_k,\beta) \notin F_\downarrow$ and $z(k,n;\delta_k,\beta) = 0$. Thus

$$\|\boldsymbol{z}(n;\beta)\|_\infty \leq \|x\|_{X_\gamma}.$$

Suppose now that $((k,\tau),(n;\beta)) \in J^\alpha_\sigma * I^\alpha$. Thus $\tau_k < \sigma < \tau$. If $\tau_k < \gamma$ then

$((k,\tau),(n;\beta)) \in J^{\alpha}_{\gamma} * I^{\alpha}$ and hence

$$|kz(k,n;\tau_k,\beta)| \leq |kx(k,n;\tau_k,\beta)| \leq \|x\|_{X_\gamma}.$$

If $\gamma \leq \tau_k$ then $(k,n;\tau_k,\beta) \notin F_{\downarrow}$ and hence $z(k,n;\tau_k,\beta) = 0$. This concludes the inductive proof that $\|z\|_{X_\sigma} \leq \|x\|_{X_\gamma}$ for every $\sigma \leq \alpha$.

Finally,

$$\sum_{k\in\mathbb{N}} |z(k;\alpha_k)| \leq \sum_{k\leq N} |x(k;\alpha_k)|.$$

This shows that $z \in S_\alpha$ and

$$\|z\|_{S_\alpha} = \max\left\{\|z\|_{X_\alpha}, \sum_{k\in\mathbb{N}} |z(k;\alpha_k)|\right\} \leq \max\left\{\|x\|_{X_\alpha}, \sum_{k\leq N} |x(k;\alpha_k)|\right\},$$

concluding the proof. ∎

Lemma 10.10. *For every $\gamma < \alpha$, S_α is dense in X_γ.*

Proof. Suppose that $x \in X_\gamma$ and $\varepsilon > 0$. We need to prove that there exists $z \in S_\alpha$ such that $\|x - z\|_{X_\gamma} \leq \varepsilon$. As $x \in X_\gamma$, there exists a finite subset $E \subseteq I^{\alpha}_{\leq\gamma}$ such that $\|\boldsymbol{x}(n;\beta)\|_\infty < \varepsilon$ for $(n;\beta) \in I^{\alpha}_{\leq\gamma} \setminus E$, and a finite subset $E' \subseteq J^{\alpha}_{\gamma} * I^{\alpha}$ such that $|kx(k,n;\tau_k,\beta)| < \varepsilon$ for $((k,\tau),(n;\beta)) \in (J^{\alpha}_{\gamma} * I^{\alpha}) \setminus E'$.

Suppose that $z \in X_\alpha$ is obtained from $x \in X_\gamma$ and

$$F = (E \cap I^{\alpha}_{\leq\gamma}) \cup \{(k,n;\gamma_k,\beta) : ((k,\tau),(n;\beta)) \in E'\}$$

as in Lemma 10.8.

Fix $(n;\beta) \in I^{\alpha}_{\leq\gamma}$. If $(n;\beta) \in F_{\downarrow}$, then $\boldsymbol{z}(n;\beta) = \boldsymbol{x}(n;\beta)$, while if $(n;\beta) \notin F_{\downarrow}$, then

$$\|\boldsymbol{z}(n;\beta) - \boldsymbol{x}(n;\beta)\|_\infty \leq \|\boldsymbol{x}(n;\beta)\|_\infty \leq \varepsilon.$$

Consider $((k,\tau),(n;\beta)) \in J^{\alpha}_{\gamma} * I^{\alpha}$. Thus $\tau_k < \gamma < \tau$ and $(n;\beta) \in I^{\alpha}_{\tau}$. If $(k,n;\tau_k,\beta) \in F$, then $kz(k,n;\tau_k,\beta) = kx(k,n;\tau_k,\beta)$. If $(k,n;\tau_k,\beta) \notin F$, then $((k,\tau);(n;\beta)) \in (J^{\alpha}_{\gamma} * I^{\alpha}) \setminus E'$ and hence

$$|kz(k,n;\tau_k,\beta) - kx(k,n;\tau_k,\beta)| \leq |kx(k,n;\tau_k,\beta)| \leq \varepsilon.$$

This concludes the proof that $\|z - x\|_{X_\gamma} \leq \varepsilon$. ∎

Lemma 10.11. *Fix $\gamma < \alpha$. If V is a neighborhood of zero in S_α, then $\overline{V}^{X_{<\gamma}} \cap X_\gamma$ contains an open neighborhood of zero in X_γ.*

Proof. Define

$$N = \max\{k \in \mathbb{N} : \alpha_k \leq \gamma\}.$$

Suppose that V is a neighborhood of zero in S_α. Without loss of generality, we can assume that

$$V = \left\{z \in S_\alpha : \|z\|_{X_\alpha} \leq \varepsilon, \sum_{k\in\mathbb{N}} |z(k;\alpha_k)| \leq \varepsilon\right\}.$$

We claim that $\overline{V}^{X_{<\gamma}} \cap X_\gamma$ contains

$$W := \left\{ x \in X_\gamma : \|x\|_{X_\gamma} \leq \varepsilon, \sum_{k \leq N} |x(k;\alpha_k)| \leq \varepsilon \right\}.$$

Indeed, suppose that $x \in W$. Let U be an open neighborhood of x in $X_{<\gamma}$. Without loss of generality, we can assume that

$$U = \{z \in X_{<\gamma} : \|x - z\|_{X_\delta} \leq \varepsilon_1\}$$

for some $\delta < \gamma$ and $\varepsilon_1 > 0$. We need to prove that $U \cap V \neq \varnothing$.

Since $x \in X_\gamma$, there exists a finite subset E of X_γ such that $\|\boldsymbol{x}(n;\beta)\|_\infty \leq \varepsilon_1$ for $(n;\beta) \in I^\alpha_{\leq\gamma} \setminus E$, and a finite subset E' of $J^\alpha_\gamma * I^\alpha$ such that $|kx(k,n;\tau_k,\beta)| \leq \varepsilon_1$ for $((k,\tau),(n;\beta)) \in (J^\alpha_\gamma * I^\alpha) \setminus E'$. Define

$$E'' = E' \cup \{((k,\tau),(n;\beta)) \in J^\alpha_\gamma * I^\alpha : (n;\beta) \in E\}.$$

Let $z \in X_\alpha$ be obtained from x and

$$F := (E \cap I^\alpha_{\leq\delta}) \cup \{(k,n;\tau_k,\beta) : ((k,\tau),(n;\beta)) \in E''\} \cup \{(k;\alpha_k) : k \leq N\}$$

as in Lemma 10.9. Then $z \in S_\alpha$ and

$$\|z\|_{X_\alpha} \leq \|x\|_{X_\gamma} \leq \varepsilon, \quad \sum_{k \leq N} |z(k;\alpha_k)| \leq \sum_{k \leq N} |x(k;\alpha_k)| \leq \varepsilon,$$

and hence $z \in V$. It remains to prove that $\|z - x\|_{X_\delta} \leq \varepsilon_1$. For $(n;\beta) \in I^\alpha_{\leq\delta}$, if $(n;\beta) \in F$ then

$$\boldsymbol{z}(n;\beta) = \boldsymbol{x}(n;\beta),$$

while if $(n;\beta) \notin F$ then $(n;\beta) \in I^\alpha_{\leq\gamma} \setminus E$ and we have

$$\|\boldsymbol{z}(n;\beta) - \boldsymbol{x}(n;\beta)\|_\infty \leq \|\boldsymbol{x}(n;\beta)\|_\infty \leq \varepsilon_1$$

by the choice of E.

For $((k,\tau),(n;\beta)) \in J^\alpha_\delta * I^\alpha$, we have $\tau_k < \delta < \tau$ and $(n;\beta) \in I^\alpha_\tau$. Suppose that $\gamma < \tau$, in which case $((k,\tau),(n;\beta)) \in J^\alpha_\gamma * I^\alpha$. If $((k,\tau),(n;\beta)) \in F$, then

$$kz(k,n;\tau_k,\beta) = kx(k,n;\tau_k,\beta);$$

if $((k,\tau),(n;\beta)) \notin F$, then $((k,\tau),(n;\beta)) \in (J^\alpha_\gamma * I^\alpha) \setminus E'$ and hence

$$|kz(k,n;\tau_k,\beta) - kx(k,n;\tau_k,\beta)| = |kx(k,n;\tau_k,\beta)| \leq \varepsilon_1.$$

Suppose now that $\tau \leq \gamma$, in which case $\tau_k < \delta < \tau \leq \gamma$. If $((k,\tau),(n;\beta)) \in F$, then

$$kz(k,n;\tau_k,\beta) = kx(k,n;\tau_k,\beta);$$

while if $((k,\tau),(n;\beta)) \notin F$, then $(n;\beta) \in I^\alpha_{\leq\gamma} \setminus E$ and hence

$$|kz(k,n;\tau_k,\beta) - kx(k,n;\tau_k,\beta)| \leq |kx(k,n;\tau_k,\beta)| \leq \|\boldsymbol{x}(n;\beta)\|_\infty \leq \varepsilon_1.$$

This concludes the proof that $\|z - x\|_{X_\delta} \leq \varepsilon_1$. ∎

Using Lemmas 10.11 and 10.10, one can prove Proposition 10.12, similarly to the way Proposition 8.7 was deduced from Lemmas 8.6 and 8.5.

Proposition 10.12. *For $\gamma < \alpha$ we have*

$$s_\gamma^{S_\alpha}(X_0) = s_\gamma^{D_\alpha}(X_0) = s_\gamma^{X_\alpha}(X_0) = s_\gamma^{X_{<\alpha}}(X_0) = X_{<(1+\gamma)}.$$

Recall that, for $(n;\beta)$ and $(m;\tau)$ in I^α, we define $(n;\beta) \le (m;\tau)$ if and only if there exist $\gamma_0 \le \gamma_1 \le \alpha$ such that $(n;\beta) \in I^\alpha_{\gamma_0}$, $(m;\tau) \in I^\alpha_{\gamma_1}$, m is a tail of n, and τ is a tail of β, i.e. for some $\ell \le d < \omega$, we have $(n;\beta) = (n_0,\dots,n_d;\beta_0,\dots,\beta_d)$, and $(m;\tau) = (n_{d-\ell},\dots,n_d;\beta_{d-\ell},\dots,\beta_d)$. In this case, we set

$$\pi^{(n;\beta)}_{(m;\tau)} := \frac{1}{n_0 \cdots n_{d-\ell-1}}.$$

Lemma 10.13. *Fix $\gamma \le \alpha$ and $(m;\tau) \in I^\alpha_\gamma$. There exists a continuous linear map Φ : $c_0(\mathbb{N}, Z) \to X_{<\gamma}$ such that $\Phi(t)(k,m;\gamma_k,\tau) = t_k$ for every $t \in c_0(\mathbb{N}, Z)$ and $k \in \mathbb{N}$.*

Proof. Fix $t \in c_0(\mathbb{N}, Z)$. For $\varepsilon > 0$, let $K_\varepsilon(t) \in \mathbb{N}$ be such that $|t_k| \le \varepsilon$ for $k > K_\varepsilon(t)$. Define $\Phi(t) := x \in Z^{I^\alpha_0}$ by setting, for $(n;\beta) \in I^\alpha_0$,

$$x(n;\beta) := \begin{cases} \pi^{(k,m;\gamma_k,\tau)}_{(n;\beta)} t_k & \text{if } (n;\beta) \le (k,m;\gamma_k,\tau) \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{10.3}$$

We now prove by induction on $\sigma < \gamma$ that $x \in X_\sigma$, and (10.3) holds for every $(n;\beta) \in I^\alpha$. Suppose that the conclusion holds for all $\delta < \sigma$.

Case $\sigma = 0$: We need to prove that $x \in c_0(I^\alpha_0, Z)$. Fix $\varepsilon > 0$. Consider

$$F = \{(n;\beta) \in I^\alpha_0 : (n;\beta) \le (k,m;\gamma_k,\tau) \text{ for some } k \le K_\varepsilon(t)\}.$$

Then $F \subseteq I^\alpha_0$ is finite and for $(n;\beta) \in I^\alpha_0 \setminus F$, either

$$x(n;\beta) = 0,$$

or $(n;\beta) \le (k,m;\gamma_k,\tau)$ for some $k > K_\varepsilon(t)$, in which case

$$|x(n;\beta)| = |\pi^{(k,m;\gamma_k,\tau)}_{(n;\beta)} t_k| \le |t_k| \le \varepsilon.$$

Case $1 \le \sigma < \gamma$: Fix $(n;\beta) \in I^\alpha_\delta$ for some $\delta \le \sigma$. If $(n;\beta) \le (k,m;\gamma_k,\tau)$ for some $k \in \mathbb{N}$, then $(\ell,n;\delta_\ell,\beta) \le (k,m;\gamma_k,\tau)$ for every $\ell \in \mathbb{N}$. Thus,

$$x(\ell,n;\delta_\ell,\beta) = \pi^{(k,m;\gamma_k,\tau)}_{(\ell,n;\delta_\ell,\beta)} t_k$$

and

$$\ell x(\ell,n;\delta_\ell,\beta) = \pi^{(k,m;\gamma_k,\tau)}_{(n;\beta)} t_k.$$

Hence, the sequence $\boldsymbol{x}(n;\beta)$ is constantly equal to $\pi^{(k,m;\gamma_k,\tau)}_{(n;\beta)} t_k$. This shows that

$$x(n;\beta) = \pi^{(k,m;\gamma_k,\tau)}_{(n;\beta)} t_k.$$

Suppose that there does not exist $k \in \mathbb{N}$ such that $(n;\beta) \leq (k,m;\gamma_k,\tau)$. Fix $\ell \in \mathbb{N}$. If $(\ell,n;\delta_\ell,\beta) \leq (k,m;\gamma_k,\tau)$ for some $k \in \mathbb{N}$, then (k,m) is a tail of (ℓ,n) and (γ_k,τ) is a tail of (δ_ℓ,β). If the length of (k,m) is strictly less than the length of (ℓ,n), then m is a tail of n and τ is a tail of β, and hence $(n;\beta) \leq (k,m;\gamma_k,\tau)$, contradicting the assumption. Therefore, $(\ell,n;\delta_\ell,\beta) = (k,m;\gamma_k,\tau)$. In particular, $(n;\beta) = (m;\tau) \in I_\gamma^\alpha$, contradicting the assumption that $(n;\beta) \in I_\delta^\alpha$ and $\delta \leq \sigma < \gamma$. Thus, the sequence $\boldsymbol{z}(n;\beta)$ is constantly zero, and hence $z(n;\beta) = 0$.

We now prove that $x \in X_\sigma$. Fix $\varepsilon > 0$. Define

$$N = \max\left\{K_\varepsilon(t), \max\{k \in \mathbb{N} : \gamma_k \leq \sigma\}\right\}.$$

Consider

$$E = \{(n;\beta) \in I_{\leq\sigma}^\alpha : (n;\beta) \leq (k,m;\gamma_k,\tau) \text{ for some } k \leq N\}.$$

If $(n;\beta) \in I_{\leq\sigma}^\alpha \setminus E$ and $x(n;\beta) \neq 0$ then, by the argument above, $(n;\beta) \leq (k,m;\gamma_k,\tau)$ for some $k > N \geq K_\varepsilon(t)$ and hence

$$|x(n;\beta)| \leq |t_k| \leq \varepsilon.$$

Consider the finite set

$$E' = \{((\ell,\rho);(n;\beta)) \in J_\sigma^\alpha * I^\alpha : (\ell,n;\rho_\ell,\beta) \in E\}.$$

If $(\ell,\rho;(n;\beta)) \in (J_\sigma^\alpha * I^\alpha) \setminus E'$ and $x(\ell,n;\rho_\ell,\beta) \neq 0$, then $\rho_\ell < \sigma < \rho$ and

$$(\ell,n;\rho_\ell,\beta) \leq (k,m;\gamma_k,\tau)$$

for some $k \in \mathbb{N}$. Since $(\ell,\rho;(n;\beta)) \notin E'$, we have $k > N$ and hence $\gamma_k > \sigma > \rho_\ell$ and

$$\pi_{(\ell,n;\rho_\ell,\beta)}^{(k,m;\gamma_k,\tau)} \leq 1/\ell.$$

Thus,

$$|\ell x(\ell,n;\rho_\ell,\beta)| \leq |\ell \pi_{(n;\beta)}^{(k,m;\gamma_k,\tau)} t_k| \leq |t_k| \leq \varepsilon$$

since $k > N \geq K_\varepsilon(t)$.

By the above, $\Phi : \mathrm{c}_0(\mathbb{N}, Z) \to X_{<\gamma}$ is a well-defined linear map. As $X_{<\gamma}$ is a Fréchetable subspace of $Z^{I_0^\alpha}$, its Borel structure is the one induced by $Z^{I_0^\alpha}$. This easily implies that Φ is Borel, and hence continuous. ∎

Lemma 10.14. *The continuous linear map $\tau : \mathrm{c}_0(\mathbb{N}, Z) \to Z^{\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}} \mapsto (nx_n)_{n\in\mathbb{N}}$, has the following properties:*

(1) $\boldsymbol{\Sigma}_2^0$ *is the complexity class of* $\tau^{-1}(\ell_1(Z))$ *in* $\mathrm{c}_0(\mathbb{N}, Z)$.

(2) $D(\boldsymbol{\Pi}_2^0)$ *is the complexity class of* $\tau^{-1}(\mathrm{bv}_0(Z))$ *in* $\mathrm{c}_0(Z)$.

(3) $\boldsymbol{\Pi}_3^0$ *is the complexity class of* $\tau^{-1}(\mathrm{c}_0(\mathbb{N}, Z))$ *and of* $\tau^{-1}(\mathrm{c}(\mathbb{N}, Z))$ *in* $\mathrm{c}_0(\mathbb{N}, Z)$.

Proof. (1) Since $\ell_1(Z)$ is $\boldsymbol{\Sigma}_2^0$ in $Z^{\mathbb{N}}$, $\tau^{-1}(\ell_1)$ is a $\boldsymbol{\Sigma}_2^0$ Polishable subgroup of $\mathrm{c}_0(\mathbb{N}, Z)$ that is not closed. Thus, $\boldsymbol{\Sigma}_2^0$ is the complexity class of $\tau^{-1}(\ell_1(Z))$.

(2) Since $\mathrm{bv}_0(Z)$ is $D(\mathbf{\Pi}^0_2)$ in $Z^{\mathbb{N}}$, and $\tau^{-1}(\mathrm{bv}_0(Z))$ is a Polishable subgroup of $\mathrm{c}_0(\mathbb{N}, Z)$, by Theorem 3.3 it suffices to prove that $\tau^{-1}(\mathrm{bv}_0(Z))$ is not $\mathbf{\Sigma}^0_2$ in $\mathrm{c}_0(\mathbb{N}, Z)$. Suppose for contradiction that $\tau^{-1}(\mathrm{bv}_0(Z)) = \bigcup_{k\in\omega} F_k$ where each $F_k \subseteq \mathrm{c}_0(\mathbb{N}, Z)$ is closed. Observe that a compatible norm on $\mathrm{bv}_0(Z)$ is given by

$$\|x\|_{\mathrm{bv}_0(Z)} = \sum_{n\in\mathbb{N}} |x_{n+1} - x_n| + \sup_{n\in\mathbb{N}} |x_n|.$$

By the Baire Category Theorem without loss of generality we can assume that

$$\{\boldsymbol{a} \in \tau^{-1}(\mathrm{bv}_0(Z)) : \|\tau((a_n))\|_{\mathrm{bv}_0} \le 2\} \subseteq F_0.$$

Define then $\boldsymbol{a}^{(N)} \in \tau^{-1}(\mathrm{bv}_0(Z))$ for $N \in \mathbb{N}$ by setting

$$a^{(N)}_n = \begin{cases} 1/n & \text{if } n \le N, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\|\tau(\boldsymbol{a}^{(N)})\|_{\mathrm{bv}_0(Z)} \le 2$ and $\boldsymbol{a}^{(N)} \in F_0$ for every $N \in \mathbb{N}$. Furthermore, the sequence $(\boldsymbol{a}^{(N)})_{N\in\mathbb{N}}$ converges in $\mathrm{c}_0(\mathbb{N}, Z)$ to the sequence $\boldsymbol{a}$ defined by $a_n = 1/n$ for every $n \in \mathbb{N}$. Since F_0 is closed in $\mathrm{c}_0(\mathbb{N}, Z)$, we must have $\boldsymbol{a} \in F_0 \subseteq \tau^{-1}(\mathrm{bv}_0(Z))$. However, $\tau(\boldsymbol{a})$ is not vanishing, and so $\tau(\boldsymbol{a}) \notin \mathrm{bv}_0(Z)$.

(3) Since $\mathrm{c}(\mathbb{N}, Z)$ is $\mathbf{\Pi}^0_3$ in $Z^{\mathbb{N}}$, and $\tau^{-1}(\mathrm{c}(\mathbb{N}, Z))$ is a Polishable subgroup of c_0, by Theorem 3.3 it suffices to prove that $\tau^{-1}(\mathrm{c}(\mathbb{N}, Z))$ is not $\mathbf{\Sigma}^0_2$. Let E_0 be the relation of tail equivalence in $2^{\mathbb{N}}$, and let $E_0^{\mathbb{N}}$ be the corresponding product equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}} = 2^{\mathbb{N}\times\mathbb{N}}$. Then $\mathbf{\Pi}^0_3$ is the potential complexity class of $E_0^{\mathbb{N}}$, for example by Lemma 5.7 and Theorem 3.3.

Thus, it suffices to define a Borel function $2^{\mathbb{N}\times\mathbb{N}} \to \mathrm{c}_0(\mathbb{N}, Z)$ that is a Borel reduction from $E_0^{\mathbb{N}}$ to the coset relation of $\tau^{-1}(\mathrm{c}(\mathbb{N}, Z))$ inside $\mathrm{c}_0(\mathbb{N}, Z)$. Fix a bijection $\langle\cdot,\cdot\rangle : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ such that if $n \le n'$ and $m \le m'$, then $\langle n, m\rangle \le \langle n', m'\rangle$. Define $2^{\mathbb{N}\times\mathbb{N}} \to Z^{\mathbb{N}}$, $x \mapsto a$, by setting $a_{\langle n,m\rangle} = \frac{1}{\langle n,m\rangle} 2^{-n} x_{n,m}$. Then the argument in [6, Lemma 8.5.3] shows that $x E_0^{\mathbb{N}} x'$ if and only if $\tau(\boldsymbol{a}) - \tau(\boldsymbol{a}') = \tau(\boldsymbol{a} - \boldsymbol{a}') \in \mathrm{c}(\mathbb{N}, Z)$, if and only if $\boldsymbol{a} - \boldsymbol{a}' \in \tau^{-1}(\mathrm{c}(\mathbb{N}, Z))$.

The same argument shows that $\mathbf{\Pi}^0_3$ is the complexity class of $\tau^{-1}(\mathrm{c}_0(\mathbb{N}, Z))$ in $\mathrm{c}_0(\mathbb{N}, Z)$. ■

The same proof as that of Corollary 8.10, with Lemma 10.14 replacing Lemma 8.9, gives the proof of Corollary 10.15 below.

Corollary 10.15. *For every $\gamma < \alpha$, X_γ is a proper subspace of $X_{<\gamma}$. The complexity class inside $X_{<\alpha}$ of S_α, D_α, and X_α is $\mathbf{\Sigma}^0_2$, $D(\mathbf{\Pi}^0_2)$, and $\mathbf{\Pi}^0_3$, respectively.*

Finally, Theorem 10.6 is proved using Corollary 10.15 and Proposition 10.12, much as Theorem 8.3 was deduced from Corollary 8.10 and Proposition 8.7.

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