



Algebra, Number Theory, Group Theory, and Algebraic Geometry. – *Airy sheaves of Laurent type: An introduction*, by NICHOLAS M. KATZ and PHAM HUU TIEN, accepted on 1 July 2025.

To Enrico Bombieri, with the utmost admiration.

ABSTRACT. – We develop the general theory of Airy sheaves of Laurent type, the local systems whose trace functions have a particular “Airy-Laurent” shape. The main goal is to provide tools for the later determination of their monodromy groups. See [Eur. J. Math. 10 (2024), article no. 65] for instances of such determinations.

KEYWORDS. – exponential sums, local systems, monodromy groups.

MATHEMATICS SUBJECT CLASSIFICATION 2020. – 11T23 (primary); 20C15, 20D06, 20G40, 22E46 (secondary).

1. INTRODUCTION

In classical analysis, Airy functions are the Fourier transforms of exponentials $e^{g(x)}$ of polynomials (originally for the polynomial $g(x) := x^3/3$), and Airy differential equations are the linear differential equations $g'(d/dt)y + ty = 0$ they satisfy. These differential equations have an irregular singularity at ∞ and have quite interesting differential Galois groups. In the seminal paper [11] of Such, he introduces their ℓ -adic finite field analogues, the local systems whose trace functions are of the form

$$t \mapsto - \sum_x \psi(g(x) + tx).$$

The local systems we are concerned with here are generalizations of these Airy local systems in several ways. We allow the “ t term” tx to be replaced by tx^a , we allow the polynomial $g(x)$ to be replaced by a Laurent polynomial $f(1/x) + g(x)$, and we allow an “outside factor” $\chi(x)$ in the sum. Using the general theory developed in this paper, the monodromy groups of some such local systems were determined in [7]. Here is a more detailed discussion.

We work in characteristic $p > 0$, and denote by $\overline{\mathbb{F}_p}$ an algebraic closure of \mathbb{F}_p . We also fix a prime $\ell \neq p$ to be able to speak of $\overline{\mathbb{Q}_\ell}$ -adic cohomology. We fix integers

$$A \geq 1, \quad B \geq 1, \quad a > B$$

about which we assume

$$p \nmid ABa.$$

We fix polynomials

$$\begin{aligned} f(x) &\in k[x], & \deg(f) &= A, & k &\text{ some finite subfield of } \overline{\mathbb{F}}_p, \\ g(x) &\in k[x], & \deg(g) &= B, & k &\text{ some finite subfield of } \overline{\mathbb{F}}_p. \end{aligned}$$

We make the assumption that both $f(x)$ and $g(x)$ are Artin–Schreier reduced: this means that in the expression $f(x) = \sum_i c_i x^i$, $g(x) = \sum_i d_i x^i$ we have $c_i = 0$, $d_i = 0$ if $p \mid i$. We define

$$\gcd_{\deg}(f) := \gcd(\{i \mid c_i \neq 0\}), \quad \gcd_{\deg}(g) := \gcd(\{i \mid d_i \neq 0\})$$

the greatest common divisor of the degrees of the monomials appearing in f , respectively, in g . We suppose

$$\gcd(a, \gcd_{\deg}(f)) = 1, \quad \gcd(a, \gcd_{\deg}(g)) = 1.$$

We also fix a (possibly trivial) multiplicative character χ of k^\times , with the convention that for $\chi \neq \mathbb{1}$, we have $\chi(0) = 0$, but $\mathbb{1}(0) = 1$. We denote by $\mathcal{G}(f, g, a, \chi)$ the lisse sheaf on \mathbb{G}_m/k whose trace function at time $t \in L^\times$, for L/k a finite extension, is

$$t \mapsto - \sum_{x \in L^\times} \psi_L(f(1/x) + g(x) + tx^a) \chi_L(x).$$

2. BASIC FACTS ABOUT $\mathcal{G}(f, g, a, \chi)$

The local system $\mathcal{G}(f, g, a, \chi)$ is lisse of rank $D = A + a$ on \mathbb{G}_m , and pure of weight one. We view it as being the Fourier transform

$$\mathrm{FT}_\psi([a]_\star(\mathcal{L}_{\psi(f(1/x)+g(x))} \otimes \mathcal{L}_\chi(x))).$$

LEMMA 2.1. *Given $A \geq 1$, $B \geq 1$, $a > B$, $p \nmid ABa$, f, g both Artin–Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = 1$, $\gcd(a, \gcd_{\deg}(g)) = 1$. Then the following statements hold for $\mathcal{G}(f, g, a, \chi)$.*

- (i) *The $I(\infty)$ -representation of $\mathcal{G}(f, g, a, \chi)$ is irreducible. It has rank $A + a$ and all slopes $A/(A + a)$.*
- (ii) *The $I(0)$ -representation of $\mathcal{G}(f, g, a, \chi)$ is the direct sum*

$$W(B, a - B) \oplus (\overline{\mathbb{Q}}_\ell)^{A+B},$$

with $W(B, a - B)$ an irreducible $I(0)$ -representation of rank $a - B$ with all slopes $B/(a - B)$.

PROOF. This is a straightforward application of Laumon's results on the local monodromy of FT_ψ . The input sheaf to FT_ψ is lisse on \mathbb{G}_m of rank a , with $I(0)$ -slopes A/a and $I(\infty)$ slopes B/a . The hypotheses $\gcd(a, \gcd_{\deg}(g)) = 1$, $\gcd(a, \gcd_{\deg}(f)) = 1$ imply, respectively, that the $I(0)$ - and $I(\infty)$ -representations of the input sheaf are irreducible, cf. [8, the proof of Lemma 2.1].

Then the $I(\infty)$ -representation of $\mathcal{G}(f, g, a, \chi)$ is $\text{FTloc}(0, \infty)(\text{rank } a, \text{slopes } A/a)$, which has rank $A + a$ and all slopes $A/(A + a)$, cf. [3, Theorem 7.4.4 (4)]. The $I(0)$ -representation of $\mathcal{G}(f, g, a, \chi)$, modulo its subspace of $I(0)$ -invariants, cf. [3, Corollary 7.4.3.1], is $\text{FTloc}(\infty, 0)(\text{rank } a, \text{slopes } B/a)$, which is the asserted $W(B, a - B)$. The asserted irreducibilities result from the irreducibilities of the input and the fact that $\text{FTloc}(0, \infty)$ and $\text{FTloc}(\infty, 0)$ are suitable equivalences of categories. ■

COROLLARY 2.2. *Hypotheses as in Lemma 2.1 suppose in addition that $a > 2B$. Then the determinant of $\mathcal{G}(f, g, a, \chi)$ is tame, so geometrically some Kummer sheaf \mathcal{L}_Λ . Moreover, if χ has odd order N , then Λ has order dividing $2N$, while if χ has even order N , then Λ has order dividing N .*

PROOF. The slopes of $\mathcal{G}(f, g, a, \chi)$ at ∞ are all < 1 , and the slopes at 0 are $B/(a - B) < 1$. Therefore, the determinant of $\mathcal{G}(f, g, a, \chi)$, a priori of finite order by Grothendieck's local monodromy theorem, is tame, hence geometrically some Kummer sheaf \mathcal{L}_Λ . Then the arithmetic determinant of $\mathcal{G}(f, g, a, \chi)$ is some constant field twist $\mathcal{L}_\Lambda \otimes \alpha^{\deg}$ of \mathcal{L}_Λ . Denote by M the order of Λ . Over any finite extension L/k containing μ_M , the trace of $\text{Frob}_{t,L}|\mathcal{L}_\Lambda$, as t runs over L^\times , attains all values in μ_M . Then we recover M as the ratios of these Frobenius traces at various points $s, t \in L^\times$. Thus, we also recover M as the same ratios of Frobenius trace on the constant field twist $\mathcal{L}_\Lambda \otimes \alpha^{\deg}$ of \mathcal{L}_Λ . Each $\text{Frob}_{t,L}|\mathcal{G}(f, g, a, \chi)$ and all its powers have traces in $\mathbb{Q}(\zeta_p, \zeta_N)$ for N the order of χ . So each Frobenius determinant lies in $\mathbb{Q}(\zeta_p, \zeta_N)$. Therefore, the geometric determinant takes values both in μ_M and in $\mathbb{Q}(\zeta_p, \zeta_N)$. But the only roots of unity in $\mathbb{Q}(\zeta_p, \zeta_N)$ lie in $\pm\mu_{pN}$. Thus, $\mu_M \leq \pm\mu_{pN}$. ■

LEMMA 2.3. *Given $A \geq 1$, $B \geq 1$, $a > B$, $p \nmid ABa$, f, g both Artin–Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = 1$, $\gcd(a, \gcd_{\deg}(g)) = 1$. Then $\mathcal{G}(f, g, a, \chi)$ is geometrically self-dual if and only if ABa is odd, both $f(x)$ and $g(x)$ are odd polynomials, and $\chi^2 = \mathbb{1}$.*

PROOF. The oddness conditions, and χ^2 trivial, imply autoduality. For $p = 2$, over even degree extensions k/\mathbb{F}_2 (coef's of f, g), after the constant field twist by $1/\sqrt{\#k}$, the traces are real (in fact in \mathbb{Q}). And when p is odd, after the constant field twist by $1/\text{Gauss}(\psi, \chi_2)$ and over even degree extensions of \mathbb{F}_p (coef's of f, g), the traces are real.

To prove the converse, we argue as follows. Since $\mathcal{G} := \mathcal{G}(f, g, a, \chi)$ is geometrically irreducible, it is self-dual if and only if $H_c^2(\mathbb{G}_m/\overline{\mathbb{F}}_p, \mathcal{G} \otimes \mathcal{G})$ is nonzero (and in fact has dimension 1). We compute this dimension as the limsup, over extensions k/\mathbb{F}_p (coef's of f, g), of the sums

$$\frac{1}{\#k(\#k - 1)} \sum_{t \in k^\times, x, y \in k} \psi_k(f(1/x) + f(1/y) + g(x) + g(y) + t(x^a + y^a)) \chi_k(xy).$$

The $t = 0$ “missing” summand is

$$\begin{aligned} \frac{1}{\#k(\#k - 1)} \sum_{x, y \in k} \psi_k(f(1/x) + f(1/y) + g(x) + g(y)) \chi_k(xy) \\ = \frac{1}{\#k(\#k - 1)} \left(\sum_{x \in k} \psi_k(f(1/x) + g(x)) \chi_k(x) \right)^2, \end{aligned}$$

which is $O(1/(\#k - 1))$, because the sum being squared is, by the Weil bound, of absolute value $\leq (A + B)\sqrt{\#k}$.

So the limsup does not change if we add this term. Then we have the limsup of

$$1/(\#k - 1) \sum_{x, y \in k, x^a + y^a = 0} \psi_k(f(1/x) + f(1/y) + g(x) + g(y)) \chi_k(xy).$$

This is then the limsup of the sum of the a sums, one for each ζ with $\zeta^a = -1$,

$$S_\zeta := 1/(\#k - 1) \sum_{x \in k} \psi_k(f(1/x) + f(1/\zeta x) + g(x) + g(\zeta x)) \chi_k(\zeta x^2).$$

Both $f(x)$ and $g(x)$ are Artin–Schreier reduced and

$$\gcd(a, \gcd_{\deg}(f)) = \gcd(a, \gcd_{\deg}(g)) = 1.$$

Then the two sums $f(x) + f(x/\zeta)$ and $g(x) + g(\zeta x)$ are each Artin–Schreier reduced. Unless both sums vanish, the S_ζ summand is $O(1/\sqrt{\#k})$, again by the Weil estimate. And even if both sums do vanish, then the sum still vanishes unless χ^2 is the trivial character. Thus, in all cases, we must have χ^2 trivial if we are to have self-duality.

Suppose now that χ^2 is trivial and \mathcal{G} is self-dual. Then for at least one ζ with $\zeta^a = -1$, both $f(x) + f(x/\zeta) = 0$ and $g(x) + g(\zeta x) = 0$. In the case $p = 2$, both f, g are odd polynomials (because both are Artin–Schreier reduced), so there is nothing to prove.

Thus, it remains to treat the case when p is odd. Suppose first that a is even. Then we claim that for any ζ with $\zeta^a = -1$, $f(x) + f(x/\zeta) \neq 0$. To see this, write $f(x) = \sum_n a_n x^n$, and define $\mathcal{E}_f := \{n \mid a_n \neq 0\}$, the set of exponents which occur in f . By hypothesis, we have

$$\gcd(a, \text{all } n \in \mathcal{E}_f) = 1.$$

We rewrite this as

$$\gcd(a, \text{all } n - a \text{ with } n \in \mathcal{E}_f) = 1.$$

If $f(x) + f(x/\zeta) = 0$, then $a_n(1 + 1/\zeta^n) = 0$ for all $n \in \mathcal{E}_f$, i.e., $\zeta^n = -1$ for all $n \in \mathcal{E}_f$, i.e., $\zeta^n = \zeta^a$ for all $n \in \mathcal{E}_f$, and finally $\zeta^{n-a} = 1$ for all $n \in \mathcal{E}_f$. Define

$$D := \gcd(\text{all } n - a \text{ with } n \in \mathcal{E}_f).$$

Then $\zeta^D = 1$. But $\gcd(a, D) = 1$, so there exist integers u, v with $au + Dv = 1$. It follows that $\zeta = (\zeta^a)^u (\zeta^D)^v = (-1)^u$. Thus, ζ is ± 1 , neither of which has $\zeta^a = -1$ if a is even.

Suppose next that a is odd. Then the above argument shows that ζ is ± 1 . But of these two choices, only $\zeta = -1$ has $\zeta^a = -1$. For this $\zeta = -1$, we have $f(x) + f(-x) = 0$, which means precisely that f is an odd polynomial. The same argument applied to g , using the fact that $\gcd(a, \text{all } n \in \mathcal{E}_g) = 1$, shows that $\zeta = -1$, hence that g is an odd polynomial. \blacksquare

LEMMA 2.4. *Given $A \geq 1, B \geq 1, a > B, p \nmid ABa$, f, g both Artin–Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = 1, \gcd(a, \gcd_{\deg}(g)) = 1$. Suppose that χ and ρ are multiplicative characters of k^\times for k/\mathbb{F}_p a finite extension containing the coefficients of both f and g . If $\chi \neq \rho$, then $\mathcal{G}(f, g, a, \chi)$ and $\mathcal{G}(f, g, a, \rho)$ are not geometrically isomorphic.*

PROOF. We argue by contradiction. Suppose that $\mathcal{G}(f, g, a, \chi)$ and $\mathcal{G}(f, g, a, \rho)$ are geometrically isomorphic. As each is geometrically irreducible, the cohomology group

$$H_c^2(\mathbb{G}_m/\overline{\mathbb{F}_p}, \mathcal{G}(f, g, a, \chi) \otimes \mathcal{G}(f, g, a, \rho)^\vee)$$

is pure of weight 2 and of dimension one. The dual $\mathcal{G}(f, g, a, \rho)^\vee$ is the (-1) -Tate twist of its complex conjugate: its trace function at $t \in L^\times$ is

$$t \in L^\times \mapsto \frac{-1}{\#L} \sum_{x \in L^\times} \psi_L(-f(1/x) - g(x) - tx^a) \bar{\rho}(x).$$

So the trace function of $\mathcal{G}(f, g, a, \chi) \otimes \mathcal{G}(f, g, a, \rho)^\vee$ is

$$t \in L^\times \mapsto \frac{-1}{\#L} \sum_{x, y \in L^\times} \psi_L(f(1/x) - f(1/y) + g(x) - g(y) + t(x^a - y^a)) \chi(x) \bar{\rho}(x).$$

The sum over t of this trace is, by the Lefschetz trace formula,

$$\text{Trace}(\text{Frob}_L | H_c^2) - \text{Trace}(\text{Frob}_L | H_c^1),$$

with H_c^2 pure of weight 2, and H_c^1 mixed of weight ≤ 1 . Thus, the dimension, namely, 1, of the relevant H_c^2 is the limsup, as L/k grows, of

$$\frac{1}{\#L} \sum_{t \in L^\times} \frac{1}{\#L} \sum_{x, y \in L^\times} \psi_L(f(1/x) - f(1/y) + g(x) - g(y) + t(x^a - y^a)) \chi(x) \bar{\rho}(x).$$

So far as the limsup is concerned, we may replace this sum over $t \in L^\times$ by the sum over all $t \in L$: indeed the $t = 0$ summand is

$$\frac{1}{(\#L)^2} \left(- \sum_{x \in L^\times} \psi_L(f(1/x) + g(x)) \chi(x) \right) \left(- \sum_{y \in L^\times} \psi_L(-f(1/y) - g(y)) \bar{\rho}(y) \right).$$

Each of the factors is $O(\sqrt{\#L})$, so this $t = 0$ term is $O(1/\#L)$, and hence does not affect the limsup.

Thus, the dimension, 1, of the H_c^2 is the limsup of

$$\begin{aligned} & \frac{1}{\#L} \sum_{t \in L} \frac{1}{\#L} \sum_{x, y \in L^\times} \psi_L(f(1/x) - f(1/y) + g(x) - g(y) + t(x^a - y^a)) \chi(x) \bar{\rho}(y) \\ &= \frac{1}{\#L} \sum_{x, y \in L^\times, x^a = y^a} \psi_L(f(1/x) - f(1/y) + g(x) - g(y) + t(x^a - y^a)) \chi(x) \bar{\rho}(y) \\ &= \sum_{\zeta \in \mu_a(\overline{\mathbb{F}}_p)} \frac{1}{\#L} \sum_{x \in L^\times} \psi_L(f(1/x) - f(1/\zeta x) + g(x) - g(\zeta x)) \chi(x) \bar{\rho}(\zeta x). \end{aligned}$$

The $\zeta = 1$ summand is

$$\frac{1}{\#L} \sum_{x \in L^\times} \chi(x) \bar{\rho}(\zeta x) = \frac{\bar{\rho}(\zeta)}{\#L} \sum_{x \in L^\times} (\chi/\rho)(x) = 0,$$

simply because χ/ρ is nontrivial. In each of the remaining summands, the Laurent polynomial $f(1/x) - f(1/\zeta x) + g(x) - g(\zeta x)$ inside the ψ is itself nonzero (because $\gcd(a, \gcd_{\deg}(f)) = 1$ and $\gcd(a, \gcd_{\deg}(g)) = 1$) and Artin–Schreier reduced, so each of these summands is $O(1/\sqrt{\#L})$. Thus, the limsup vanishes, the desired contradiction. ■

LEMMA 2.5. *Let X/\mathbb{F}_q be smooth and geometrically connected of dimension $d \geq 1$, $\ell \neq p$, K/\mathbb{Q} a finite extension, and L/K a finite Galois extension. Suppose that \mathcal{F} and \mathcal{G} are nonzero lisse $\overline{\mathbb{Q}}_\ell$ -sheaves on X . Suppose that both \mathcal{F} and \mathcal{G} are geometrically irreducible and have all their Frobenius traces in L . Suppose further that for every $\sigma \in \text{Gal}(L/K)$, there exist lisse sheaves \mathcal{F}^σ and \mathcal{G}^σ on X whose trace functions are the σ -conjugates of those of \mathcal{F} and of \mathcal{G} . If \mathcal{F} and \mathcal{G} are geometrically isomorphic, then \mathcal{F}^σ and \mathcal{G}^σ are geometrically isomorphic.*

PROOF. If \mathcal{F} and \mathcal{G} are geometrically isomorphic, then because each is geometrically irreducible, there exists an α^{\deg} twist such that we have an arithmetic isomorphism

$$\mathcal{F} \cong \mathcal{G} \otimes \alpha^{\deg}.$$

This implies that for every finite extension k/\mathbb{F}_q , and every $t \in X(k)$, we have an equality

$$\text{Trace}(\text{Frob}_{t,k} | \mathcal{F}) = \alpha^{\deg(k/\mathbb{F}_q)} \text{Trace}(\text{Frob}_{t,k} | \mathcal{G}).$$

Because \mathcal{F} is not the zero sheaf, for some k_0/\mathbb{F}_q and some $t \in X(k_0)$, $\text{Trace}(\text{Frob}_{t,k_0} | \mathcal{F})$ is nonzero. Then $\text{Trace}(\text{Frob}_{t,k_0} | \mathcal{G})$ must also be nonzero, and we recover $\alpha^{\deg(k_0/\mathbb{F}_q)}$ as the ratio of nonzero traces of \mathcal{F} and of \mathcal{G} . As these traces lie in L , it follows that $\alpha^{\deg(k_0/\mathbb{F}_q)}$ lies in L . Extending scalars from \mathbb{F}_q to k_0 , we reduce to the case when $\alpha \in L^\times$. Then we simply apply σ to the above equality of traces to obtain

$$\text{Trace}(\text{Frob}_{t,k} | \mathcal{F}^\sigma) = \sigma(\alpha)^{\deg(k/\mathbb{F}_q)} \text{Trace}(\text{Frob}_{t,k} | \mathcal{G}^\sigma);$$

i.e., we have an arithmetic isomorphism

$$\mathcal{F}^\sigma \cong \mathcal{G}^\sigma \otimes \sigma(\alpha)^{\deg},$$

and hence the desired geometric isomorphism. \blacksquare

PROPOSITION 2.6. *Given $A \geq 1$, $B \geq 1$, $a > B$, $p \nmid ABA$, let \mathcal{F} and \mathcal{G} be local systems on $\mathbb{G}_m/\overline{\mathbb{F}_p}$, whose local monodromies are of the form*

$$\mathcal{F}_{I(0)} \cong (A + B)\mathbb{1} \oplus W_{\mathcal{F}}, \quad \mathcal{G}_{I(0)} \cong (A + B)\mathbb{1} \oplus W_{\mathcal{G}},$$

with $W_{\mathcal{F}}, W_{\mathcal{G}}$ both irreducible of rank $a - B$ and totally wild, and

$$\mathcal{F}_{I(\infty)} \cong V_{\mathcal{F}}, \quad \mathcal{G}_{I(\infty)} \cong V_{\mathcal{G}},$$

with $V_{\mathcal{F}}, V_{\mathcal{G}}$ both of rank $A + a$ with all slopes < 1 . Let \mathcal{L} be a rank one local system on $\mathbb{G}_m/\overline{\mathbb{F}_p}$, such that $\mathcal{F} \cong \mathcal{G} \otimes \mathcal{L}$. Then $\mathcal{L} \cong \overline{\mathbb{Q}_\ell}$.

PROOF. We first show that \mathcal{L} is tame at ∞ . Indeed, if it were not, then $V_{\mathcal{G}} \otimes \mathcal{L}$ has all slopes equal to $\text{Swan}_\infty(\mathcal{L}) \geq 1$, while $V_{\mathcal{F}}$ has all slopes < 1 . Once we have this, it suffices to show that $\mathcal{L}_{I(0)}$ is trivial. Suppose first that $a - B > 1$. Then

$$\mathcal{G}_{I(0)} \otimes \mathcal{L}_{I(0)} \cong (A + B)\mathcal{L}_{I(0)} \oplus (\text{irreducible of rank } > 1).$$

So the one-dimensional constituents are each $\mathcal{L}_{I(0)}$. But the one-dimensional constituents of $\mathcal{F}_{I(0)}$ are each $\mathbb{1}$. Suppose next that $a - B = 1$. Then in both $\mathcal{F}_{I(0)}$ and $\mathcal{G}_{I(0)}$, the trivial constituents are in the majority. But after tensoring with $\mathcal{L}_{I(0)}$, the $\mathcal{L}_{I(0)}$ constituents are in the majority. Hence, $\mathcal{L}_{I(0)}$ is trivial. \blacksquare

COROLLARY 2.7. *For \mathcal{F} as in Proposition 2.6 above, suppose \mathcal{L} is a rank one local system on $\mathbb{G}_m/\overline{\mathbb{F}}_p$, such that $\mathcal{F}^\vee \cong \mathcal{F} \otimes \mathcal{L}$. Then $\mathcal{L} \cong \overline{\mathbb{Q}}_\ell$.*

PROOF. Indeed, both \mathcal{F} and \mathcal{F}^\vee have the shapes of local monodromies of the Proposition. ■

PROPOSITION 2.8. *Suppose $p \nmid ABa$, f, g are both Artin–Schreier reduced, and*

$$\gcd(a, \gcd_{\deg}(f)) = 1, \quad \gcd(a, \gcd_{\deg}(g)) = 1.$$

Then the $I(0)$ -representation of $\mathcal{G}(f, g, a, \chi)$ is tensor indecomposable under each of the following conditions.

- (a) *The rank $A + a \neq 4$.*
- (b) *$A + a = 4$ and $p = 2$.*
- (c) *$A + a = 4$, $p \neq 2$, and $(A, B, a) \neq (1, 1, 3)$.*

PROOF. Indeed, the $I(0)$ -representation is the direct sum $T \oplus W$ of a nonzero tame part and an irreducible wild part. In rank $\neq 4$, the result follows from [6, Corollary 10.4]. In the case of rank 4, the tame part has rank $A + B \geq 2$, so in characteristic $p = 2$ we may again apply [6, Corollary 10.4]. To apply [6, Corollary 10.4] with p odd and rank 4, we must avoid the case $A + B = 2$, i.e., the case $A = 1 = B$ and $a = 3$. ■

PROPOSITION 2.9. *Suppose $p \nmid ABa$, f, g are both Artin–Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = 1, \gcd(a, \gcd_{\deg}(g)) = 1$. Suppose that $\mathcal{G}(f, g, a, \chi)$ is tensor indecomposable for $I(0)$. Let us denote*

$$D := A + a, \quad t := A + B, \quad w := a - B,$$

the rank, the dimension of the tame part T , and the dimension of the wild part W of the $I(0)$ -representation $V = T \oplus W$. Then $\mathcal{G}(f, g, a, \chi)$ is not tensor induced over $I(0)$ under each of the following conditions.

- (a) *D is not a power.*
- (b) *$w = 1$.*
- (c) *$t - w > \sqrt{D}$.*
- (d) *$p \nmid w$ and $w < D - \sqrt{D}$.*

PROOF. Case (a) is trivial.

To treat case (b), suppose $w = 1$, and V is tensor induced: $V = U_1 \otimes \cdots \otimes U_n$ with $n \geq 2$, $\dim(U_i) = d \geq 2$, and $I(0)$ acts through $\mathrm{GL}_d(\mathbb{C}) \wr \mathcal{S}_n$. As $p \nmid aB$ and

$a - B = w = 1$, $p > 2$. Since W has dimension $w = 1$, some element $\gamma \in P(0)$ must act on W as a scalar $\zeta \neq 1$, an N th root of unity with $N > 1$ a p -power. By [8, Lemma 3.2 (ii) with $a = 1$], γ is tensor indecomposable, so it must induce an n -cycle while permuting the n tensor factors of V . By the formula for tensor induction [1],

$$|\text{Trace}(\gamma|_V)| \leq d \leq D/2 \leq D - 2$$

since $D = d^n \geq 4$. On the other hand,

$$|\text{Trace}(\gamma|_V)| = |D - 1 + \zeta| \geq D - 2,$$

with equality only when $\zeta = -1$, which is impossible since $p > 2$.

To treat case (c), use the $I(0)$ -tensor indecomposability of V to apply [8, part (ii) of Lemma 3.5]. It shows the existence of an element $h \in I(0)$ with $|\text{Trace}(h|_V)| \leq \sqrt{D}$ if V is tensor induced. But $V = t\mathbb{1} + W$, and any $h \in I(0)$, being of finite order, has $|\text{Trace}(h|_W)| \leq w$. Hence, $\sqrt{D} \geq |\text{Trace}(h|_V)| \geq t - w$, a contradiction.

For case (d), we may assume $w > 1$ by (b), and then use the fact that an element $\gamma \in I(0)$ which is generator of $I(0)/P(0)$ has spectrum on W consisting of all the w th roots of some root of unity ρ (because when $p \nmid w$, W is the Kummer induction $[w]_*\mathcal{L}$ of some rank one \mathcal{L} , and γ acts by cyclically permuting the w factors of the induction: because γ has finite order on V , ρ is itself a root of unity). Then we apply [8, Lemma 3.2 (i)] (with its $a = w$) to see that γ is tensor indecomposable in the $I(0)$ -representation if $w < D - \sqrt{D}$. Then we repeat the argument of case (b): if V is n -tensor induced, then

$$|\text{Trace}(\gamma|_V)| \leq d = D^{1/n} \leq \sqrt{D}.$$

But $\text{Trace}(\gamma|_W) = 0$ since $w > 1$, and hence $\sqrt{D} \geq |\text{Trace}(\gamma|_V)| = t = D - w$, i.e., $w \geq D - \sqrt{D}$, a contradiction. \blacksquare

THEOREM 2.10. *Suppose that $p \nmid ABa(A + a)(a - B)$, f and g are both Artin-Schreier reduced, and $\gcd(a, \gcd_{\text{deg}}(f)) = \gcd(a, \gcd_{\text{deg}}(g)) = 1$. Then $\mathcal{G}(f, g, a, \chi)$ is primitive on $\mathbb{G}_m/\overline{\mathbb{F}_p}$, under each of the following conditions.*

- (a) $w := a - B$ is not of the form $p^s - 1$ for any $s \geq 1$.
- (b) $w = p^s - 1$ and $A \neq 1$.
- (c) $w = p^s - 1$, $A = 1$, and $\chi \neq \chi_2$ (for χ_2 the quadratic character).
- (d) $w = p^s - 1$, $A = 1$, $\chi = \chi_2$, ABa is odd, each of f, g is an odd polynomial, and $B < 2p$.
- (e) $w = p^s - 1$, $A = 1$, $\chi = \chi_2$, and $\mathcal{G}(f, g, a, \chi)$ has infinite G_{geom} .

(f) $w = p^s - 1$, $A = 1$, $\chi = \chi_2$, $p \geq 5$, each of $f, g \in \mathbb{F}_p[x]$, with $g(x) = \sum_{i=0}^B a_i x^i$, and either

$$B \equiv \frac{p-1}{2} \pmod{p-1}$$

or

$$\sum_{i: i \equiv \frac{p-1}{2} \pmod{p-1}} a_i \neq 0.$$

PROOF. We argue by contradiction. Suppose $\mathcal{G}(f, g, a, \chi) = \pi_* \mathcal{H}$ for some finite étale $\pi : U \rightarrow \mathbb{G}_m$ of degree $d > 1$ and some local system \mathcal{H} on U . Then $d \times \text{rank}(\mathcal{H}) = \text{rank}(\mathcal{G}(f, g, a, \chi)) = A + a$ is prime to p . Also U is geometrically connected; otherwise, $\pi_* \mathcal{H}$ is not irreducible. Denote by X the complete nonsingular model of U , and denote by $\pi : X \rightarrow \mathbb{P}^1$ the finite flat map on the complete curves. Let

$$C := \pi^{-1}(0), \quad E = \pi^{-1}(\infty),$$

of cardinalities c, e , respectively.

For each point $x \in E$, denote by

$$\pi_x : \text{Spec}((K_{X,x})^\wedge) \rightarrow \text{Spec}((K_{\mathbb{P}^1, \infty})^\wedge)$$

the induced map of the spec's of completed function fields. Then for $\mathcal{G} := \mathcal{G}(f, g, a, \chi)$, we have

$$\mathcal{G}|_{I(\infty)} = \bigoplus_{x \in E} \pi_{x*}(\mathcal{H}_{I(x)}).$$

But $\mathcal{G}|_{I(\infty)}$ is irreducible; hence, there is precisely one point in E , call it ∞_{up} , and

$$\mathcal{G}|_{I(\infty)} = \pi_{\infty_{up}*}(\mathcal{H}_{I(\infty_{up})}),$$

with $\mathcal{H}_{I(\infty_{up})}$ irreducible (because its direct image is irreducible). Because ∞_{up} is the unique point lying over ∞ , the degree of $\pi_{\infty_{up}}$ is precisely $d := \text{deg}(\pi)$, which is a divisor of $A + a$. Looking at degrees, we thus have

$$d \times \text{rank}(\mathcal{H}) = \text{rank}(\mathcal{G}).$$

Therefore, $\text{deg}(\pi_{\infty_{up}}) = d$ is prime to p ; hence, $\pi_{\infty_{up}}$ is tame. By [4, Lemma 1.6.4.1], it follows that

$$\text{Swan}_{\infty_{up}}(\mathcal{H}) = \text{Swan}_{\infty}(\mathcal{G}).$$

Similarly, we have

$$\mathcal{G}|_{I(0)} = \bigoplus_{x \in C} \pi_{x*}(\mathcal{H}_{I(x)}),$$

while

$$\mathcal{G}|_{I(0)} = W_{B, a-B} \oplus (\overline{\mathbb{Q}_\ell})^{A+B},$$

with $W_{B,a-B}$ irreducible of rank $w := a - B$ with all slopes $B/(a - B)$. There is precisely one point $x_0 \in C$ whose $\pi_{x_0 \star}(\mathcal{H}_{I(x_0)})$ contains $W_{B,a-B}$ as a summand. More precisely, we have

$$\pi_{x_0 \star}(\mathcal{H}_{I(x_0)}) = W_{B,a-B} \oplus (\overline{\mathbb{Q}_\ell})^n, \quad \text{for some } n \geq 0.$$

We first consider the case $n = 0$. Then $\mathcal{H}_{I(x_0)}$ is irreducible. Moreover, it cannot be tame; i.e., it cannot be a Kummer sheaf \mathcal{L}_χ : if it were, then by Frobenius reciprocity its direct image contains all \mathcal{L}_ρ with $\rho^{\deg(\pi_{x_0})} = \chi$, whereas its direct image is totally wild. Looking at degrees, we have

$$\deg(\pi_{x_0}) \times \text{rank}(\mathcal{H}) = \text{rank}(W_{B,a-B}) = a - B.$$

As $p \nmid (a - B)$, we see that π_{x_0} has degree prime to p . Again by [4, Lemma 1.6.4.1], it follows that

$$\text{Swan}_{x_0}(\mathcal{H}) = \text{Swan}_0(\mathcal{G}) = B.$$

In this $n = 0$ case, we now argue as follows. On the one hand, for $\mathcal{G} := \mathcal{G}(f, g, a, \chi)$, for the Euler–Poincaré characteristic, we have

$$\text{EP}(U, \mathcal{H}) = \text{EP}(\mathbb{G}_m, \mathcal{G}) = -\text{Swan}_0(\mathcal{G}) - \text{Swan}_\infty(\mathcal{G}) = -B - A.$$

But

$$\begin{aligned} \text{EP}(U, \mathcal{H}) &= \text{EP}(U) \text{rank}(\mathcal{H}) - \sum_{x \in C} \text{Swan}_x(\mathcal{H}) - \text{Swan}_{\infty \text{up}}(\mathcal{H}) \\ &= \text{EP}(U) \text{rank}(\mathcal{H}) - B - \sum_{x \in C, x \neq x_0} \text{Swan}_x(\mathcal{H}) - A. \end{aligned}$$

Subtracting these two expressions for $\text{EP}(U, \mathcal{H})$, we find that

$$\text{EP}(U) \text{rank}(\mathcal{H}) = \sum_{x \in C, x \neq x_0} \text{Swan}_x(\mathcal{H}).$$

In particular, $\text{EP}(U) \text{rank}(\mathcal{H}) \geq 0$, and hence $\text{EP}(U) \geq 0$. As U is the complement of at least two points (one in D and at least one in C) in a complete nonsingular curve, call it X , we have $\text{EP}(U) = 2 - 2g_X - 1 - \#C \geq 0$. On the other hand, $2 - 2g_X - 1 - \#C \leq 0$, with equality only if $g_X = 0$ and $\#C = 1$. Because $\#C = 1$, $\deg(\pi_{x_0})$ must be $d = \deg(\pi)$. Then the entire $I(0)$ -representation is wild, a contradiction, since the $I(0)$ -representation has an $A + B \geq 2$ dimensional trivial part. (Alternatively, as the referee pointed out, one could use the fact that for $x \in C \setminus \{x_0\}$, $\pi_{x \star}(\mathcal{H}_{I(x)})$ is tame, hence by [4, Lemma 1.6.4.1], \mathcal{H} is tame at each such x , and so $\text{EP}(U) = 0$ by the formula displayed above.)

Suppose next that $n \geq 1$. Then $\pi_{x_0 \star}(\mathcal{H}_{I(x_0)})$ contains $\overline{\mathbb{Q}_\ell}$, which we write as $\mathcal{L}_\mathbb{1}$. Then by Frobenius reciprocity, $\mathcal{H}_{I(x_0)}$ contains $\mathcal{L}_\mathbb{1}$. Then $\deg(\pi_{x_0})$ cannot be divisible by any prime to p integer $r > 1$, for otherwise $\pi_{x_0 \star}(\mathcal{H}_{I(x_0)})$ contains $\pi_{x_0 \star}(\mathcal{L}_\mathbb{1})$, which contains all \mathcal{L}_ρ with $\rho^r = \mathbb{1}$. This is impossible because the entire tame part of the $I(0)$ -representation of \mathcal{G} is copies of $\mathcal{L}_\mathbb{1}$. Thus, $\deg(\pi_{x_0}) = p^s$ for some $s \geq 0$. If $s = 0$, i.e., if $\deg(\pi_{x_0}) = 1$ is prime to p , then $\text{Swan}_{x_0}(\mathcal{H}) = \text{Swan}_0(\mathcal{G}) = B$, and we conclude as in the $n = 0$ case above.

Suppose next that $n \geq 1$ and $\deg(\pi_{x_0}) = p^s$ with $s \geq 1$. Then by Frobenius reciprocity, $\pi_{x_0 \star}(\mathcal{L}_\mathbb{1})$ contains $\mathcal{L}_\mathbb{1}$ just once and contains no \mathcal{L}_ρ for any nontrivial ρ (because it only contains \mathcal{L}_ρ if $\rho^{\deg(\pi_{x_0})} = \mathbb{1}$). Therefore,

$$\pi_{x_0 \star}(\mathcal{L}_\mathbb{1}) = \mathcal{L}_\mathbb{1} \oplus (\text{totally wild of rank } p^s - 1).$$

If $\mathcal{H}_{I(x_0)}$ were not simply $\mathcal{L}_\mathbb{1}$, any other irreducible constituent would either be tame (in which case its direct image would also have a wild part of rank $p^s - 1$), or would be wild, in which case its direct image would be totally wild.

Thus, in this $n \geq 1$ case, we have $n = 1$, $\text{rank}(\mathcal{H}) = 1$, $\mathcal{H}_{I(x_0)} = \mathcal{L}_\mathbb{1}$, and

$$\pi_{x_0 \star}(\mathcal{H}_{I(x_0)}) = \pi_{x_0 \star}(\mathcal{L}_\mathbb{1}) = \mathcal{L}_\mathbb{1} \oplus (\text{totally wild of rank } p^s - 1).$$

So the wild part of the $I(0)$ -representation of \mathcal{G} has dimension $w = p^s - 1$.

We now continue with the analysis of the case when $w = p^s - 1$. Looking at what remains of the $I(0)$ -representation, we find

$$\mathcal{L}_\mathbb{1}^{A+B-1} = \bigoplus_{x \in C, x \neq x_0} \pi_{x \star}(\mathcal{H}_{I(x)}).$$

Each individual direct image $\pi_{x \star}(\mathcal{H}_{I(x)})$ is then a sum of $\mathcal{L}_\mathbb{1}$. Being tame, it follows that \mathcal{H} is tame at each $x \neq x_0$ in C . Then \mathcal{H} must be $I(x)$ -trivial at each such x ; otherwise, its direct image contains various \mathcal{L}_ρ with nontrivial ρ . Then each π_x for $x \neq x_0$ must have degree 1: it cannot have degree divisible by a prime to p integer $r > 1$ because that introduces nontrivial tame pieces in the direct image, and it cannot have degree a strictly positive power of p because that introduces nonzero wild parts in the direct image. Thus, at each $x \neq x_0$ in C , the degree of π_x is 1. From the above displayed equation

$$\mathcal{L}_\mathbb{1}^{A+B-1} = \bigoplus_{x \in C, x \neq x_0} \pi_{x \star}(\mathcal{H}_{I(x)}),$$

we then see that

$$\#C = A + B,$$

and that \mathcal{H} is lisse of rank one outside of the single point ∞_{up} . Thus,

$$\text{EP}(U, \mathcal{H}) = \text{EP}(\mathbb{G}_m, \mathcal{G}) = -\text{Swan}_0(\mathcal{G}) - \text{Swan}_\infty(\mathcal{G}) = -B - A.$$

At the same time, remembering that \mathcal{H} has rank one, we have

$$\mathrm{EP}(U, \mathcal{H}) = \mathrm{EP}(U) - \mathrm{Swan}_{\infty up}(\mathcal{H}) = \mathrm{EP}(U) - A.$$

Thus, $\mathrm{EP}(U) - A = -B - A$, and hence

$$\mathrm{EP}(U) = -B.$$

In terms of the complete nonsingular model X of U , this gives

$$-B = \mathrm{EP}(U) = 2 - 2g_X - \#D - \#C = 2 - 2g_X - 1 - (A + B);$$

hence, $2 - 2g_X - 1 - A = 0$, i.e., $-2g_X = A - 1$. This can only hold if $g_X = 0$ and $A = 1$. Putting ∞up at ∞ , \mathcal{H} is lisse of rank one on \mathbb{A}^1 , with $\mathrm{Swan}_{\infty}(\mathcal{H}) = 1$. Thus, \mathcal{H} is $\mathcal{L}_{\psi(\alpha x)}$ for some $\alpha \neq 0$ in $\overline{\mathbb{F}}_p$. Putting x_0 at 0, the morphism π is a polynomial $H(x) \in \overline{\mathbb{F}}_p[x]$ which has degree $A + a = 1 + a = 1 + B + w = p^s + B$, which has 0 as a root of multiplicity p^s , and which has B simple zeros, each of which is nonzero. Thus, we obtain a geometric isomorphism

$$\mathcal{G}(f, g, a, \chi) \cong [H(x)]_{\star} \mathcal{L}_{\psi(\alpha x)}.$$

Over a large enough finite extension k/\mathbb{F}_p (namely, one which contains α and the coefficients of each of the polynomials f, g, H , and with $\chi^{\#k-1} = \mathbb{1}$) both of $\mathcal{G}(f, g, a, \chi)$ and $[H(x)]_{\star} \mathcal{L}_{\psi(\alpha x)}$ are geometrically irreducible and geometrically isomorphic local systems on \mathbb{G}_m/k . Therefore, there exists some $\gamma \in \overline{\mathbb{Q}}_{\ell}^{\times}$ for which we have an *arithmetic* isomorphism

$$(2.10.1) \quad \mathcal{G}(f, g, a, \chi) \otimes \gamma^{\mathrm{deg}} \cong [H(x)]_{\star} \mathcal{L}_{\psi(\alpha x)}.$$

Recall that $\mathcal{G}(f, g, a, \chi)$ is, arithmetically, the Fourier transform

$$\mathcal{G}(f, g, a, \chi) := \mathrm{FT}_{\psi}([a]_{\star} (\mathcal{L}_{\psi(f(1/x)+g(x))} \otimes \mathcal{L}_{\chi}(x))),$$

and hence

$$\mathcal{G}(f, g, a, \chi) \otimes \gamma^{\mathrm{deg}} := \mathrm{FT}_{\psi}([a]_{\star} (\mathcal{L}_{\psi(f(1/x)+g(x))} \otimes \mathcal{L}_{\chi}(x) \otimes \gamma^{\mathrm{deg}})).$$

Applying the inverse Fourier transform $\mathrm{FT}_{\overline{\psi}}$ to equation (2.10.1), we get an arithmetic isomorphism

$$[a]_{\star} (\mathcal{L}_{\psi(f(1/x)+g(x))} \otimes \mathcal{L}_{\chi}(x) \otimes \gamma^{\mathrm{deg}}) \cong \mathrm{FT}_{\overline{\psi}}([H(x)]_{\star} \mathcal{L}_{\psi(\alpha x)}).$$

We next prove that this cannot happen if χ has order ≥ 3 . The key point is that

$$\mathrm{Gal}(\mathbb{Q}(\chi, \zeta_p)/\mathbb{Q}(\zeta_p)) = \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q}).$$

So we may choose

$$\sigma \in \text{Gal}(\mathbb{Q}(\chi, \zeta_p)/\mathbb{Q}(\zeta_p))$$

so that the σ -conjugate system to $\mathcal{G}(f, g, a, \chi)$ is

$$\mathcal{G}(f, g, a, \chi)^\sigma = \mathcal{G}(f, g, a, \chi^\sigma),$$

while

$$([H(x)]_\star \mathcal{L}_{\psi(\alpha x)})^\sigma = [H(x)]_\star \mathcal{L}_{\psi(\alpha x)}.$$

Applying Lemma 2.5, we find that $[H(x)]_\star \mathcal{L}_{\psi(\alpha x)}$ is isomorphic to both $\mathcal{G}(f, g, a, \chi)$ and to $\mathcal{G}(f, g, a, \chi^\sigma)$. But if χ has order ≥ 3 , there exists σ for which χ^σ is any character of the same order as χ , and in particular there exists σ for which $\chi^\sigma \neq \chi$. For such a σ , $\mathcal{G}(f, g, a, \chi)$ and $\mathcal{G}(f, g, a, \chi^\sigma)$ are not geometrically isomorphic, by Lemma 2.4.

For $\chi = 1$, the “traces nowhere vanishing” argument of [8, the proof of Proposition 4.4] shows that $\mathcal{G}(f, g, a, 1)$ is always primitive.

We now deal with the case where $A = 1$, $w = p^s - 1$, $\chi = \chi_2$ has order 2, and $\mathcal{G}(f, g, a, \chi_2) \cong [H(x)]_\star \mathcal{L}_{\psi(\alpha x)}$ with H a polynomial of degree $B + p^s$, with 0 as a root of multiplicity p^s and with B simple roots, each nonzero. For an $I(0)$ -representation V , $[H(x)]_\star V$ as $I(0)$ -representation is the induction through H viewed as lying in $\overline{\mathbb{F}_p}[[x]]$, call it H^{fml} . We apply this to $V := \mathcal{L}_{\psi(\alpha x)}$, which is trivial as $I(0)$ -representation. We recall from [5, Corollary 6.4.5 (2)] that we may compute

$$\text{Swan}_0([H^{fml}(x)]_\star \mathcal{L}_{\psi(\alpha x)}) = \text{Swan}_0([H^{fml}(x)]_\star \mathcal{L}_1)$$

as follows. Expand $H^{fml}(x)$:

$$H^{fml}(x) = x^{p^s} \left(\sum_{m \geq 0} \alpha_m x^m \right).$$

Then $\text{Swan}_0([H^{fml}(x)]_\star \mathcal{L}_1)$ is the least prime to p integer m with $\alpha_m \neq 0$.

On the other hand, the $I(0)$ -representation of $[H(x)]_\star \mathcal{L}_{\psi(\alpha x)}$ is the direct sum of $[H^{fml}(x)]_\star \mathcal{L}_{\psi(\alpha x)}$ with B copies of \mathcal{L}_1 , so

$$\text{Swan}_0([H(x)]_\star \mathcal{L}_{\psi(\alpha x)}) = \text{Swan}_0([H^{fml}(x)]_\star \mathcal{L}_{\psi(\alpha x)}) (= \text{Swan}_0([H^{fml}(x)]_\star \mathcal{L}_1)).$$

But H is a polynomial of degree $B + p^s$. Thus, in $\overline{\mathbb{F}_p}[[x]]$, $H = H^{fml}$, and its expansion is

$$H(x) = x^{p^s} \left(\sum_{m=0}^B a_m x^m \right),$$

with a_0, a_B both nonzero. Moreover, if $1 \leq m \leq B - 1$ is nonzero mod p , then $a_m = 0$ (otherwise, Swan_0 would be this lower m).

Suppose now that ABa is odd and that both f, g are odd polynomials. Then $\mathcal{G}(f, g, a, \chi_2)$ is self-dual (in fact orthogonally self-dual). So if $\mathcal{G}(f, g, a, \chi_2) \cong [H(x)]_* \mathcal{L}_{\psi(\alpha x)}$, then

$$\mathcal{H} := [H(x)]_* \mathcal{L}_{\psi(\alpha x)}$$

is self-dual. As \mathcal{H} is pure of weight zero and geometrically irreducible, its autoduality is equivalent to having $\dim(H_c^2(\mathbb{G}_m/\overline{\mathbb{F}}_p, \mathcal{H}^{\otimes 2})) = 1$. This dimension is the limsup over larger and larger extension L of any chosen finite extension k/\mathbb{F}_p which contains the coefficients of f, g, H , of the complex absolute value of

$$\begin{aligned} (1/\#L) \sum_{t \in L^\times} (\text{Trace}(\text{Frob}_{t,L} | \mathcal{H}))^2 &= (1/\#L) \sum_{t \in L^\times} \sum_{x, y \in L: H(x)=t=H(y)} \psi_L(\alpha x) \psi_L(\alpha y) \\ &= (1/\#L) \sum_{x, y \in L: H(x)=H(y) \neq 0} \psi_L(\alpha(x+y)). \end{aligned}$$

The “missing” term with $t = 0$ is $(1/\#L) (\sum_{x \in L: H(x)=0} \psi_L(\alpha x))^2$, which is at most $\deg(H)^2/\#L$, so does not affect the limsup. So the dimension of this H_c^2 is the limsup of

$$(1/\#L) \sum_{x, y \in L: H(x)=H(y)} \psi_L(\alpha(x+y)).$$

The affine curve $H(x) = H(y)$ is smooth outside the point $(0, 0)$. Indeed, its singularities are the points on the curve where $dH(x)/dx = 0 = dH(y)/dy$. From the explicit form of H above, we see that

$$dH(x)/dx = a_B(p^s + B)x^{p^s+B-1}, \quad dH(y)/dy = a_B(p^s + B)y^{p^s+B-1}.$$

The polynomial $H(x) - H(y)$ has the factorization

$$H(x) - H(y) = (x - y)\Delta_H, \quad \text{with } \Delta_H := (H(x) - H(y))/(x - y).$$

The polynomial Δ_H is not divisible by $x - y$; indeed its leading term is

$$a_B \prod_{\zeta \in \mu_{p^s+B}, \zeta \neq 1} (x - \zeta y).$$

The intersection of the two loci $x - y = 0$ and $\Delta_H = 0$ is the single point $(0, 0)$. Thus, the curve $\Delta_H = 0$ is lisse outside the point $(0, 0)$ (because this open set of $\Delta_H = 0$ is the complement of $x = y$ in $H(x) = H(y)$).

The sum of $\psi_L(\alpha(x+y))$ over the locus $x = y$ vanishes. So our limsup is the limsup of

$$(1/\#L) \sum_{x, y \in L: \Delta_H=0} \psi_L(\alpha(x+y)).$$

We will show that this limsup is in fact 0 provided that $B < 2p$. Suppose first that $B < p$. Then in the expansion of H , there can be no middle terms: we must have

$$H(x) = x^{p^s} (a_0 + a_B x^B).$$

Then

$$\Delta_H = a_0(x - y)^{p^s - 1} + a_B \prod_{1 \neq \zeta \in \mu_{p^s + B}} (x - \zeta y).$$

This is the finite part of the projective curve of equation

$$a_0 z^B (X - Y)^{p^s - 1} + a_B \prod_{1 \neq \zeta \in \mu_{p^s + B}} (X - \zeta Y) = 0,$$

which has $p^s + B - 1$ points at ∞ . If we invert $X - Y$, then in coordinates

$$z := Z/(X - Y), \quad x := X/(X - Y), \quad \text{and thus } Y/(X - Y) = x - 1,$$

this curve becomes

$$a_0 z^B + a_B \prod_{1 \neq \zeta \in \mu_{p^s + B}} (x - \zeta(x - 1)) = 0.$$

This affine curve, call it \mathcal{C} , is defined by this polynomial, which is an Eisenstein polynomial in z for any of the factors $(x - \zeta(x - 1))$ with $1 \neq \zeta \in \mu_{p^s + B}$. In particular, it is Eisenstein for the factor $2x - 1$ (present because $p^s + B$ is even, as B was odd). Thus, \mathcal{C} is geometrically irreducible; hence, $\Delta_H = 0$ is geometrically irreducible. The points on \mathcal{C} with $z = 0$ are the points at ∞ on $\Delta_H = 0$, and z has a simple pole on $\Delta_H = 0$ at each of its zeros in \mathcal{C} . In particular, $2x - 1$ has a pole of order B at the zero of z over $2x - 1 = 0$. Over $\Delta_H = 0$, we are summing $\mathcal{L}_{\psi(\alpha((X+Y)/(X-Y)))} = \mathcal{L}_{\psi(\alpha(2x-1))}$, which has Swan = B at the zero of z in \mathcal{C} over $2x - 1$. In particular, $\mathcal{L}_{\psi(\alpha((X+Y)/(X-Y)))}$ is not geometrically constant. Hence, this sum is $O(1/\sqrt{\#L})$, and the limsup is 0.

Suppose now that $2p > B > p$. Then in the expansion of H , there can be a middle term:

$$H(x) = x^{p^s} (a_0 + a_p x^p + a_B x^B) = a_0 x^{p^s} + a_p x^{p+p^s} + a_B x^{p^s+B}.$$

In this case, we write

$$p + p^s := pN,$$

and

$$\begin{aligned} \Delta_H &= a_0(x - y)^{p^s - 1} + a_p(x^N - y^N)^{p-1}((x^N - y^N)/(x - y)) \\ &\quad + a_B \prod_{1 \neq \zeta \in \mu_{p^s + B}} (x - \zeta y). \end{aligned}$$

Because $N = 1 + p^{s-1}$ is even, the factor $(x^N - y^N)/(x - y)$ is divisible by $x + y$. Now we repeat the above argument. The curve $\Delta_H = 0$ is the finite part of the projective curve of equation

$$a_0 Z^B (X - Y)^{p^s - 1} + a_p Z^{B-p} (X^N - Y^N)^{p-1} ((X^N - Y^N)/(X - Y)) \\ + a_B \prod_{1 \neq \zeta \in \mu_{p^s + B}} (X - \zeta Y) = 0,$$

which has $p^s + B - 1$ points at ∞ . If we invert $X - Y$, then in coordinates

$$z := Z/(X - Y), \quad x := X/(X - Y), \quad \text{and thus } Y/(X - Y) = x - 1,$$

we obtain the affine curve \mathcal{C} of equation

$$a_0 z^B + a_p z^{B-p} (x^N - (x-1)^N)^{p-1} (x^N - (x-1)^N) \\ + a_B \prod_{1 \neq \zeta \in \mu_{p^s + B}} (x - \zeta(x-1)) = 0.$$

The curve \mathcal{C} is defined by this polynomial, which is (again) an Eisenstein polynomial in z for the factor $(2x - 1)$. From this point on, we repeat verbatim the proof in the case $B < p$ above.

In this $w = p^s - 1$, $A = 1$ case, if $\mathcal{G}(f, g, a, \chi_2)$ is induced, then it is induced from a rank one local system $\mathcal{L}_{\psi(\alpha x)}$, which has finite G_{geom} , and hence $\mathcal{G}(f, g, a, \chi)$ itself has finite G_{geom} .

In the $w = p^s - 1$, $A = 1$ case with $f, g \in \mathbb{F}_p[x]$, $p \geq 5$ and $g(x) = \sum_i a_i x^i$, we will use the hypothesis that either $B \equiv \frac{p-1}{2} \pmod{(p-1)}$ or

$$\sum_{i: i \equiv \frac{p-1}{2} \pmod{(p-1)}} a_i \neq 0$$

to show that G_{geom} is infinite. For this, it suffices to exhibit a point $t \in \mathbb{F}_p^\times$ where

$$\text{Trace}(\text{Frob}_{t, \mathbb{F}_p} | \mathcal{G}(f, g, a, \chi_2))$$

is not divisible by $\text{Gauss}(\psi, \chi_2)$ as an algebraic integer. (Recall that $\mathcal{G}(f, g, a, \chi_2)$ has Frobenius traces in $\mathbb{Z}[\zeta_p]$ and is pure of weight one. Pass to $\mathcal{G}_0(f, g, a, \chi_2) := \mathcal{G}(f, g, a, \chi_2) \otimes (\text{Gauss}(\psi, \chi_2))^{-\text{deg}}$, which is pure of weight zero with traces in $\mathbb{Z}[\zeta_p][1/p]$. This twist \mathcal{G}_0 has arithmetic determinant of finite order. Indeed, any Frobenius determinant on \mathcal{G}_0 at a point $t \in \mathbb{F}_q^\times$ is an element of $\mathbb{Z}[\zeta_p][1/p]$ which is a unit for all places $\lambda \nmid p$ of $\mathbb{Q}(\zeta_p)$ (because \mathcal{G}_0 is part of a compatible system) and has complex absolute value 1 after all complex embeddings. As $\mathbb{Q}(\zeta_p)$ has a unique place over p , it

follows (product formula) that this determinant has absolute value 1 everywhere, so is a root of unity in $\mathbb{Q}(\zeta_p)$, so it has order dividing $2p$. Thus, the arithmetic determinant of \mathcal{E}_0 has finite order. Then the finiteness of G_{geom} is equivalent to \mathcal{E}_0 having all Frobenius traces algebraic integers.)

To show that $\text{Trace}(\text{Frob}_{t, \mathbb{F}_p} | \mathcal{G}(f, g, a, \chi_2))$ is not divisible by $\text{Gauss}(\psi, \chi_2)$ in $\mathbb{Z}[\zeta_p]$, we use a p -adic calculation. Define

$$\pi := \zeta_p - 1.$$

Then $\text{ord}_p(\pi) = 1/(p-1)$, while $\text{ord}_p(\text{Gauss}(\psi, \chi_2)) = 1/2$. For $p \geq 5$, we have $1/2 > 1/(p-1)$. So we need only to find a Frobenius trace $\text{Trace}(\text{Frob}_{t, \mathbb{F}_p} | \mathcal{G}(f, g, a, \chi_2))$ which is divisible by π but not by π^2 . This amounts to computing this Frobenius trace modulo π^2 . For any $x \in \mathbb{F}_p$,

$$\psi(x) = \zeta_p^x = (1 + \pi)^x \equiv 1 + \pi x \pmod{\pi^2},$$

and for any $x \in \mathbb{F}_p^\times$,

$$\chi_2(x) \equiv x^{(p-1)/2} \pmod{p}.$$

So for any Laurent polynomial $L(x) = \sum_i a_i x^i \in \mathbb{F}_p[x, 1/x]$, and any $x \in \mathbb{F}_p^\times$, we have

$$\chi_2(x)\psi(L(x)) \equiv x^{(p-1)/2}(1 + \pi)^{L(x)} \equiv x^{(p-1)/2}(1 + \pi L(x)) \pmod{\pi^2}.$$

Expanding out $L(x) = \sum_i a_i x^i$,

$$\begin{aligned} \sum_{x \in \mathbb{F}_p^\times} \chi_2(x)\psi(L(x)) &\equiv \sum_{x \in \mathbb{F}_p^\times} x^{(p-1)/2} \left(1 + \pi \sum_i a_i x^i \right) \pmod{\pi^2} \\ &\equiv \sum_{x \in \mathbb{F}_p^\times} \chi_2(x) + \pi \sum_i a_i \sum_{x \in \mathbb{F}_p^\times} x^{a_i + (p-1)/2} \pmod{\pi^2}. \end{aligned}$$

The sum $\sum_{x \in \mathbb{F}_p^\times} \chi_2(x)$ vanishes. The sum $\sum_{x \in \mathbb{F}_p^\times} x^{a_i + (p-1)/2}$ vanishes modulo p unless the exponent $a_i + (p-1)/2$ is a multiple of $p-1$, in which case this sum is $-1 \pmod{p}$. Thus,

$$- \sum_{x \in \mathbb{F}_p^\times} \chi_2(x)\psi(L(x)) \equiv \pi \sum_{i: i \equiv \frac{p-1}{2} \pmod{p-1}} a_i \pmod{\pi^2}.$$

In $\mathcal{G}(f, g, a, \chi_2)$, the relevant Laurent polynomial is $f(1/x) + g(x) + tx^a$. Here $f(1/x) = a_{-1}/x$, $g(x) = \sum_{i=0}^B a_i x^i$. Because $p \geq 5$, the $1/x$ term contributes 0. If B is not $(p-1)/2 \pmod{p-1}$, the tx^a term contributes 0, simply because $a = B + p^s - 1$ is not $(p-1)/2 \pmod{p-1}$, no matter what the value of $t \in \mathbb{F}_p^\times$. So

for such B , we are done; if $\sum_{i \leq B: i \equiv \frac{p-1}{2}} a_i \neq 0$, then we may choose any $t \in \mathbb{F}_p^\times$ at which to take the trace.

If, on the other hand, B is $(p-1)/2 \pmod{p-1}$, then the exponent a in tx^a , which is $a = B + p^s - 1$, is $(p-1)/2 \pmod{p-1}$. So the tx^a term contributes t , and

$$-\sum_{x \in \mathbb{F}_p^\times} \chi_2(x) \psi(L(x)) \equiv \pi \left(t + \sum_{i \leq B: i \equiv \frac{p-1}{2} \pmod{p-1}} a_i \right) \pmod{\pi^2}.$$

We may always choose $t \in \mathbb{F}_p^\times$ so that the innermost sum is nonzero mod p . \blacksquare

Here is an extension of the previous Theorem 2.10 to the special case $B = 1$, where we drop the hypothesis that $p \nmid (a - B)$.

THEOREM 2.11. *Suppose that $p \nmid ABa(A + a)$, f and g are both Artin–Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = \gcd(a, \gcd_{\deg}(g)) = 1$. Suppose that $B = 1$. Then $\mathcal{G}(f, g, a, \chi)$ is primitive on $\mathbb{G}_m/\overline{\mathbb{F}}_p$.*

PROOF. Repeat verbatim the first four paragraphs of the proof of Theorem 2.10, down to the point

$$\deg(\pi_{x_0}) \times \text{rank}(\mathcal{H}) = \text{rank}(W_{B,a-B}) = a - B$$

in the discussion of the $n = 0$ case. Because $B = 1$, $W_{B,a-B}$ has $\text{Swan}_0(W_{B,a-B}) = B = 1$. In this $n = 0$ case, \mathcal{H} is totally wild, and

$$\pi_{x_0 \star} \mathcal{H} \cong W_{B,a-B}.$$

By [4, Lemma 1.6.4.1], we have

$$\text{Swan}_0(\pi_{x_0 \star} \mathcal{H}) = \text{Swan}_{x_0}(\mathcal{H}) + \text{rank}(\mathcal{H}) \text{Swan}_0(\pi_{x_0 \star} \mathcal{L}_1).$$

In this equality, the left-hand side is $\text{Swan}_0(\pi_{x_0 \star} \mathcal{H}) = \text{Swan}_0(W_{B,a-B}) = 1$, while on the right, $\text{Swan}_{x_0}(\mathcal{H}) \geq 1$ and $\text{Swan}_0(\pi_{x_0 \star} \mathcal{L}_1) \geq 0$. Therefore, we must have

$$\text{Swan}_{x_0}(\mathcal{H}) = 1 \quad \text{and} \quad \text{Swan}_0(\pi_{x_0 \star} \mathcal{L}_1) = 0.$$

Because $\text{Swan}_0(\pi_{x_0 \star} \mathcal{L}_1) = 0$, we must have $\deg(\pi_{x_0}) := D$ prime to p . Indeed, if $\pi_{x_0 \star} \mathcal{L}_1$ is tame, then $\pi_{x_0 \star} \mathcal{L}_1$ has rank D and, being tame, is given by

$$\pi_{x_0 \star} \mathcal{L}_1 = \bigoplus_{\chi \text{ with } \chi^D = 1} \mathcal{L}_\chi.$$

Thus, there are precisely D characters of order dividing D ; hence, D is prime to p .

Once we have $\deg(\pi_{x_0}) := D$ prime to p , repeat the rest of the $n = 0$ case EP argument to get a contradiction.

Suppose now that $n \geq 1$. Exactly as in the proof of Theorem 2.10, we see that $\deg(\pi_{x_0}) = p^s$ for some $s \geq 0$. If $s = 0$, i.e., if $\deg(\pi_{x_0}) = 1$ is prime to p , then $\text{Swan}_{x_0}(\mathcal{H}) = \text{Swan}_0(\mathcal{G}) = B$, and we conclude as in the $n = 0$ case above.

We further see that when $s \geq 1$, we have $n = 1$, $\text{rank}(\mathcal{H}) = 1$, $\mathcal{H}_{I(x_0)} = \mathcal{L}_{\mathbb{1}}$, and

$$\pi_{x_0\star}(\mathcal{H}_{I(x_0)}) = \pi_{x_0\star}(\mathcal{L}_{\mathbb{1}}) = \mathcal{L}_{\mathbb{1}} \oplus (\text{totally wild of rank } p^s - 1).$$

But $\deg(\pi_{x_0})$ divides the rank of $\pi_{x_0\star}(\mathcal{H}_{I(x_0)})$, which is $1 + (a - B) = a$ (because $B = 1$). But $p \nmid a$, so $\deg(\pi_{x_0})$ cannot be p^s with $s \geq 1$. ■

Here is an extension of Theorem 2.10 to the special case $A = 1$, where we (partially) drop the hypothesis that $p \nmid (A + a)(a - B)$.

THEOREM 2.12. *Suppose that $p \nmid ABa$, the polynomials f and g are both Artin-Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = \gcd(a, \gcd_{\deg}(g)) = 1$. Suppose that $A = 1$ and that $A + a = n_0 p^e$ with $e \geq 0$ and $1 \leq n_0 < p$. Then $\mathcal{G}(f, g, a, \chi)$ is primitive on $\mathbb{G}_m/\overline{\mathbb{F}}_p$.*

PROOF. Because $A = 1$, the $I(\infty)$ -representation of $\mathcal{G}(f, g, a, \chi)$ is totally wild of rank $A + a = 1 + a$, with all slopes $A/(A + a) = 1/(a + 1)$. By Pink's argument [2, Lemma 11], if this $I(\infty)$ -representation is induced, it is Kummer induced of some prime to p degree $D > 1$. As this D divides the rank $1 + a$, we see that $D|n_0$, and hence $D < p$. Thus, $\mathcal{G}(f, g, a, \chi) = \pi_{\star}\mathcal{H}$ for some lisse \mathcal{H} on a finite étale connected

$$\pi : U \rightarrow \mathbb{G}_m$$

of degree D .

On the complete nonsingular model X of U , there is a unique point x_{∞} lying over ∞ simply because $\mathcal{G}_{I(\infty)}$ is irreducible.

Now consider the unique point $x_0 \in X$ over 0 for which $\pi_{x_0\star}\mathcal{H}$ contains $W_{B,a-B}$ as $I(0)$ -representation. The degree d_0 of π_{x_0} is $\leq D$, hence is $< p$, hence is prime to p . Thus, we have

$$\pi_{x_0\star}(\mathcal{H}_{I(x_0)}) = W_{B,a-B} \oplus (\overline{\mathbb{Q}}_{\ell})^n, \quad \text{for some } n \geq 0.$$

Because $d_0 := \deg(\pi_{x_0})$ is prime to p , we have

$$\text{Swan}_{x_0}(\mathcal{H}_{I(x_0)}) = \text{Swan}_0(W_{B,a-B} \oplus (\overline{\mathbb{Q}}_{\ell})^n) = B.$$

At any other point x_i lying over 0, the degree $d_i := \deg(\pi_{x_i})$ is again $\leq D < p$, hence is prime to p . At each such point, $\pi_{x_i}(\mathcal{H}_{I(x_0)})$ is a trivial $I(0)$ -representation.

This first implies that $\mathcal{H}_{I(x_i)}$ is tame, and then that both $d_i = 1$ (otherwise, the Kummer direct image of any \mathcal{L}_χ by $[d_i]$ will not be entirely trivial) and that $\mathcal{H}_{I(x_i)}$ is just the direct sum $\overline{\mathbb{Q}_\ell}^{\text{rank}(\mathcal{H})}$.

Now we give the EP argument. On the one hand, we have

$$\text{EP}(U, \mathcal{H}) = \text{EP}(\mathbb{G}_m, \mathcal{G}) = -\text{Swan}_0(\mathcal{G}) - \text{Swan}_\infty(\mathcal{G}) = -B - A,$$

while we also have

$$\begin{aligned} \text{EP}(U, \mathcal{H}) &= \text{EP}(U) \text{rank}(\mathcal{H}) - \sum_{x_i \text{ over } 0} \text{Swan}_{x_i}(\mathcal{H}) - \text{Swan}_{x_\infty}(\mathcal{H}) \\ &= \text{EP}(U) \text{rank}(\mathcal{H}) - B - A. \end{aligned}$$

Comparing the two expressions for $\text{EP}(U, \mathcal{H})$, we find $\text{EP}(U) \text{rank}(\mathcal{H}) = 0$, and hence $\text{EP}(U) = 0$. But

$$\text{EP}(U) = 2 - 2g_X - \#\{x_i \text{ over } 0\} - 1;$$

hence, $2g_X = 1 - \#\{x_i \text{ over } 0\}$. Hence, $g_X = 0$ and there is precisely one point over 0, as well as precisely one point over ∞ . Thus, in suitable coordinates, U is \mathbb{G}_m , $x_\infty = \infty$, $x_0 = 0$, and π is the D 'th power map. At $x_0 = 0$, \mathcal{H} cannot be totally wild (otherwise, $[D]_\star(\mathcal{H})$ would be totally wild at 0), so it must contain some \mathcal{L}_χ . Then, $[D]_\star(\mathcal{H})$ contains $[D]_\star(\mathcal{L}_\chi)$, which cannot be $I(0)$ -trivial unless $D = 1$ (and $\chi = \mathbb{1}$). Thus, $D = 1$, contradiction. ■

Recall from [10, Chapter 1] the notion of $(\mathbf{S}+)$ for the G_{geom} of a local system in its given representation: that it is irreducible, primitive, tensor indecomposable, and not tensor induced.

COROLLARY 2.13. *Suppose that $p \nmid ABa$, the polynomials f and g are both Artin-Schreier reduced, and $\gcd(a, \gcd_{\text{deg}}(f)) = \gcd(a, \gcd_{\text{deg}}(g)) = 1$. Suppose that $A = 1$ and that $A + a = n_0 p^e$ with $0 \leq e \leq 1$ and $1 \leq n_0 < p$. If $e = 0$, suppose further that $A + a$ is not a power. Then $\mathcal{G}(f, g, a, \chi)$ satisfies $(\mathbf{S}+)$.*

PROOF. Indeed, the $I(\infty)$ -representation is tensor indecomposable, cf. Lemma 3.4 later on. Furthermore, if $e = 1$, then $D = A + a$ cannot be a power. Thus, in all cases, $\mathcal{G}(f, g, a, \chi)$ cannot be tensor induced. ■

3. ELEMENTS WITH SPECIAL SPECTRA AND TENSOR INDUCTION

Let $V = \mathbb{C}^d$. We will say an element $g \in \text{GL}(V)$ has *quasi-simple spectrum*, and write that g is a *qsp-element*, if g is diagonalizable, and has at most one repeated eigenvalue but at least two distinct eigenvalues.

PROPOSITION 3.1. *Let $V = V_1 \otimes \cdots \otimes V_n$ be a tensor product of $n \geq 2$ \mathbb{C} -vector spaces each of dimension $d \geq 2$. Suppose $g \in (\mathrm{GL}(V_1) \otimes \cdots \otimes \mathrm{GL}(V_n)) \rtimes \mathfrak{S}_n$ induces a nontrivial permutation π on the set of n tensor factors V_i and that g has simple or quasi-simple spectrum, and finite order on V . Then the following statements hold.*

- (i) *Suppose $d \geq 3$. Then π is either an n -cycle or a 2-cycle.*
- (ii) *If $d = 2$, then π is either an n -cycle, a 2-cycle, a 3-cycle, or a disjoint product of a 2-cycle and a 3-cycle.*

PROOF. Write $\pi = \sigma_1 \sigma_2 \dots \sigma_l$ as a product of disjoint cycles of non-increasing lengths

$$(3.1.1) \quad k_1 \geq k_2 \geq \cdots \geq k_l \geq 1.$$

If $l = 1$, then π is an n -cycle, and we are done. Hence, we will assume $l \geq 2$, and so $\dim(V) = d^n \geq 4$.

First we note that if $g = X \otimes Y$ is tensor decomposable, then both X and Y have simple spectra. Indeed, if X say of size $s \times s$ with $s > 1$ has only a single eigenvalue, then each of the eigenvalues of g repeats $\geq s$ times, contrary to the assumption that g has a simple eigenvalue. Hence, X has at least two distinct eigenvalues $\alpha_1 \neq \alpha_2$. Now if Y admits a multiple eigenvalue $\beta_1 = \beta_2$, then $\alpha_1 \beta_1$ and $\alpha_2 \beta_1$ are two distinct multiple eigenvalues of g , again a contradiction. Hence, Y has a simple spectrum, and similarly does X .

Suitably conjugating g in $\mathrm{GL}(V)$, we may assume that

$$\pi = (1, 2, \dots, k_1)(k_1 + 1, k_1 + 2, \dots, k_1 + k_2) \dots \left(\sum_{i=1}^{l-1} k_i + 1, \sum_{i=1}^{l-1} k_i + 2, \dots, n \right).$$

Now we can write $g = X \otimes Y$, where

$$X \in \mathrm{GL}(V_1 \otimes V_2 \otimes \cdots \otimes V_{k_1 + \dots + k_{l-1}})$$

permutes the $n - k_l$ tensor factors V_1, \dots, V_{n-k_l} , inducing the permutation

$$(1, 2, \dots, k_1)(k_1 + 1, k_1 + 2, \dots, k_1 + k_2) \dots \left(\sum_{i=1}^{l-2} k_i + 1, \sum_{i=1}^{l-2} k_i + 2, \dots, n - k_l \right),$$

and

$$Y \in \mathrm{GL}(V_{n-k_l+1} \otimes V_{n-k_l+2} \otimes \cdots \otimes V_n)$$

inducing the k_l tensor factors cyclically.

By the previous remark, both X and Y have simple spectra, and we may rescale X and Y so that both have finite order. Also, since π is nontrivial, we have $k_1 \geq 2$ and

$k_l \leq k_1$ by (3.1.1). Assume first that $d \geq 3$. Then, applying [10, Proposition 5.2.3], we see that $k_1 = 2, k_2 = \dots = k_{l-1} = 1$. Now if $k_l = 1$, then we arrive at (i). If $k_l = 2$, then we must have $l = 2$ by (3.1.1). In this case, the proof of [10, Lemma 5.2.2] shows that g has at least two distinct multiple eigenvalues on V , a contradiction.

Assume now that $d = 2$. Again applying [10, Proposition 5.2.3], we have that either

- (a) $k_1 = 2$ and $k_2 = \dots = k_{l-1} = 1$, or
- (b) $k_1 = 3$ and $k_2 = \dots = k_{l-1} = 1$, or
- (c) $k_1 = 3, k_2 = 2$, and $k_3 = \dots = k_{l-1} = 1$.

In the case of (a), we cannot have $(l, k_l) = (2, 2)$ again by [10, Lemma 5.2.2]. So $k_l = 1$, and we arrive at (ii).

Suppose we are in the case of (b). If $k_l = 1$, then we arrive at (ii). If $k_l = 2$, then $l = 2$ by (3.1.1), and (ii) holds again. If $k_l = 3$, then $l = 2$ by (3.1.1), and the proof of [10, Lemma 5.2.2] shows that g has at least two distinct multiple eigenvalues on V (namely, $\gamma\delta$ and $\gamma\delta\zeta_3$ in its notation), a contradiction.

Finally, assume we are in the case of (c). If $k_l = 1$, then we arrive at (ii). If $k_l = 2$, then $l = 3$ by (3.1.1), and the proof of [10, Lemma 5.2.2] shows that g has at least two distinct multiple eigenvalues on V , again a contradiction. ■

We rule out the case of n -cycle of Proposition 3.1 in a more special situation.

PROPOSITION 3.2. *Let $r \geq 2$ be a prime and let $V = V_1 \otimes \dots \otimes V_r$ be a tensor product of r \mathbb{C} -vector spaces each of dimension $d \geq 2$. Suppose $g \in (\mathrm{GL}(V_1) \otimes \dots \otimes \mathrm{GL}(V_r)) \rtimes \mathbb{S}_r$ induces an r -cycle on the set of r tensor factors V_i . Assume in addition that g is conjugate to*

$$\mathrm{diag}\left(\underbrace{1, \dots, 1}_{t \text{ times}}, \alpha, \alpha\zeta, \alpha\zeta^2, \dots, \alpha\zeta^{w-1}\right),$$

where $t \geq 2, w \geq 1, \alpha \in \mathbb{C}^\times$ is a root of unity, $\zeta = \exp(2\pi i/w)$, and $(\alpha, w) \neq (1, 1)$. Then $d = r = 2$, and either $(t, w, \alpha) = (3, 1, -1)$ or $t = w = 2$ and $\alpha = \pm 1$.

PROOF. (a) The assumptions imply that g is a qsp-element of finite order, with 1 being the only multiple eigenvalue. Again conjugating g suitably in $\mathrm{GL}(V)$, we may assume that

$$g : V_1 \mapsto V_2 \mapsto \dots \mapsto V_r \mapsto V_1.$$

In particular, g^r induces a semisimple element of $\mathrm{GL}(V_1)$, and thus we can find a basis (e_1^1, \dots, e_d^1) of V_1 in which g^r acts as $\mathrm{diag}(x_1, x_2, \dots, x_d)$ for some roots of unity $x_i \in \mathbb{C}^\times$. Defining $e_j^i = g^{i-1}(e_j^1)$ for $2 \leq i \leq r$ and $1 \leq j \leq d$, we see that (e_1^i, \dots, e_d^i) is a basis of V_i . Now arguing as in the proof of [10, Proposition 5.2.1],

we see that the spectrum of g can be written (counting multiplicities) as

$$(3.2.1) \quad \text{Spec}(g) = \underbrace{\{1, \dots, 1\}}_{t \text{ times}} \sqcup Z = X \sqcup Y,$$

where $X = \{x_1, x_2, \dots, x_d\}$, and Y consists of $(d^r - d)/r$ r -tuples, each being all the r th roots of some $x_{i_1}x_{i_2} \dots x_{i_r}$ with $1 \leq i_1, i_2, \dots, i_r \leq d$ being not all the same, and

$$Z := \{\alpha, \alpha\zeta, \alpha\zeta^2, \dots, \alpha\zeta^{w-1}\}.$$

In particular, Y is stable under the multiplication by the subgroup μ_r of \mathbb{C}^\times .

Suppose $d = r = 2$. Then (3.2.1) shows that

$$x_1 + x_2 = \text{Trace}(g) = t + \alpha \sum_{i=1}^w \zeta^i.$$

If $w = 2$, then we get $t = 2$ and then $2 = x_1 + x_2$, which implies $x_1 = x_2 = 1$ for the roots of unity x_1, x_2 . In this case, $X = \{1, 1\}$ and $Y = \{1, -1\}$, and so $\alpha = \pm 1$. If $w = 1$, then we get $t = 3$ and then $3 = x_1 + x_2 - \alpha$ which implies $x_1 = x_2 = 1 = -\alpha$ for the roots of unity x_1, x_2, α .

(b) We will now assume $(d, r) \neq (2, 2)$, so that

$$(3.2.2) \quad (d^r - d)/r \geq 2.$$

In particular, Y contains at least two μ_r -cosets of $\beta, \gamma \in \mathbb{C}^\times$ (counting multiplicities).

First we show that

$$(3.2.3) \quad r|w.$$

Suppose that $r \nmid w$. Then $|\{\beta, \beta\zeta_r\} \cap Z| \leq 1$, and similarly $|\{\gamma, \gamma\zeta_r\} \cap Z| \leq 1$. It follows from (3.2.1) that at least one of $\beta, \beta\zeta_r$ is 1, which means that the μ_r -coset of β is μ_r . Similarly, the μ_r -coset of γ is μ_r . Thus, Y contains ζ_r twice, and hence $\zeta_r \neq 1$ is a multiple eigenvalue of g , a contradiction.

In particular, the elements in Z sum up to zero, and so

$$(3.2.4) \quad \text{Trace}(g) = t.$$

Next we show that the multi-set Y contains 1 at most once. Indeed, if Y contains 1 at least twice, then since Y is μ_r -stable, Y contains ζ_r at least twice, again a contradiction.

(c) Suppose that X contains 1 at least twice. Then, without loss we may assume $x_1 = x_2 = 1$. Now if $x_j \neq 1$ for some $j > 2$, then Y contains δ at least twice for

$$\delta^r = x_1^{r-1}x_j = x_2^{r-1}x_j \neq 1,$$

and thus $\delta \neq 1$ is a multiple eigenvalue of g , a contradiction. It follows that

$$x_1 = x_2 = \cdots = x_d = 1,$$

which means that g^r acts trivially on V_1 and hence $g^r = \text{id}_V$. The formula for tensor induction [1] and (3.2.4) then show that

$$t = \text{Trace}(g) = d.$$

Note that, in this case, Y consists of $(d^r - d)/r$ copies of μ_r and X contains 1 exactly d times. So the multiplicity of 1 as an eigenvalue of g is

$$d + (d^r - d)/r.$$

But this multiplicity is at most $t + 1 = d + 1$, so we arrive at $(d^r - d)/r \leq 1$, contrary to (3.2.2).

We have therefore shown that X contains 1 at most once. But in (c) we showed that Y also contains 1 at most once. On the other hand, the multiplicity of 1 in $\text{Spec}(g)$ is at least $t \geq 2$. So we conclude that $t = 2$, and each of X and Y contains 1 exactly once. In such a case, the μ_r -invariance of Y implies that $\zeta_r \in Y$. Since $\zeta_r \neq 1$, ζ_r belongs to the set Z which is μ_w -invariant. By (3.2.3), Z is also μ_r -invariant, and hence $1 = (\zeta_r)(\zeta_r)^{-1}$ belongs to Z . But then the multiplicity of 1 in $\text{Spec}(g)$ becomes 3, a contradiction. ■

Next we will prove an auxiliary result on finite permutation groups.

LEMMA 3.3. *Let p be a prime, and let $J = P \rtimes C$ be a transitive subgroup of \mathbf{S}_n with $n > 1$ such that P is a transitive normal p -subgroup and $C = \langle \gamma \rangle$ is a cyclic p' -group. Suppose that every element in the coset γP is either trivial, a 2-cycle, a 3-cycle, or a disjoint product of a 2-cycle and a 3-cycle. Then one of the following statements holds.*

- (i) $p = n = |J| = |P| = 2$.
- (ii) $p = n = |J| = |P| = 3$.
- (iii) $p = n = 3$, $P = C_3$, and $J = \mathbf{S}_3$. Furthermore, γ is a 2-cycle.
- (iv) $n = 4$, $p = 2$, $P = C_2^2$, and $J = \mathbf{A}_4$. Furthermore, γ is a 3-cycle.

PROOF. Let ρ denote the corresponding permutation character of J . Then the transitivity of J means that

$$(3.3.1) \quad \sum_{x \in J} \rho(x) = |J|.$$

Also, since P is transitive, we have

$$(3.3.2) \quad \sum_{x \in P} \rho(x) = |P|, \quad \text{and} \quad n = p^c \leq |P| \text{ for some } c \in \mathbb{Z}_{\geq 1}.$$

(a) First consider the case $J = P$. Then either $p = 2$ and every nontrivial element x in P is a 2-cycle, in which case $\rho(x) = n - 2$, or $p = 3$ and every nontrivial element x in P is a 3-cycle, in which case $\rho(x) = n - 3$. Using (3.3.2), in the former case we have

$$|P| = \sum_{x \in P} \rho(x) = n + (|P| - 1)(n - 2) = 2 + |P|(n - 2),$$

i.e., $|P|(n - 3) = -2$. As $|P| \geq 2$, we must have that $n = 2$ and hence $|P| = 2$, as stated in (i). In the latter case we have

$$|P| = \sum_{x \in P} \rho(x) = n + (|P| - 1)(n - 3) = 3 + |P|(n - 3),$$

i.e., $|P|(n - 4) = -3$. As $|P| \geq 3$, we conclude that $n = 3$ and hence $|P| = 3$, as stated in (ii).

(b) From now on we will assume that $J > P$, i.e., $\gamma \notin P$. By assumption, $\rho(x) \geq n - 5$ for all $x \in \gamma P$; furthermore, $x^6 = 1$, so $J/P \hookrightarrow C_6$. It follows from (3.3.1) and (3.3.2) that

$$(3.3.3) \quad 6|P| \geq |J| = \sum_{x \in J} \rho(x) = \sum_{x \in P} \rho(x) + \sum_{x \in \gamma P} \rho(x) \geq (n - 4)|P|,$$

whence $p^c = n \leq 10$.

Assume in addition that $p \geq 5$. Then in fact we have $n = p \in \{5, 7\}$ and hence $P \cong C_p$. Now

$$\mathbf{N}_{S_p}(P) = P \rtimes \langle \sigma \rangle,$$

where σ is a $(p - 1)$ -cycle; in particular, any $1 \neq \sigma^i$ has a unique fixed point. As $P \triangleleft J$ and $\gamma \notin P$, we have $1 \neq \sigma^j \in \gamma P$ for some $j \in \mathbb{Z}$. Thus, σ^i is either a 2-cycle, a 3-cycle, or a disjoint product of a 2-cycle and a 3-cycle, none of which can have exactly one fixed point.

Now we consider the case $p = 3$. As $\gamma \neq 1$ is a 3'-element, it must be a 2-cycle, and thus $\gamma^2 = 1$. It follows that $J/P = C_2$, so instead of (3.3.3) we now have

$$2|P| = |J| = \sum_{x \in J} \rho(x) = \sum_{x \in P} \rho(x) + \sum_{x \in \gamma P} \rho(x) \geq (n - 4)|P|.$$

Thus, $3^c = n \leq 6$. It follows that $n = 3$, $P = C_3$ and $J = S_3$ as $J > P$, and we arrive at (iii).

Finally, let $p = 2$. As $\gamma \neq 1$ is a $2'$ -element, it must be a 3-cycle, and thus $\gamma^3 = 1$. It follows that $J/P = C_3$. Furthermore, any element $x \in J$ belongs to γP if and only if $x^{-1} \in \gamma^{-1}P$, and so we also have $\rho(y) \geq n - 5$ for all $y \in \gamma^{-1}P$. So instead of (3.3.3) we now have

$$3|P| = |J| = \sum_{x \in J} \rho(x) = \sum_{x \in P} \rho(x) + \sum_{x \in \gamma P} \rho(x) + \sum_{x \in \gamma^{-1}P} \rho(x) \geq (2n - 9)|P|.$$

Thus, $2^c = n \leq 6$. The case $n = 2$ is impossible since $J > P \geq C_2$. So $n = 4$. Since the subgroup P of a Sylow 2-subgroup of S_4 , which is dihedral of order 8, is normalized by the 3-cycle γ , we conclude that $P \cong C_2^2$ and so $J = A_4$, as stated in (iv). ■

Now we establish some basic lemmas about tensor indecomposability and lack of tensor induction for $I(\infty)$ of ℓ -adic local systems. Recall that a representation ρ of a group G on a vector space V is said to be tensor decomposable if there exists a vector space isomorphism $V \cong V_1 \otimes V_2$ with both $\dim(V_i) \geq 2$ such that the image $\rho(g) \in \text{GL}(V)$ of each element $g \in G$ can be written as $A_1 \otimes A_2$ with each $A_i \in \text{GL}(V_i)$. This notion of tensor decomposability of a representation ρ does not in general imply that ρ is a tensor product of representations $\rho_1 \otimes \rho_2$.

LEMMA 3.4. *Let \mathcal{F} be an irreducible $I(\infty)$ -representation of rank $D \geq 2$ all of whose slopes are N/D with $N \geq 1$ and $\gcd(N, D) = 1$. Suppose further that $p^2 \nmid D$. Then \mathcal{F} is tensor indecomposable.*

PROOF. By (the $I(\infty)$ -version of) [9, Proposition 2.2], if \mathcal{F} is tensor decomposable, it is also linearly tensor decomposable, and hence we can write it as $\mathcal{A} \otimes \mathcal{B}$ where both \mathcal{A}, \mathcal{B} are $I(\infty)$ -representations of dimensions ≥ 2 . Because $p^2 \nmid D$, at least one of \mathcal{A}, \mathcal{B} has dimension prime to p , say \mathcal{A} has dimension prime to p . By the argument proving [9, Proposition 2.2(ii)], we may do so in such a way that \mathcal{A} has $G_{\text{geom}} \leq \text{SL}_{\dim(\mathcal{A})}$, and then infer that both \mathcal{A}, \mathcal{B} have all slopes $\leq N/D$. Each of \mathcal{A}, \mathcal{B} is irreducible (otherwise, their tensor product is reducible). Let λ be the unique slope of \mathcal{A} (unique because \mathcal{A} is $I(\infty)$ -irreducible). Then for $d := \dim(\mathcal{A})$, $d\lambda \in \mathbb{Z}$. This integrality shows that $\lambda < N/D$; indeed, if $\lambda = N/D$, then $dN/D \in \mathbb{Z}$ with $d < D$, impossible because $\gcd(N, D) = 1$. Similarly, \mathcal{B} has unique slope $\mu < N/D$, and hence $\mathcal{A} \otimes \mathcal{B}$ has slopes $\leq \sup(\lambda, \mu) < N/D$, contradiction. ■

LEMMA 3.5. *Let \mathcal{F} be an $I(\infty)$ -representation of rank $D \geq 2$ all of whose slopes are N/D with $N \geq 1$ and $\gcd(N, D) = 1$. Then \mathcal{F} is $I(\infty)$ -irreducible.*

PROOF. Indeed, any nonzero irreducible subrepresentation V has all slopes N/D , and the product $\dim(V) \times N/D \in \mathbb{Z}$, impossible if $\dim(V) < D$. ■

Combining Lemmas 3.4 and 3.5, we get the following corollary.

COROLLARY 3.6. *Suppose $\gcd(a, A) = 1$ and $p^2 \nmid D := A + a$. Then $\mathcal{G}(f, g, a, \chi)$ is both $I(\infty)$ -irreducible and $I(\infty)$ -tensor indecomposable.*

PROOF. Here the $I(\infty)$ -slopes are $A/(A + a)$ with $\gcd(A, a + A) = 1$. ■

LEMMA 3.7. *Let \mathcal{F} be an irreducible $I(\infty)$ -representation of rank $D \geq 2$ all of whose slopes are N/D with $N \geq 1$ and $\gcd(N, D) = 1$. Suppose further that $p^2 \nmid D$. Suppose that $D = d^n$ with $n \geq 2$, $n < p$, and $\gcd(n, D) = 1$. Then \mathcal{F} is not n -tensor induced.*

PROOF. If \mathcal{F} were n -tensor induced, the map $I(\infty) \mapsto S_n$, giving the action on the tensor factors, is trivial on $P(\infty)$ simply because $P(\infty)$ is a pro- p group while S_n for $n < p$ has order prime to p . So the image of $I(\infty)$ is a cyclic subgroup of S_n , generated by the image π of a chosen element $\gamma \in I(\infty)$ which generates $I(\infty)/P(\infty)$. We first claim that π is an n -cycle. If not, write it as a product of disjoint cycles to see that γ preserves a tensor decomposition, and (hence) that every power of γ , times any element of $P(\infty)$, preserves this same tensor decomposition. Thus, the entire group $I(\infty)$ preserves this tensor decomposition. By Lemma 3.4, this contradicts the tensor indecomposability of \mathcal{F} . Once γ induces an n -cycle, γ (and then the entire group $I(\infty)$) preserves the tensor decomposition of the Kummer pullback $[n]^*\mathcal{F}$. But this Kummer pullback $[n]^*\mathcal{F}$ has rank D and all slopes nN/D , so it is irreducible when $\gcd(n, D) = 1$ (because then $\gcd(nN, D) = 1$) and hence is tensor indecomposable, the desired contradiction. ■

LEMMA 3.8. *Suppose \mathcal{F} is an $I(\infty)$ -representation of the form $T \oplus W$, with T tame of rank $t \geq 1$ and with W irreducible of rank $w \geq 1$ with all slopes m/w with $m \geq 1$ and $\gcd(m, w) = 1$. Suppose further that $t + w \neq 4$. Suppose that $D := t + w$, the rank of \mathcal{F} , is a power $D = d^n$ with $n \geq 2$, $n < p$, and $\gcd(n, w) = 1$. Then \mathcal{F} is not n -tensor induced.*

PROOF. By [6, Corollary 10.4], \mathcal{F} is tensor indecomposable. If it were n -tensor induced, then precisely as in the proof of Lemma 3.7, the image of γ must be an n -cycle. Then $[n]^*\mathcal{F}$ is tensor decomposed. But $[n]^*\mathcal{F} = [n]^*T \oplus [n]^*W$. Here $[n]^*T$ is tame of the same rank t , and $[n]^*W$ has rank w and all slopes nm/w . Then $[n]^*W$ is irreducible by Lemma 3.5, and by [6, Corollary 10.4], $[n]^*\mathcal{F}$ is tensor indecomposable, the desired contradiction. ■

LEMMA 3.9 (Compare to [9, Lemma 3.2]). *Suppose $A, a \geq 1$, and \mathcal{F} an $I(\infty)$ -representation of rank $D := A + a$ all of whose slopes are $A/(A + a)$. Suppose that \mathcal{F} is tensor indecomposable over $I(\infty)$. Suppose that \mathcal{F} is n -tensor induced for*

some $n \geq 2$. Consider the map $\phi : I(\infty) \rightarrow \mathbb{S}_n$ giving the action on the tensor factors. If $(n-2)A < a$, then ϕ is trivial on $P(\infty)$, and the image of ϕ is the cyclic group generated by an n -cycle. Moreover, n is prime to p .

PROOF. To show that ϕ is trivial on $P(\infty)$, view $\mathbb{S}_n \leq \mathbb{O}_{n-1}$ by the deleted permutation representation. It suffices to show that $\phi : I(\infty) \rightarrow \mathbb{O}_{n-1}$ has $\text{Swan}_\infty < 1$. Note that

$$\text{Swan}_\infty \leq (n-1)(\text{the largest slope of } \mathcal{F}) = (n-1)A/(A+a).$$

Thus, $\text{Swan}_\infty < 1$ is the condition

$$(n-1)A < A+a, \quad \text{i.e.,} \quad (n-2)A < a.$$

Let $\gamma \in I(\infty)$ be a generator of $I(\infty)/P(\infty)$. Then the image of ϕ is the cyclic subgroup of \mathbb{S}_n generated by $\phi(\gamma)$. If $\phi(\gamma)$ were not an n -cycle, it (and every power of it, and hence every element of $I(\infty)$) would preserve some given tensor decomposition of \mathcal{F} , contradicting the tensor indecomposability of \mathcal{F} over $I(\infty)$. Because $I(\infty)/P(\infty)$ has (pro) order prime to p , its image under ϕ has order prime to p . ■

REMARK 3.10. In the above Lemma 3.9, the condition $(n-2)A < a$ is always satisfied for $n = 2$. For the extreme case $A = 1$, the condition is $n < a + 2$, which is satisfied for all $n \geq 2$. (Indeed, the rank $D = a + 1$ must be an n th power, say $a + 1 = d^n$. Now if $n \geq a + 2 > a + 1$, then $n > d^n$, which is false for all $n \geq 1$ and all $d \geq 2$: worst case being $d = 2$, for which $2^n \geq n + 1$.) If in this $A = 1$ case we also had $p^2 \nmid (a + 1)$ and both $\gcd(n, 1 + a) = 1$ and $n < p$ whenever $1 + a = d^n$, then we would rule out \mathcal{F} being tensor induced, by using Lemma 3.7. (But already for $n = 2$, we have a problem when $1 + a$ is an even square.)

THEOREM 3.11. Suppose $p \nmid ABA$, f, g are both Artin–Schreier reduced, and

$$\gcd(a, \gcd_{\deg}(f)) = 1, \quad \gcd(a, \gcd_{\deg}(g)) = 1.$$

Suppose that $\mathcal{G} = \mathcal{G}(f, g, a, \chi)$ is n -tensor induced as a representation of $\pi_1(\mathbb{G}_m/\overline{\mathbb{F}}_p)$ for some $n \geq 2$. Assume in addition that $p \nmid w = a - B$ and $D = a + A > 4$. Then all the following conditions hold.

- (i) Either $n = p = 3$ or $(D, n, p) = (16, 4, 2)$.
- (ii) \mathcal{G} is tensor decomposable over $I(\infty)$.
- (iii) If in addition $\gcd(a, A) = 1$, then $p^n \mid D$.

PROOF. By assumption, $G = G_{\text{geom}}$ stabilizes a tensor-induced decomposition $V = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ of the underlying representation V , with $d := \dim(V_i)$ and $D = d^n$.

Let $\pi : G \rightarrow S_n$ denote the permutation representation of G while acting on the n tensor factors of V . By Proposition 2.8, \mathcal{G} is tensor indecomposable over $I(0)$; hence, $\pi(I(0))$ is a transitive subgroup of S_n . Furthermore, since $D > 4$, we must have $w > 1$ by Proposition 2.9 (b).

Fix a p' -generator γ of $I(0)$ over $P(0)$, and write $\pi(I(0)) = J = P \rtimes C$, where $P = \pi(P(0))$ and $C = \langle \pi(\gamma) \rangle$. By Lemma 2.1, the condition $p \nmid w$ implies that the action of γ on V has spectrum

$$(3.11.1) \quad \text{diag}(\underbrace{1, \dots, 1}_{t \text{ times}}, \alpha, \alpha\zeta, \alpha\zeta^2, \dots, \alpha\zeta^{w-1}),$$

where $t = A + B \geq 2$, $w = a - B \geq 1$, $\alpha \in \mathbb{C}^\times$ is a root of unity, $\zeta = \exp(2\pi i/w)$. As $w > 1$, $\gamma|_V$ is a qsp-element. In fact, this also holds for any element in the coset $\gamma P(0)$ for the same reason.

Since $J \leq S_n$ is transitive and $P \triangleleft J$, P acts on the set $\{V_1, V_2, \dots, V_n\}$ with $e \geq 1$ orbits $\Omega_1, \dots, \Omega_e$, all of length n/e and permuted cyclically by $\pi(\gamma)$. Suppose that $e > 1$. Letting U_j be the tensor product of the V_i in Ω_j for $1 \leq j \leq e$, we see that γ permutes the e tensor factors of the decomposition $V = U_1 \otimes U_2 \otimes \dots \otimes U_e$ cyclically, say

$$U_1 \mapsto U_2 \mapsto U_3 \mapsto \dots \mapsto U_e \mapsto U_1.$$

Choosing a prime divisor r of e , we see that γ permutes the r sets

$$\Delta_j := \{U_i \mid 1 \leq i \leq r, i \equiv j \pmod{r}\}, \quad 1 \leq j \leq r,$$

cyclically. Letting W_j be the tensor product of the U_i in Δ_j , $1 \leq j \leq r$, we now have that the element γ with spectrum (3.11.1) permutes the r tensor factors of the decomposition $V = W_1 \otimes W_2 \otimes \dots \otimes W_r$ cyclically. But this is impossible by Proposition 3.2 and the assumption that $D > 4$.

Thus, $P \leq S_n$ is a transitive subgroup; in particular, $n = p^c$ for some $c \geq 1$. By Proposition 3.2 applied to any $\gamma' \in \gamma P(0)$, $\pi(\gamma')$ cannot be an n -cycle (because $D > 4$). Applying Proposition 3.1 to γ' , we see that any element in the coset $\pi(\gamma)P$ is either trivial, a 2-cycle, a 3-cycle, or a disjoint product of a 2-cycle and a 3-cycle. Hence, we can apply Lemma 3.3 to J .

In the cases of Lemmas 3.3 (i) or 3.3 (ii), $\pi(\gamma)P = P$, and so some element $\gamma_1 \in \gamma P(0)$ has $\pi(\gamma_1)$ being an n -cycle, a contradiction. Thus, we are in the case of Lemma 3.3 (iii), whence $n = p = 3$, or of Lemma 3.3 (iv), whence $(n, p) = (4, 2)$. Observe that in the latter case $D = 16$. Indeed, in this case $\pi(\gamma)$ is a 3-cycle, so we may write the action of γ as $X \otimes Y$, where $X \in \text{GL}(V_1 \otimes V_2 \otimes V_3)$ permutes V_1, V_2, V_3 cyclically, and $Y \in \text{GL}(V_4)$. By the proof of Proposition 3.1, X has simple spectrum. Applying Proposition 3.1 to X , we see that $d = 2$ and hence $D = 16$. Thus, we have proved (i).

In these two remaining cases, we now show that

$$(3.11.2) \quad a > (n - 2)A.$$

Assume we are in the case of Lemma 3.3 (iii), so that $\pi(\gamma)$ is a 2-cycle and $D = d^3$. Since $\gamma|_V$ has finite order and flips, say, V_1 and V_2 , the formula for tensor induction [1] shows that

$$|\text{Trace}(\gamma|_V)| \leq d^2 \leq D/2.$$

Now, (3.11.1) implies that $\text{Trace}(\gamma|_V) = t$, whence $t + w = D \geq 2t$, and so

$$a - B = w \geq t = A + B,$$

implying (3.11.2).

Next suppose we are in the case of Lemma 3.3 (iv), so that $\pi(\gamma)$ is a 3-cycle and $D = d^4 = 16$. Since $\gamma|_V$ has finite order and permutes, say, V_1 , V_2 , and V_3 , cyclically, the formula for tensor induction [1] shows that

$$|\text{Trace}(\gamma|_V)| \leq d^2 = D/4.$$

Now, (3.11.1) implies that $\text{Trace}(\gamma|_V) = t$, whence $t + w = D \geq 4t$, and so

$$a - B = w \geq 3t = 3A + 3B,$$

implying (3.11.2).

Thus, we have proved (3.11.2). Now, if \mathcal{G} is tensor indecomposable over $I(\infty)$, then the equality $n = p$ contradicts Lemma 3.9. So \mathcal{G} is tensor decomposable over $I(\infty)$, proving (ii).

Assume in addition that $\gcd(a, A) = 1$. Then the tensor decomposability over $I(\infty)$ of \mathcal{G} implies by Lemma 3.4 that $p|D$. But $D = d^n$, so $p^n|D$, establishing (iii). ■

COROLLARY 3.12. *Suppose $p \nmid ABa$, f , g are both Artin–Schreier reduced, and $\gcd(a, \gcd_{\deg}(f)) = 1$, $\gcd(a, \gcd_{\deg}(g)) = 1$. Suppose in addition that $p \nmid w = a - B$ and $D = a + A > 4$. If $\mathcal{G}(f, g, a, \chi)$ is primitive (e.g., by Theorem 2.10), then it satisfies (S+) if either $p \geq 5$ or $p^2 \nmid D$.*

REMARK 3.13. In cases when $p|w$, there are other ways to prove (S+). We can sometimes apply Theorems 2.11 or 2.12 to prove primitivity, and we can sometimes apply Propositions 2.8 and 2.9 to prove the absence of tensor induction.

ACKNOWLEDGMENTS. – The authors are grateful to the referee for a very careful reading of the paper and a number of helpful comments and suggestions.

FUNDING. – The second author gratefully acknowledges the support of the NSF (grant DMS-2200850) and the Joshua Barlaz Chair in Mathematics.

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Received 6 November 2024

Nicholas M. Katz
Department of Mathematics, Princeton University
Princeton, NJ 08544, USA
nmk@math.princeton.edu

Pham Huu Tiep
Department of Mathematics, Rutgers University
Piscataway, NJ 08854, USA
tiep@math.rutgers.edu