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Algebraic Geometry and Number Theory. – *Cubic surfaces with infinite, discrete automorphism group*, by János Kollár and David Villalobos-Paz, accepted on 1 July 2025.

Abstract. – We prove that the automorphism group of an affine, cubic surface with equation $xyz = g(x, y)$ contains $\mathbb{Z}$ as a finite index subgroup. These equations were first studied by Jacobsthal (1939) and Mordell (1952).

Keywords. – cubic surface, automorphism group, integral point.

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1. Introduction

While the automorphism group of a smooth, projective, cubic surface $\overline{S}$ is always finite, the automorphism group of a smooth, affine, cubic surface S can be infinite. The Markov-type equations $xyz = x^2 + y^2 + z^2 + c$ give the best known examples.

In general, non-linear automorphisms of a smooth, cubic surface $S \subset \mathbb{A}^3$ seem to be controlled by the singularities of the pair $(\overline{S}, \overline{B})$, where $\overline{B} := \overline{S} \setminus S$ is the *curve at infinity*. If $\overline{B}$ is smooth, then every automorphism of S is linear, and $\operatorname{Aut}(S)$ is finite. If $\overline{B}$ has a unique singular point and $\overline{S}$ is smooth, then $\operatorname{Aut}(S)$ is finite but contains a non-linear involution; see [19, Paragraph 18]. However, if $\overline{B}$ has 2 or 3 singular points and $\overline{S}$ is smooth, then $\operatorname{Aut}(S)$ is infinite. These are the Markov-type surfaces; see Paragraph 11. No other smooth, affine, cubic surface with infinite, discrete automorphism group seems to have been known.

For cubic surfaces $S_g := (xyz = g(x, y))$, the curve at infinity is typically an irreducible, nodal cubic, and $\overline{S}_g$ has an A_2 singularity at the node. We show that, over any field, all of these have an infinite automorphism group and describe all isomorphisms between them. The answer—given in Theorem 28—turns out to be quite subtle, and less complete over $\mathbb{Z}$. The following are representative examples.

Example 1. For $S_0 := (xyz = x^3 + y^3 + 1) \subset \mathbb{A}^3_{\mathbb{Z}}$, the following hold.

(1) (Generators) $\operatorname{Aut}_{\mathbb{Z}}(S_0)$ is generated by 3 involutions

$$\tau : \quad (x, y, z) \mapsto (y, x, z),$$

$$\begin{aligned} \sigma_x : \quad & (x,y,z) \mapsto \big(x, xz - y^2, xz^2 - y^2 z - x^2 y\big), \\ \sigma_y : \quad & (x,y,z) \mapsto \big(yz - x^2, y, yz^2 - x^2 z - xy^2\big). \end{aligned}$$

(2) (Structure) $\mathrm{Aut}_{\mathbb{Z}}(S_0) \cong \mathbb{Z} \rtimes \mathbb{Z}/2$.

(3) (Fundamental domain) $\{(x,y,z) : x \le y \le (x^3+1)^{1/2}\}$ is a fundamental domain for the action of $\mathrm{Aut}_{\mathbb{Z}}(S_0)$ on $S_0 \cap (\mathbb{R}^+)^3$.

EXAMPLE 2. Set $T_{n,m} := (xyz = x^3 + y^3 + nx + my + 1)$ for $n, m \in \mathbb{Z} \setminus \{0\}$. Then

$$\mathrm{Aut}_{\mathbb{Z}}(T_{n,m}) = \mathrm{Aut}_{\mathbb{C}}(T_{n,m}) \cong \begin{cases} \mathbb{Z} & \text{if } n \neq m, \\ \mathbb{Z} \rtimes \mathbb{Z}/2 & \text{if } n = m. \end{cases}$$

In their simplest form, the coordinate functions of the generator of $\mathbb{Z}$ are given by polynomials of degrees 13, 34, and 55, containing 110, 998, and 2881 monomials; see Remark 32 (4).

We also study equations of the form $xyz = g(x,y)$ for higher degree polynomials g, which were also considered in [29, 34, 35].

THEOREM 3. *Let R be an integral domain, $g(x,y) \in R[x,y]$, and set*

$$S_g := \big(xyz = g(x,y)\big) \subset \mathbb{A}^3_R. \tag{3.1}$$

Assume that g has bidegree (n,m) with $n, m \ge 3$, and the coefficients of $x^n, y^m, 1$ are units in R. Then there is an explicit, infinite order automorphism σ_g of S_g, defined over R, to be constructed in Paragraphs 26–27.

PARAGRAPH 4 (Properties of σ_g). If g has bidegree $(3,3)$, then no 1-dimensional, closed subscheme of S_g is invariant under σ_g. Thus, σ_g is loxodromic (as in [6]) and in many cases this implies that integral points are Zariski dense; see Paragraph 50. Most likely the same holds for all $n, m \ge 3$.

By contrast, if g has bidegree $(2,2)$, then S_g is a Markov type surface (Paragraph 11) and σ_g preserves the pencil of curves $(\lambda x + \mu y = 0)$; thus, σ_g is parabolic. Integral points are sometimes not Zariski dense; see [30, p. 297] and [35].

We believe that $\langle \sigma_g \rangle$ has finite index in $\mathrm{Aut}_R(S_g)$, and in most cases $\mathrm{Aut}_R(S_g) = \langle \sigma_g \rangle$. We prove these for cubic polynomials in Notation 29 and Corollary 32.

PARAGRAPH 5 (Steps of the proof). For Example 1, the involutions are constructed in Paragraph 21. It is then easy to see that the subgroup $\langle \sigma_x, \sigma_y, \tau \rangle$ is isomorphic to D_∞ (Paragraph 22), and its fundamental domain is as claimed (Paragraph 23). Most of the paper is then devoted to proving that there are no other automorphisms. This is obtained in Corollary 31, as a special case of Theorem 28, which studies the surfaces (3.1) for cubic polynomials $g(x, y)$. The proof of Theorem 28 is outlined in Paragraph 33 and completed in Paragraph 46.

For Example 2 and Theorem 3, we construct the analogs of σ_x, σ_y in Paragraph 26 and prove Theorem 3 in Paragraph 27. We have a complete description of all isomorphisms only for $\deg g = 3$; see Theorem 28.

PARAGRAPH 6 (Previous results on automorphisms). [39, Lemma 4.2] showed the existence of infinite order automorphisms of $P_0 := \mathbb{P}^2 \setminus (xyz = x^3 + y^3)$. Explicit, degree 8 polynomials defining these automorphisms are given in [31, Section 6]. These maps were generalized to other Del Pezzo surfaces in [26].

It was observed in [19] that suitable variants of these maps are Geiser-type involutions (Paragraph 20), occurring in pairs $\sigma_\pm$. Since $\sigma_\pm$ are involutions,

$$\sigma_\pm^2(x{:}y{:}z) = \Phi_\pm(x, y, z) \cdot (x{:}y{:}z),$$

for some polynomials $\Phi_\pm$. Working over $\mathbb{Z}$ suggests that the $\Phi_\pm$ should be powers of $xyz - x^3 - y^3$. We computed, and indeed

$$\Phi_\pm(x, y, z) = (xyz - x^3 - y^3)^{21}.$$

The integral points of P_0 are given by $(x, y, z) \in \mathbb{Z}^3$ that satisfy $xyz - x^3 - y^3 = \pm 1$. Thus, the $\sigma_\pm$ map integral points of P_0 to integral points.

The cubic surface S_0 in Example 1 is the universal cover of P_0. The covering map, given by $(x, y, z) \mapsto (x{:}y{:}z)$, has degree 3, and the $\sigma_\pm$ lift to automorphisms of S_0. Surprisingly, in the latter incarnation one can take a 'twisted square root' of the $\sigma_\pm$; these are the automorphisms σ_x, σ_y in Example 1 (1). By explicit computation,

$$\sigma_+ = \sigma_x \circ \tau \circ \sigma_x \quad \text{and} \quad \sigma_- = \sigma_y \circ \tau \circ \sigma_y. \tag{6.1}$$

Building on these results, [19] classifies all smooth Del Pezzo surfaces $\bar{S}$ and nodal curves $\bar{B} \subset \bar{S}$ such that $\operatorname{Aut}(\bar{S} \setminus \bar{B})$ is infinite. In all cases, $\deg \bar{S} \geq 4$.

Formulas (52)–(53) in [34] also lead to σ_g in Theorem 3 though there they are viewed as maps between different surfaces over $\mathbb{Z}$.

In many papers, the question of unexpected isomorphisms is approached from the other direction, as the search for unexpected compactifications. The best studied example is $\mathbb{A}^2$. For compactifications with Du Val singularities, see [28], while [33] classifies even more general cases.

The constructions can be generalized to give isomorphisms between some other affine Del Pezzo surfaces and exclude other possibilities. Currently, not all cases are covered, but we consider the following likely.

CONJECTURE 7. *Let S be a smooth, affine, cubic surface whose automorphism group is infinite and discrete. Then S is*

(1) *either a Markov-type surface (Paragraph 11),*

(2) *or the curve at infinity $\overline{B}$ is nodal, and $\overline{S}$ is singular at the node.*

Integral points

[17] studied $xyz = g(x, y)$ when g is symmetric in x, y. [29] shows that $xyz = ax^3 + by^3 + c$ always has infinitely many integer solutions if $|ab| > 1$, for example,

$$x = bu^8 + 3abc^2u^5 + 3a^2bc^4u^2 + c, \quad y = u^3 + ac^2, \quad \text{with } u := a^3bc^5 + 1.$$

For S_0, this gives $(365, 9, 14803)$.

[29] and [30, Section 30.3] state (with barely a hint of proof) that the same holds for all $xyz = g(x, y)$. This turns out to be not completely true; see [17] when $\deg g = 2$. Mordell's example was turned into a proof in [34, 35], settling the majority of cases. The cases $\deg g = 3$ are fully treated in [20]. It is not known whether there are higher degree exceptional cases.

For the growth rate of the number of solutions, we propose the following.

CONJECTURE 8. *The number of integral points on S_0 satisfying $1 \le x, y \le N$ grows like $c \log^2 N$.*

Numerical searches suggest that the true value of c should be close to 1. We prove in Corollary 48 (5) that the number of integral points is at least $\varepsilon \log N \log\log N$ for some (very small) $\varepsilon > 0$.

For the upper bound, note first that the number of integral points satisfying $-N \le x, y \le N$ is asymptotically $6N$, but almost all of these points lie on 3 lines; see Example 51. If we remove these lines, the number of points should grow like $cN^{1/2}$, but again most of these points should lie on some other curves $\mathbb{A}^1 \to S_0$.

Thus, the logarithmic upper bound in Conjecture 8 implies that, for every $\mathbb{A}^1 \to S_0$ defined over $\mathbb{Z}$, the image has finite intersection with $(\mathbb{R}^+)^3$. In all examples that we know, the image is disjoint from $(\mathbb{R}^+)^3$; see Section 8 for details.

For all equations as in (3.1), [4, 14, 38] suggest that the number of integral points on S_g, not lying on the image of any curve $\mathbb{A}^1 \to S_g$, and satisfying $-N \le x, y \le N$, should grow like a power of $\log N$.

Unfortunately, Conjecture 8 and [4, 38] suggest 3 different exponents for $\log N$; see Paragraph 49 for a discussion.

Composing the lines in S_0 with $\mathrm{Aut}_{\mathbb{Z}}(S_0)$ gives infinitely many $\mathbb{A}^1 \to S_0$ defined over $\mathbb{Z}$, given by higher and higher degree polynomials. Thus, we need to remove infinitely many curves in order to have a chance for logarithmic growth rate if negative values of x, y are allowed.

2. Cubic surfaces

Notation 9. From now on, $S := (h(x, y, z) = 0) \subset \mathbb{A}^3$ denotes an affine cubic surface. Its projective closure is denoted by $\overline{S} \subset \mathbb{P}^3$, and $\overline{B} := \overline{S} \setminus S$ is the *curve at infinity*. It is defined by the homogeneous cubic part $h_3(x, y, z)$ of $h(x, y, z)$.

To avoid degenerate cases, we assume that $\overline{S}$ has at worst Du Val singularities. Equivalently, $\overline{S}$ is normal and not a cone.

Paragraph 10 (Linear automorphisms). For an affine cubic surface S, the group of *linear automorphisms*, denoted by $\mathrm{Aut}^{\mathrm{lin}}(S)$, is the largest subgroup of $\mathrm{Aut}(\overline{S})$ that maps S to itself. Equivalently, it maps the plane at infinity to itself. This is thus a subgroup of PGL_4.

The automorphism group of a smooth, projective cubic surface is finite; all possibilities are listed in [9, Section 9.5].

If $\overline{S}$ has Du Val singularities, then $\mathrm{Aut}(\overline{S})$ can be positive dimensional. All these are listed in [8]. There are 9 isolated cubics and a 1-dimensional family. For example, the surface $S_{14} := (x_0x_2^2 = x_1^3 + x_3x_0^2)$ has an E_6 singularity at (0:0:0:1) and admits a $\mathbb{G}_m$-action

$$(x_0{:}x_1{:}x_2{:}x_3) \mapsto (x_0{:}\lambda^2x_1{:}\lambda^3x_2{:}\lambda^6x_3).$$

Thus, all 3 of the affine surfaces $S_{14} \setminus (x_i = 0)$ (for $i = 0, 1, 2$) are smooth with a $\mathbb{G}_m$-action. As another example, $(xyz = 1)$ is isomorphic to $\mathbb{G}_m^2$, and its automorphism group is $\mathbb{G}_m^2 \rtimes \mathrm{GL}_2(\mathbb{Z})$.

For $S_0 := (xyz = x^3 + y^3 + 1)$, the obvious automorphism is the involution $\tau : (x : y : z) \mapsto (y : x : z)$.

If k contains a 3rd root of unity $\varepsilon \neq 1$, then $\mathrm{Aut}^{\mathrm{lin}}(S_0)$ also has a subgroup $\mathbb{Z}/3 \oplus \mathbb{Z}/3$ generated by the diagonal matrices $\mu = (\varepsilon, \varepsilon, \varepsilon)$ and $\rho := (\varepsilon, \varepsilon^2, 1)$. In this case, $\mathrm{Aut}^{\mathrm{lin}}(S_0)$ has order 18.

Paragraph 11 (Markov-type surfaces). Let $S \subset \mathbb{A}^3$ be a cubic surface with projective closure $\overline{S} \subset \mathbb{P}^3$. Let $p \in \overline{S} \setminus S$ be a point at infinity. Projecting S from p gives a birational map $\pi_p : \overline{S} \dashrightarrow \mathbb{P}^2$ if p is a double point, and a double cover if p is a smooth point. In the latter case let $\tau_p : \overline{S} \dashrightarrow \overline{S}$ be the corresponding Galois involution. Note that τ_p contracts $\overline{S} \cap T_p\overline{S}$ to p, and τ_p gives an automorphism of S iff the tangent plane $T_p\overline{S}$ is the plane at infinity, equivalently, iff p is a singular point of $\overline{B}$.

If $p = (0{:}0{:}1{:}0)$, then the equation of S can be written as $z^2 + c_2(x, y)z + c_3(x, y) = 0$, where c_i has degree $\le i$. Then

$$\tau_p : (x, y, z) \mapsto \big(x, y, c_2(x, y) - z\big). \tag{11.1}$$

This is a non-linear automorphism if $\deg c_2 = 2$, but a linear one if $\deg c_2 \le 1$.

We call S a *Markov-type* surface if $\bar{B}$ has 2 or 3 nodes, and $\bar{S}$ is smooth at 2 or 3 of them. (Other authors have different definitions.)

For the classical Markov equation $3xyz = x^2 + y^2 + z^2$—discussed in [24,25]—the curve at infinity is $(xyz = 0)$, with 3 singular points. Each of them gives an involution; they generate a subgroup isomorphic to $\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$. The automorphism group seems to have been fully determined only in [10,32]. Automorphism groups and integral points of Markov-type surfaces are further studied in [7,12].

If the curve at infinity is the union of a line and a conic meeting at 2 points, then we get the free product of 2 involutions. In all other cases, the curve at infinity has at most 1 singular point, giving a single, order 2 subgroup of $\mathrm{Aut}(S)$.

3. CUBIC SURFACES WITH AN A_2 SINGULARITY

PARAGRAPH 12. Let R be an integral domain with quotient field K. We study cubic surfaces $S \subset \mathbb{A}^3_R$ such that

(1) the curve at infinity $\bar{B}_K$ is geometrically irreducible, nodal, and

(2) $\bar{S}_K$ has an A_2 singularity at the node.

In suitable coordinates, the equation of $\bar{S}$ is $g_2(x, y)z = g_3(x, y, w)$, where g_i is homogeneous of degree i. Then Paragraph 12 (2) holds iff $g_3(0, 0, 1) \neq 0$ and Paragraph 12 (1) holds where $g_2(x, y)$ and $g_3(x, y, 0)$ are relatively prime.

NOTATION 13. Let R be an integral domain. We let $P_n(R)$ denote the set of polynomials $a(t) := a_n t^n + \cdots + a_0$ satisfying $a_n \neq 0 \neq a_0$. Two such polynomials $a(t)$ and $b(t)$ are $\mathbb{G}_m$*-equivalent* iff $b(t) = \lambda_1 a(\lambda_2 t)$ for some $\lambda_1, \lambda_2 \in R^\times$.

Let $P_n^\times(R) \subset P_n(R)$ denote the subset where a_n, a_0 are units. Thus, $P_n^\times(R) = P_n(R)$ if R is a field.

DEFINITION 14. Let k be a field and consider a singularity Z defined by an equation $g_2(x_1, \dots, x_n) +$ (higher terms) where g_2 is a homogeneous quadric of rank 2. We say that Z is *split* if g_2 is the product of 2 linear forms over k. We use this notion for nodes ($n = 2$) and for A_m singularities ($n = 3$ and $m \geq 2$).

PARAGRAPH 15 (Normal form for the equation). Let k be a field and $\bar{S} \subset \mathbb{P}^3_{xyzw}$ a cubic surface as in Paragraph 12.

If the A_2 singularity is split, we can choose $g_2 = xy$. Then, by changing z, we can eliminate from g_3 all terms divisible by xy. That is, we have a cubic form g_3 restricted to $(xy = 0)$. We write these as

$$(a_3x^3 + a_2x^2w + a_1xw^2 + a_0w^3) + (b_3y^3 + b_2y^2w + b_1yw^2 + b_0w^3) - cw^3,$$

where $c = a_0 = b_0$. As we noted in Paragraph 12, a_3, b_3, a_0, b_0 are all non-zero. If S is smooth, then $a(x)$ and $b(y)$ have no multiple roots. We have thus proved the following.

CLAIM 15. *Let k be a field and S an affine cubic surface as in Paragraph 12 with a split A_2 singularity at infinity. Then one can write its equation in the form*

$$S_{a,b} := \big(xyz = a(x) + b(y) - c\big) \subset \mathbb{A}^3,$$

where $a(t), b(t) \in P_3(k)$ and $a(0) = b(0) = c$. ■

The only linear isomorphisms between these normal forms are given by

$$(x, y, z) \mapsto (\lambda_x x, \lambda_y y, \lambda_z z) \quad \text{or} \quad (\lambda_y y, \lambda_x x, \lambda_z z),$$

for $\lambda_x, \lambda_y, \lambda_z \in k^\times$. ■

Over other rings, we take Claim 15 as our definition.

DEFINITION 16. Let R be an integral domain, and $a(t), b(t) \in P_3^\times(R)$. Then $a_0 = a(0)$ and $b_0 = b(0)$. Set

$$S_{a,b} := \big(xyz = b_0 a(x) + a_0 b(y) - a_0 b_0\big) \subset \mathbb{A}^3_R. \tag{16.1}$$

Since a_0, b_0 are units, we can divide by $a_0 b_0$, and change z to $a_0 b_0 z$ to get the equivalent form

$$S_{a,b} := \big(xyz = a_0^{-1} a(x) + b_0^{-1} b(y) - 1\big) \subset \mathbb{A}^3_R. \tag{16.2}$$

PARAGRAPH 17 (Blow-up representation). Projecting from the singular point to the $(x{:}y{:}w)$-plane contracts the 6 lines

$$\begin{aligned} &(x = b_3 y^3 + b_2 y^2 w + b_1 y w^2 + b_0 w^3 = 0),\\ &(y = a_3 x^3 + a_2 x^2 w + a_1 x w^2 + a_0 w^3 = 0). \end{aligned} \tag{17.1}$$

Let β_i be the roots of $b(t)$ and α_i the roots of $a(t)$.

We can thus obtain $S_{a,b}$ by starting with $\mathbb{A}^2_{uv}$, blowing up the 6 points $(\alpha_i, 0), (0, \beta_i)$, and then removing the birational transform of $(uv = 0)$. The map is given by

$$x = u^2 v, \quad y = uv^2, \quad z = a(u) + b(v) - c, \quad w = uv.$$

If the 6 points α_i, β_i are different, we get a smooth affine surface. If 2 (resp. 3) of them coincide, then the affine surface has an A_1 (resp. A_2) singularity. However, the singularity at infinity stays A_2. These extra A_1 or A_2 singularities on $S_{a,b}$ play no role in our investigations.

The birational transform of $(w = 0)$ becomes the curve at infinity $\overline{B}$.

Thus, the points $\beta_\infty := (0{:}1{:}0)$ and $\alpha_\infty := (1{:}0{:}0)$ get identified to form the node at infinity.

The 6 exceptional curves of the blow-up become the 6 lines through the singular point. The birational transforms of the lines $\langle \alpha_\infty, \beta_i \rangle$ or $\langle \beta_\infty, \alpha_i \rangle$ become conics that are tangent to one of the branches of $\overline{B}$ at the node.

Paragraph 18 (Lines on $S_{a,b}$). There are 6 lines on $S_{a,b}$ passing through the singular point. If α is a root of $a(x)$, then $(\alpha, 0, t)$ is a line on $S_{a,b}$. Similarly, if β is a root of $b(x)$, then we get $(0, \beta, t)$.

The residual intersection of $S_{a,b}$ with the plane spanned by these 2 lines is another line, giving 9 lines not passing through the singular point.

Thus, we get a k-line on $S_{a,b}$ not passing through the singular point iff both $a(x)$ and $b(y)$ have a root in k.

Paragraph 19 (Moduli). Using the blow-up representation (Paragraph 17), we see that the moduli of projective cubics with an A_2 singularity are isomorphic to the moduli of 3 + 3 points on a pair of lines $(uv = 0) \subset \mathbb{P}^2_{uvw}$.

We are working with affine cubics; this corresponds to also fixing the line $(w = 0)$. Thus, we have $(uv = 0) \subset \mathbb{P}^2_{uvw}$ with marked points (0:1:0) and (1:0:0). The automorphism group is now reduced to the maps

$$(u{:}v{:}w) \mapsto (\lambda u{:}\mu v{:}w) \text{ and } (\mu v{:}\lambda u{:}w).$$

4. Constructing the involutions

For introductions, see [11], [9, Chapter 8] or [19, Section 4].

Paragraph 20 (Geiser involutions of singular Del Pezzo surfaces). A degree 2 Del Pezzo surface T has a natural double cover structure $\pi : T \to \mathbb{P}^2$, ramified along a degree 4 curve R. The Geiser involution is the corresponding Galois involution.

The preimage E_L of a line $L \subset \mathbb{P}^2$ is a curve of arithmetic genus 1.

Assume now that T is singular at a point p. Then R is singular at $\pi(p)$. Let $L \ni \pi(p)$ be a line. Then E_L is singular at p, and we expect it to be a rational curve with a node at p. The Geiser involution interchanges the 2 branches of E_L at p. All such curves E_L form a pencil in $|-K_T|$. This pencil is base-point-free on $T \setminus \{p\}$, its general fiber is $\mathbb{G}_m$, and the special fibers, consisting of pairs of lines, are isomorphic to $(uv = 0) \subset \mathbb{A}^2_{uv}$. The Geiser involution interchanges the u and v axes.

Our plan is to identify these pencils on S_0, and write down formulas for σ_x, σ_y. Since S_0 is a degree 3 Del Pezzo surface, we need to blow it up at 1 point to get a degree 2 Del Pezzo surface. The only 'natural' point on $\overline{S}_0$ is the singular point. So we look for pencils of rational curves whose base point is the singular point. Once we find these, it turns out that going from $\overline{S}_0$ to the degree 2 Del Pezzo surface involves the resolution of the singularity, 2 blow-ups, and 3 blow-downs.

PARAGRAPH 21 (Computing σ_x, σ_y). Projecting $S_0 := (xyz = x^3 + y^3 + 1)$ to the (x, y)-plane as in Paragraph 17 shows that the open subset lying over $(xy \neq 0)$ is isomorphic to $\mathbb{G}_m \times \mathbb{G}_m$, and the rest consists of 6 lines over the points $(x = y^3 + 1 = 0)$ and $(y = x^3 + 1 = 0)$.

Thus, projection to the x- or y-axis gives an anticanonical pencil, whose general fiber is $\mathbb{G}_m$, and the special fibers consist of pairs of lines. The latter are over the points $(x^3 + 1 = 0)$ resp. $(y^3 + 1 = 0)$.

These projections are good candidates for the pencils associated with the Geiser involutions in Paragraph 20. Thus, for the projection to the y-axis, in the (x, y)-coordinates we are looking for an involution of the form

$$(x, y) \mapsto \left(\tfrac{c(y)}{x}, y\right).$$

For fixed y, this is an involution of $\mathbb{G}_m \times \{y\}$ if $c(y) \neq 0$ but degenerates at the roots of $c(y)$. Since over the roots of $y^3 + 1$ we have reducible fibers, it is reasonable to expect that $c(y) = y^3 + 1$. This is a good guess since $xyz = x^3 + y^3 + 1$ gives that

$$\tfrac{c(y)}{x} = \tfrac{y^3+1}{x} = yz - x^2.$$

Thus, we get a regular involution

$$(x, y) \mapsto (\bar{x}, \bar{y}) := \left(yz - x^2, y\right) = \left(\tfrac{y^3+1}{x}, y\right). \tag{21.1}$$

We have no choice for $\bar{z}$, but it should be regular. Using that $\bar{y}^3 + 1 = y^3 + 1 = x\bar{x}$, we get that

$$\begin{aligned} \bar{z} &= \frac{\bar{x}^3 + \bar{y}^3 + 1}{\bar{x}\bar{y}} = \frac{\bar{x}^3 + x\bar{x}}{\bar{x}\bar{y}} = \frac{\bar{x}^2 + x}{y} = \frac{y^2z^2 - 2x^2yz + x(x^3 + 1)}{y} \\ &= \frac{y^2z^2 - 2x^2yz + xy(xz - y^2)}{y} = yz^2 - x^2z - xy^2. \end{aligned} \tag{21.2}$$

Thus, on S_0 we obtain the involutions

$$\begin{aligned} \sigma_x &: (x, y, z) \mapsto \left(x, xz - y^2, xz^2 - y^2z - x^2y\right) = \left(x, \tfrac{x^3+1}{y}, \tfrac{(xz-y^2)^2+y}{x}\right), \\ \sigma_y &: (x, y, z) \mapsto \left(yz - x^2, y, yz^2 - x^2z - xy^2\right) = \left(\tfrac{y^3+1}{x}, y, \tfrac{(yz-x^2)^2+x}{y}\right). \end{aligned} \tag{21.3}$$

PARAGRAPH 22 (Structure of $\mathrm{Aut}_{\mathbb{Z}}(S_0)$). We determine the structure of the subgroup generated by σ_x, σ_y and linear automorphisms. (We see in Theorem 28 that this is in fact the whole $\mathrm{Aut}_{\mathbb{Z}}(S_0)$.) We use the *infinite dihedral group* $D_\infty := \mathbb{Z}/2 * \mathbb{Z}/2$.

First, σ_x, σ_y are involutions; hence, they generate a dihedral group. There are many ways to see that $\sigma_x\sigma_y$ has infinite order; see, for example, (36.1). So $\langle \sigma_x, \sigma_y\rangle \cong D_\infty$. Next, τ interchanges σ_x, σ_y, so

$$\big(\langle\sigma_x\rangle * \langle\sigma_y\rangle\big) \rtimes \langle\tau\rangle \cong (\mathbb{Z}/2 * \mathbb{Z}/2) \rtimes \mathbb{Z}/2.$$

This is, however, also isomorphic to D_∞, for example, with generators σ_x and τ.

If k contains a 3rd root of unity $\varepsilon \neq 1$, then, as noted in Paragraph 10, $\mathrm{Aut}^{\mathrm{lin}}(S_0)$ contains a subgroup H of order 9 generated by $\mu = (\varepsilon, \varepsilon, \varepsilon)$ and $\rho := (\varepsilon, \varepsilon^2, 1)$. We have

$$\sigma_x\mu = \rho\sigma_x, \quad \sigma_x\rho = \mu\sigma_x, \quad \tau\mu = \mu\tau, \quad \tau\rho = \rho^2\tau.$$

Thus, H is a normal subgroup.

Also note that the normalizer of $\langle\mu\rangle$ descends to

$$S_0/\langle\mu\rangle \cong P_0 = \mathbb{P}^2 \setminus (xyz = x^3 + y^3).$$

We see that

$$\sigma_x\tau\sigma_x\mu = \sigma_x\tau\rho\sigma_x = \sigma_x\rho^2\tau\sigma_x = \mu^2\sigma_x\tau\sigma_x,$$

so indeed $\sigma_x\tau\sigma_x$ descends. This was one of our starting points in (6.1).

PARAGRAPH 23 (Fundamental domain). The rational forms of the involutions (21.3)

$$\left(x, \tfrac{x^3+1}{y}, \tfrac{(xz-y^2)^2+y}{x}\right) \quad \text{and} \quad \left(\tfrac{y^3+1}{x}, y, \tfrac{(yz-x^2)^2+x}{y}\right)$$

show that σ_x, σ_y map the positive octant $S_0 \cap (\mathbb{R}^+)^3$ to itself. Applying τ if necessary, we may assume that $x \le y$, and $(x^3+1)/y < y$ iff $x^3 + 1 < y^2$. Thus, a fundamental domain for the $\mathrm{Aut}(S_0)$-action on $S_0 \cap (\mathbb{R}^+)^3$ is

$$x \le y \le (x^3+1)^{1/2}.$$

Following the recipe of (21.2) for a general $S_{a,b}$ results in a $\bar{z}$ that is regular if $a_3 = a_0$, but not otherwise. Next, we discuss the geometric reason behind this. It will then show the correct generalization of the formulas (21.3).

PARAGRAPH 24 (Geometric construction of σ_x, σ_y). First, take the minimal resolution of the singularity of $\overline{S}_{a,b}$. We give 2 versions of the dual graph

$$C_y — E_y — E_x — C_x \quad \text{and} \quad C_y — (-2) — (-2) — C_x$$

where E_x, E_y denote the exceptional curves, and C_x, C_y the birational transforms of the local branches of $\bar{B}$. The left side shows the names, the right side the self-intersections.

Then we blow up twice the point on C_x that is also on the exceptional set. We get

$$\begin{array}{ccccccccccc} E'_y & - & E'_x & - & E_1 & & (-2) & - & (-3) & - & (-2) \\ | & & & & | & \text{and} & | & & & & | \\ C' & & \rule{2cm}{0.4pt} & & E_2 & & (-1) & & \rule{2cm}{0.4pt} & & (-1) \end{array} \tag{24.1}$$

Thus, (24.1) has a numerical left-right symmetry. The 2 blow-ups creating E_1, E_2 result in a surface with $(K^2) = 1$, but it is not a weak Del Pezzo surface since there is a (-3)-curve.

Note that the (-3)-curve E'_x is the birational transform of E_x, and by Paragraph 17, it is intersected by the birational transforms of 3 lines; denote these by L'_1, L'_2, L'_3.

If we contract one of the lines L'_i, then (24.1) becomes

$$\begin{array}{ccccc} (-2) & - & (-2) & - & (-2) \\ | & & & & | \\ (-1) & & \rule{2cm}{0.4pt} & & (-1) \end{array} \tag{24.2}$$

Now we have a degree 2 weak Del Pezzo surface, and the Geiser-type involution interchanges the left and right sides.

Remarkably, the end result is independent of which line we choose, but for now choose say L'_3.

To compute the action of σ_x on the birational transform E''_x of E_x, we use the coordinate $t = x/w$. Then $E_y \cap E_x = (t = 0)$ and $E_x \cap C_x = (t = \infty)$. The 3 lines L'_i meet E_x at the 3 roots of $a(x) = a_3x^3 + a_2x^2 + a_1x + a_0$; call these α_i.

The involution σ_x then acts by $t \mapsto c/t$ for some constant c. The Geiser involution maps lines to lines; thus, it interchanges L''_1 and L''_2; hence, it interchanges α_1 and α_2. This gives that $c = \alpha_1\alpha_2$. Thus,

$$\alpha_3 \mapsto \tfrac{\alpha_1\alpha_2}{\alpha_3}.$$

Note that $\sigma_x(\alpha_3) = \alpha_3$ iff $\alpha_1\alpha_2\alpha_3 = 1$, that is, iff $a_3 = a_0$. This explains our stated observation that the formula in (21.2) defines an involution if $a_3 = a_0$, but not otherwise.

Once we understand that σ_x need not be an automorphism of $S_{a,b}$, but an isomorphism between two different surfaces $S_{a,b}$ and $S_{a',b'}$, we choose the coordinates t, t' on the surfaces independent of each other. By Paragraph 19, they are well defined only up to scaling; thus, we can as well choose $t' = 1/t$. That is, on the roots we get the map

$$(\alpha_1, \alpha_2, \alpha_3) \mapsto \left(\tfrac{1}{\alpha_1}, \tfrac{1}{\alpha_2}, \tfrac{1}{\alpha_3}\right).$$

The inverses of the roots satisfy the equation $\bar{a}(t) = 0$, where the coefficients are in reversed order (Notation 13).

Building on these considerations—and after some trial and error—the algebraic construction of the general σ_x and σ_y works out very cleanly.

NOTATION 25. For an integral domain R, let $P_n(R)$ and $P_n^{\times}(R)$ be as in Notation 13. For $a(t) \in P_n(R)$, define $a^*(t)$ by the formula $a(t) = ta^*(t) + a_0$. We also set

$$\bar{a}(t) := \tfrac{1}{a_0^{n-1}a_n} t^n a\bigl(\tfrac{a_0}{t}\bigr)$$

though the reason for this becomes clear only in (26.8).

Note that $a \mapsto \bar{a}$ is an involution on $P_n^{\times}(R)$. Furthermore, $\bar{a}_n = \frac{1}{a_0^{n-2}a_n}$ is in R for $n \geq 3$ iff a_n, a_0 are units in R.

PARAGRAPH 26 (Algebraic construction of σ_x and σ_y). These computations are implicit in [17, 29], and equivalent to [34, Formulas (52)–(53)]. Start with a surface

$$S_{a,b} = \bigl(xyz = a(x) + b(y) - c\bigr) \subset \mathbb{A}^3_{xyz}, \tag{26.1}$$

where $a(x) = a_n x^n + \cdots + a_0$ and $b(y) = b_m y^m + \cdots + b_0$. We assume that $a_0 = b_0 = c$, and a_n, b_m, c all non-zero. As in Paragraph 24, we want σ_y to send (x, y) to

$$(\bar{x}, \bar{y}) = \bigl(b(y)/x, y\bigr) = (yz - a^*(x), y), \tag{26.2}$$

and we hope that the new surface has equation

$$S_{\bar{a},b} = \bigl(xyz = \bar{a}(x) + b(y) - c\bigr) \tag{26.3}$$

for some not yet known $\bar{a}(x)$, which must satisfy $\bar{a}_0 = a_0$.

Using $b(y) = \bar{x}x$, we get that

$$\begin{aligned} \bar{z} &= \frac{\bar{a}(\bar{x}) + b(\bar{y}) - c}{\bar{x}\bar{y}} = \frac{\bar{a}(\bar{x}) - c + b(y)}{\bar{x}y} \\ &= \frac{\bar{x}\bar{a}^*(\bar{x}) + x\bar{x}}{\bar{x}y} = \frac{\bar{a}^*(\bar{x}) + x}{y}, \end{aligned} \tag{26.4}$$

and it should be regular. Now use $\bar{x} = yz - a^*(x)$. So

$$\bar{a}^*(\bar{x}) + x = \bar{a}^*\bigl(yz - a^*(x)\bigr) + x = (\text{divisible by } y) + \bar{a}^*\bigl(-a^*(x)\bigr) + x. \tag{26.5}$$

The only known polynomial in x that is divisible by y is

$$a(x) = xyz - b(y) - c = y\bigl(xz - b^*(y)\bigr). \tag{26.6}$$

Thus, our hope is that $a(x)$ divides $\bar{a}^*(-a^*(x)) + x$. A general $a(x)$ has distinct roots, so we need that each root of $a(x)$ is a root of $\bar{a}^*(-a^*(x)) + x$.

Let α be a root of $a(x)$. Then $0 = a(\alpha) = \alpha a^*(\alpha) + a_0$, so $-a^*(\alpha) = a_0/\alpha$. Substituting, we get that

$$\bar{a}^*\big(-a^*(\alpha)\big) + \alpha = \bar{a}^*\big(\tfrac{a_0}{\alpha}\big) + \alpha = \tfrac{\alpha}{a_0}\big(\tfrac{a_0}{\alpha}\bar{a}^*\big(\tfrac{a_0}{\alpha}\big) + a_0\big) = \tfrac{\alpha}{a_0} \cdot \bar{a}\big(\tfrac{a_0}{\alpha}\big), \tag{26.7}$$

where we used $\bar{a}_0 = a_0$. Thus, we are done if $\frac{a_0}{\alpha}$ is a root of $\bar{a}$. This tells us that $\bar{a}(x)$ should be a constant times $x^n a(\frac{a_0}{x})$. Since $\bar{a}_0 = a_0$, we must have

$$\bar{a}(x) := \frac{1}{a_0^{n-1} a_n} x^n a\big(\tfrac{a_0}{x}\big). \tag{26.8}$$

Thus, if a_n, a_0 are units, then σ_y gives an R-isomorphism between $S_{a,b}$ and $S_{\bar{a},b}$.

If the a_n, a_0 are not units, then the formulas (26.2) for $(\bar{x}, \bar{y})$ are in $R[x, y, z]$, but the formula for $\bar{z}$ involves the denominator $a_0^{n-2} a_n$.

Interchanging x, y in the construction gives σ_x.

Paragraph 27 (Proof of Theorem 3). If a_n, b_m and $a_0 = b_0$ are units in R, then the composition of the isomorphisms σ_x, σ_y constructed in Paragraph 26 gives the automorphism

$$\sigma_{a,b} : S_{a,b} \xrightarrow{\sigma_x} S_{\bar{a},b} \xrightarrow{\sigma_y} S_{\bar{a},\bar{b}} \xrightarrow{\sigma_x} S_{a,\bar{b}} \xrightarrow{\sigma_y} S_{a,b}.$$

We need to show that it has infinite order. For this it is enough to show that the birational maps

$$\sigma'_x : (x, y) \mapsto \big(x, a(x)y^{-1}\big) \quad \text{and} \quad \sigma'_y : (x, y) \mapsto \big(b(y)x^{-1}, y\big)$$

generate an infinite subgroup of $\mathrm{Bir}(\mathbb{A}^2)$. Now observe that if $\deg a, \deg b \geq 3$ and $\max\{|x|, |y|\}$ is large, then either σ'_x or σ'_y increases $\max\{|x|, |y|\}$. ■

5. Isomorphisms

We are now ready to describe all isomorphisms in the cubic case.

Theorem 28. *Let R be an integral domain, and consider the set of all cubic surfaces $\{S_{a,b} : a, b \in P_3^\times(R)\}$ with $P_3^\times$ as in Notation 13 and Definition 16.*

Then the groupoid of isomorphisms between the surfaces $S_{a,b}$ is generated by

(1) *linear isomorphisms (Claim 15),*

(2) $\sigma_y : S_{a,b} \cong S_{\bar{a},b}$ *(Paragraph 26) for all $a, b \in P_3^\times(R)$, and*

(3) $\sigma_x : S_{a,b} \cong S_{a,\bar{b}}$ *(Paragraph 26) for all $a, b \in P_3^\times(R)$.*

Next, we give some consequences over $\mathbb{Z}$.

NOTATION 29. Assume now that we work over $\mathbb{Z}$, and let $a(t), b(t) \in P_3^\times(\mathbb{Z})$. We can assume that $a_0 = b_0 = 1$. By changing the sign of x, y, z, we can also assume that $a_3 = b_3 = 1$. Thus, we have

$$a(t) := t^3 + a_2t^2 + a_1t + 1, \quad b(t) := t^3 + b_2t^2 + b_1t + 1,$$

and

$$\bar{a}(t) := t^3 + a_1t^2 + a_2t + 1, \quad \bar{b}(t) := t^3 + b_1t^2 + b_2t + 1.$$

There are 8 surfaces in play

$$S_{a,b}, S_{\bar{a},b}, S_{a,\bar{b}}, S_{\bar{a},\bar{b}}, S_{b,a}, S_{\bar{b},a}, S_{b,\bar{a}}, S_{\bar{b},\bar{a}}. \tag{29.1}$$

We always have the obvious linear isomorphism $\tau : S_{a',b'} \cong S_{b',a'}$, so the first 4 surfaces are mapped by τ to the last 4.

With these choices, $\sigma_y : S_{a,b} \cong S_{\bar{a},b}$ is given by

$$\big(yz - (x^2 + a_2x + a_1), y, yz^2 - z(x^2 + a_2x + a_1) - (x + a_2)(y^2 + b_2y + b_1)\big),$$

and $\sigma_x : S_{a,b} \cong S_{a,\bar{b}}$ is given by

$$\big(x, xz - (y^2 + b_2y + b_1), xz^2 - z(y^2 + b_2y + b_1) - (y + b_2)(x^2 + a_2x + a_1)\big).$$

Using the above notation and assumptions, we have 3 corollaries.

COROLLARY 30. *If $S_{a,b} \cong S_{a',b'}$ for some $a', b' \in P_3^\times(\mathbb{Z})$, then $S_{a',b'}$ is one of the 8 surfaces in (29.1).* ■

COROLLARY 31. *If $a(t) = b(t) = t^3 + 1$, then the 8 surfaces in (29.1) are all isomorphic to S_0, and $\mathrm{Aut}_{\mathbb{Z}}(S_0) = \langle \sigma_x, \sigma_y, \tau \rangle$.* ■

COROLLARY 32. *Assume in addition that $a_2 \neq a_1$, $b_2 \neq b_1$ and*

$$(a_2, a_1) \neq (b_2, b_1) \neq (a_1, a_2).$$

Then

(1) *all linear isomorphisms between the surfaces (29.1) are given by τ, and*

(2) $\mathrm{Aut}_{\mathbb{Z}}(S_{a,b}) = \mathrm{Aut}_{\mathbb{C}}(S_{a,b})$ *is infinite, cyclic, and generated by the composition*

$$\sigma_{a,b} : S_{a,b} \xrightarrow{\sigma_x} S_{\bar{a},b} \xrightarrow{\sigma_y} S_{\bar{a},\bar{b}} \xrightarrow{\sigma_x} S_{a,\bar{b}} \xrightarrow{\sigma_y} S_{a,b}.$$ ■

REMARK 32(3). One should think of $\mathrm{Aut}_{\mathbb{Z}}(S_{a,b}) \cong \mathbb{Z}$ as an index 8 subgroup of $\mathrm{Aut}_{\mathbb{Z}}(S_0) \cong D_\infty$.

REMARK 32 (4). We computed the coordinate functions of $\sigma_{a,b}$ using Macaulay2. They are polynomials of degrees 13, 34, and 55, having 178, 3485, and 15314 monomials. Simplifying the formulas using a Gröbner basis, the degrees stay the same, but the number of monomials drops to 110, 998, and 2881. We also computed some very special cases, for example, $(a_2, a_1, b_2, b_1) = (0, 1, 0, 2)$ or $(a_2, a_1, b_2, b_1) = (0, 1, 0, 1)$. The number of monomials stays the same.

PARAGRAPH 33 (Plan of the proof of Theorem 28). We try to follow the proof in [19].

Given any isomorphism $\psi : S_{a,b} \cong S_{a',b'}$, we want to show that there is a $\phi \in \langle \sigma_x, \sigma_y \rangle$ such that the composite

$$\phi \circ \psi : S_{a,b} \cong S_{a',b'} \cong S_{a'',b''} \quad \text{is linear.} \tag{33.1}$$

Note that σ_x, σ_y are defined over R, and an isomorphism is linear over R iff it is linear over K. Thus, once we prove (33.1) over the quotient field of R, it holds over R as well. We may even replace the quotient field by its algebraic closure.

Thus, assume from now on that $R = K$ is an algebraically closed field, and let $A \subset \overline{S}_{a,b}$ be a smooth hyperplane section with only 1 place at infinity as in Example 34 (1).

Given any isomorphism $\psi : S_{a,b} \cong S_{a',b'}$, we consider the curve $\psi_*(A) \subset \overline{S}_{a',b'}$. It has only 1 place at infinity.

Next we use isomorphisms σ_x, σ_y to simplify the singularity of any curve C that has only 1 place at infinity. A similar question was considered in [19, Corollary 7]. Although [19, Corollary 7] applies only to smooth Del Pezzo surfaces of degree > 3, extrapolating from [19, Corollary 7 (7.1)] suggests that, for suitable ϕ,

$$\operatorname{mult}_s\bigl(\phi_*(C)\bigr) \text{ might be at most } \tfrac{1}{3}\deg\bigl(\phi_*(C)\bigr). \tag{33.2}$$

The presence of the A_2 singularity changes the computation. The first part goes even better; see Proposition 36. Applying it to $C := \psi_*(A)$, we can achieve that

$$\operatorname{mult}_s(\phi \circ \psi)_*(A) \le \tfrac{1}{4}\deg\bigl((\phi \circ \psi)_*(A)\bigr). \tag{33.3}$$

On the other hand, the curve $(\phi \circ \psi)_*(A)$ still passes through the A_2 singularity; hence, the pair $(\overline{S}_{a'',b''}, \varepsilon \cdot (\phi \circ \psi)_*(A))$ is not canonical (Paragraph 41) for any $\varepsilon > 0$. This is bad news since the Noether–Fano method (Section 7) would need $(\overline{S}_{a'',b''}, \varepsilon \cdot (\phi \circ \psi)_*(A))$ to be canonical for $\varepsilon_0 := 3/\deg((\phi \circ \psi)_*(A))$.

Instead, we are only able to obtain the discrepancy (Paragraph 41) inequality

$$\operatorname{discrep}\bigl(\overline{S}_{a'',b''}, \varepsilon_0 \cdot (\phi \circ \psi)_*(A)\bigr) \ge -\tfrac{1}{2}. \tag{33.4}$$

(Canonical would be discrep ≥ 0.) The traditional Noether–Fano method needs strong control of the singularities of $(\phi \circ \psi)_*(A)$ but makes no assumption about the indeterminacy locus of $\phi \circ \psi$. In the present setting, we know that the indeterminacy locus is contained in the boundary $\overline{B}$, and we are able to use this to relax the condition on the singularities. A result of this type is proved in Theorem 44.

We put the pieces together in Paragraph 46. The end result is that $(\phi \circ \psi)_*(A)$ lies either in $|-K|$ or in $|-2K|$. In the first case, $\phi \circ \psi$ is linear, as needed. The second case is excluded using Lemma 38 (3).

DEFINITION 34. Let C be an irreducible curve, with projective closure $\overline{C}$. We say that C has *one place at infinity* if the normalization of $\overline{C}$ has only one geometric point over $\overline{C} \setminus C$. Equivalently, $\overline{C}$ has a single geometric point at infinity, and it is a cusp.

EXAMPLE 34 (1). Let $\overline{S}$ be a cubic surface whose curve at infinity $\overline{B}$ is nodal. Let $p \in \overline{B}$ be a flex with flex tangent line L_p. For a general plane $H \supset L_p$, the intersection $A := S \cap H$ is a smooth, elliptic curve with one place at infinity.

The nodal cubic $\overline{B}$ need not have a flex over the base field k, but it has 3 flexes over the algebraic closure. So there is always a flex over a suitable cubic extension of k.

6. TRANSFORMING CURVES

NOTATION 35. Let k be a field and $S = S_{a,b}$ an affine surface as in Definition 16. Let $\overline{S}$ be its projective closure, $\bar{s} \in \overline{S}$ the unique singular point at infinity, and $\overline{B} \subset \overline{S}$ the curve at infinity. Let $\overline{T} \to \overline{S}$ denote the minimal resolution, with exceptional curve $E = E_x \cup E_y$.

If C has one place at infinity, we use $\operatorname{mult}_E(C)$ to denote the multiplicity of the birational transform of C on $\overline{T}$ at its intersection point with E. Note that $\operatorname{mult}_E(C) = 0$ if $\bar{s} \notin \overline{C}$. Also, $\operatorname{mult}_{\bar{s}} \overline{C} \geq \operatorname{mult}_E C$, but they need not be equal.

The following is a key step toward proving Theorem 28.

PROPOSITION 36. *Let $C \subset S$ be a curve with one place at infinity. Then there is a $\phi \in \langle \sigma_x, \sigma_y \rangle$ giving $\phi : (S, C) \dashrightarrow (S^{\mathrm{m}}, C^{\mathrm{m}})$ such that either* $\deg C^{\mathrm{m}} < 3$, *or* $\operatorname{mult}_E C^{\mathrm{m}} \leq \frac{1}{4} \deg C^{\mathrm{m}}$.

PROOF. The transformations σ_x, σ_y both involve 2 blow-ups followed by 2 contractions. This is exactly the $r = 2$ special case of [19, Paragraph 29]. Using its notation, $p = \operatorname{mult}_E(C)$ and q is the intersection multiplicity of the birational transform of C with the birational transform of $\overline{B}$. So $\deg C = p + q$. We check in Lemma 38 that $p < q$. Thus, $B \overset{p,q}{\rule[0.5ex]{1.5em}{0.4pt}} B$ is transformed to

$$\begin{array}{ll} B \overset{p,2p-q}{\rule[0.5ex]{1.5em}{0.4pt}} B & \text{if } q \leq 2p, \\ B \overset{q-2p,\,p+2(q-2p)}{\rule[0.5ex]{1.5em}{0.4pt}} B & \text{if } q \geq 2p. \end{array} \tag{36.1}$$

Equivalently,

$$\deg\bigl(\sigma_x(C)\bigr) = \begin{cases} 4\,\mathrm{mult}_E(C) - \deg(C) & \text{if } \mathrm{mult}_E(C) \geq \frac{1}{3}\deg(C), \\ 3\deg(C) - 8\,\mathrm{mult}_E(C) & \text{otherwise.} \end{cases}$$

In the first case, $4\,\mathrm{mult}_E(C) - \deg(C) < \deg(C)$ iff $\mathrm{mult}_E(C) < \frac{1}{2}\deg(C)$. This holds by Lemma 38 (2) unless $\deg C < 3$.

In the second case, $3\deg(C) - 8\,\mathrm{mult}_E(C) < \deg(C)$ iff $\mathrm{mult}_E(C) > \frac{1}{4}\deg(C)$.

We can thus lower the degree using the transformations σ_x, σ_y, except when we have reached $(S^{\mathrm{m}}, C^{\mathrm{m}})$ such that either $\mathrm{mult}_E C^{\mathrm{m}} \leq \frac{1}{4}\deg C^{\mathrm{m}}$ or $\deg C^{\mathrm{m}} < 3$. ■

Lemma 37. *Let T be a projective, weak Del Pezzo surface of degree 3, and $C \subset T$ an irreducible curve that is not contained in an effective anticanonical divisor. Then* $\mathrm{mult}_c C \leq \frac{1}{2}\deg C$ *for every smooth point $c \in T$.*

Proof. Since $\dim|-K_T| = 3$, there is an anticanonical divisor F_c that is singular at c. By assumption, $C \not\subset F_c$. Thus,

$$2\,\mathrm{mult}_c C \leq \mathrm{mult}_c F_c \cdot \mathrm{mult}_c C \leq (F_c \cdot C) = \deg C. \qquad ■$$

Lemma 38 (Curves with 1 place at infinity). *Let $C \subset \overline{S}$ be a curve of degree ≥ 3 with 1 place at infinity at the node. Let $\pi : \overline{T} \to \overline{S}$ be the minimal resolution with exceptional curves E_1, E_2. Let C_T be the birational transform of C on $\overline{T}$. Then*

(1) *C_T meets $E_1 \cap E_2$ at a point of $\overline{B}_T$,*

(2) *$(C_T \cdot (E_1 + E_2)) < (C_T \cdot \overline{B}_T)$, and*

(3) *there is no such curve in $|-2K_{\overline{S}}|$.*

Proof. If (1) does not hold, then C_T is disjoint from $\overline{B}_T$, which is the pull-back of a line in the blow-up model (Paragraph 17). Thus, C_T is one of the 6 exceptional curves of the blow-up model, so C is a line through the singular point.

Assume next that the common point is $\bar{s} := E_2 \cap \overline{B}_T$. Set $p := (C_T \cdot (E_1 + E_2))$ and $q := (C_T \cdot \overline{B}_T)$. Blow up $\bar{s}$. Denote the exceptional curve by E_3, and let C_3, B_3 be the birational transforms $C_T, \overline{B}_T$.

If $p \geq q$, then C_3 is disjoint from B_3 and has intersection number p with E_3. Note that $(B_3^2) = 0$, so it moves in a basepoint-free pencil, with section E_3. Then C_3 is a subcurve of a member of the pencil; hence, $(C_3 \cdot E_3) \leq 1$. So $p = 1$ and $\deg C \leq 2$. These cases are listed in Paragraph 17.

If $C \in |-2K_{\overline{S}}|$, then C is Cartier, so $\pi^*C = C_T + \frac{2p}{3}E_2 + \frac{p}{3}E_1$ and $\deg C = (C_T \cdot (E_1 + E_2)) + (C_T \cdot \overline{B}_T)$. If $\deg C \leq 6$, then $p < 3$ by (2); hence, $\frac{p}{3}$ is not an integer. ■

Corollary 39. *Let k be a field and $S := (zg_2(x, y) = g_3(x, y))$ an affine cubic surface as in Paragraph 12. If $g_2(x, y)$ is irreducible over k, then every k-automorphism of S is linear.*

Proof. By Definition 34, after possibly a degree 3 extension k'/k, there is a curve $C_1 \subset S$ with 1 place at infinity. Then $g_2(x, y)$ is still irreducible over k'. If there is a non-linear k'-automorphism, then the image of C_1 is a curve $C \subset \bar{S}$ of degree ≥ 3 with 1 place at infinity at the node. By Lemma 38 (1), then C_T meets $E_1 \cap E_2$ at a k' point of $\bar{B}_T$. If the node is unsplit, this intersection is a degree 2 point over k'. ■

7. Noether–Fano method

We start with some general comments on closed graphs of birational maps.

Paragraph 40. Let T_i be proper surfaces with rational singularities and $D_i \subset T_i$ geometrically irreducible curves. Let $\phi^\circ : T_1 \setminus D_1 \cong T_2 \setminus D_2$ be an isomorphism that does not extend to an isomorphism of T_1 and T_2.

Let $\bar{T}$ be the normalization of the closed graph of ϕ° with projections $p_i : \bar{T} \to T_i$. Let $\bar{D}_i$ be the birational transform of D_i on $\bar{T}$.

Let $E_i \subset \bar{T}$ be a p_i-exceptional curve. Then E_i is not p_{3-i}-exceptional; hence, p_{3-i} must map E_i birationally onto D_{3-i}. Thus, $E_i = \bar{D}_{3-i}$ is the unique exceptional curve of p_i.

Since the T_i have rational singularities, the E_i are smooth rational curves, contracted to a point $t_i \in T_i$. Since $\bar{D}_i \to D_i$ is an isomorphism away from t_i, we conclude that t_i is the only possible singular point of D_i.

Assume now that the D_i are nodal and the pairs (T_i, D_i) are log canonical. As in [19, Paragraph 17], we get that $\bar{D}_1, \bar{D}_2$ meet at 2 points on $\bar{T}$, both cyclic quotients. On the minimal resolution of $\bar{T}$, the exceptional curves and the birational transforms of the D_i form a cycle of rational curves.

These are strong restrictions but still leave many possibilities. In order to exploit Proposition 36, we need a few definitions and results on discrepancies. We state these for surfaces, which is the only case that we use. For general introductions, see [21, Section 2.3] or [18, Section 2.1].

Paragraph 41 (Discrepancies for surface pairs). Let S be a normal surface and $\pi : T \to S$ a proper, birational morphism. We are interested in the local behavior of π over a point $s \in S$, so assume that π is an isomorphism over $S \setminus \{s\}$, and T is normal. We can write $K_T \sim_{\mathbb{Q}} \pi^* K_S + E$ where E is π-exceptional (with rational coefficients).

Next, let $\Delta = \sum a_i D_i$ be a finite, $\mathbb{Q}$-linear combination of distinct curves on S, and Δ_T its birational transform on T. Then $\Delta_T \sim_{\mathbb{Q}} \pi^*\Delta - F$ where F is π-exceptional. We can formally write

$$K_T + \Delta_T \sim_{\mathbb{Q}} \pi^*(K_S + \Delta) + E - F. \tag{41.1}$$

The coefficient of an exceptional curve E_j in $E - F$ is called the *discrepancy* of E_j, denoted by $a(E_j, S, \Delta)$. We set $a(D_i, S, \Delta) := -a_i$.

The infimum over all exceptional curves and all proper, birational morphisms is called the *discrepancy* of (S, Δ), denoted by $\operatorname{discrep}(S, \Delta)$.

If Δ is effective, the pair (S, Δ) is called *canonical* if $\operatorname{discrep}(S, \Delta) \geq 0$, and *log canonical* if $\operatorname{discrep}(S, \Delta) \geq -1$.

We need 2 results. The first is a special case of [21, Lemma 2.30]. The second follows from a direct combination of [21, Theorem 5.50] and [18, Lemma 2.5]. For surfaces, it can be proved directly by computing one blow-up, and using induction.

Claim 41 (2). $\operatorname{discrep}(S, \Delta) = \min\{\operatorname{discrep}(T, \Delta_T + F - E), a(E_j, S, \Delta)\}$, *where* E_j *runs through all* π*-exceptional curves.* ■

Claim 41 (3). *Let* S *be a smooth surface,* $D_0 \subset S$ *a smooth curve, and* Δ *an effective* $\mathbb{Q}$*-divisor such that* $(D_0 \cdot \Delta) \leq 1$. *Then, in a neighborhood of* D_0,

$$\operatorname{discrep}(S, d_0 D_0 + \Delta) \geq -d_0. \qquad ■$$

We can now translate Proposition 36 into a discrepancy statement. First, consider a general version.

Lemma 42. *Let* $(t \in T)$ *be an* A_2 *singularity and* $\pi : T' \to T$ *its minimal resolution with exceptional curves* $E_1, E_2 \subset T'$. *Let* $C \subset T$ *be a curve and* C' *its birational transform on* T'. *Assume that* $(E_1 \cdot C') = 0$, $(E_2 \cdot C') = r$, *and* C' *has a cusp of multiplicity* r *at* $E_2 \cap C'$. *Then, for* $0 \leq c \leq \frac{1}{r}$, *the discrepancy of* (T, cC) *is* $-2cr/3$ *(in a neighborhood of* t*).*

Proof. First, note that $c\pi^*C \sim_{\mathbb{Q}} cC' + (2cr/3)E_2 + (cr/3)E_1$. Using Claim 41 (2), it remains to show that discrepancy of $(T', cC' + (2cr/3)E_2 + (cr/3)E_1)$ is at least $-2cr/3$. This follows from Claim 41 (3). ■

Corollary 43. *Let* $C^m \subset S^m$ *be as in Proposition 36, and assume that* C^m *is smooth away from* $\operatorname{Sing} \overline{S}^m$. *Then the discrepancy of* $(\overline{S}^m, \frac{3}{\deg C^m} C^m)$ *is at least* $-\frac{1}{2}$.

Proof. Set $r = \operatorname{mult}_E C$ and apply Lemma 42 with $c = \frac{3}{\deg C^m}$. ■

In order to study automorphisms of open varieties, we need a variant of the Noether–Fano method. See [22, Section 5.1] for a description of the classical version, and [19, Section 8] for related results. We state the general case but then use it only for surfaces.

THEOREM 44. *Let* X_i *be proper, normal varieties,* $D_i \in |-K_{X_i}|$ *irreducible divisors, and* $\phi^\circ : (X_1 \setminus D_1) \cong (X_2 \setminus D_2)$ *an isomorphism. Let* $D_1 \neq A_1 \in |-K_{X_1}|$ *be a divisor and set* $A_2 := \phi^\circ_*(A_1)$. *Assume that* $A_2 \in |-mK_{X_2}|$ *and the discrepancy of* $(X_2, \frac{1}{m}A_2)$ *is at least* $-1+\varepsilon$. *Then* $m\varepsilon \leq 1$.

COMMENT 44 (1). The traditional Noether–Fano method does not involve the divisors D_i but assumes that $(X_2, \frac{1}{m}A_2)$ is canonical; that is, we can choose $\varepsilon = 1$. It gives that $m = 1$; hence, we get an isomorphism $X_1 \cong X_2$, provided the $-K_{X_i}$ are ample.

In our case, Corollary 43 only gives that $\varepsilon \geq \frac{1}{2}$. However, the D_i put strong restrictions on the indeterminacy loci of ϕ and ϕ^{-1}, and we can use these to push the proof through.

PROOF. Let Y be the normalization of the closure of the graph of ϕ° with projections $p_i : Y \to X_i$. Let A_Y be the birational transform of the A_i on Y. Write $K_Y \sim_{\mathbb{Q}} p_i^* K_{X_i} + E_i$ and $A_Y = p_i^* A_i - F_i$. For any $c \in \mathbb{Q}$, we have

$$K_Y + cA_Y \sim_{\mathbb{Q}} p_i^*\bigl(K_{X_i} + cA_i\bigr) + E_i - cF_i. \tag{44.2}$$

Now choose $c = \frac{1}{m}$. Then

$$K_{X_2} + \tfrac{1}{m}A_2 \sim_{\mathbb{Q}} 0 \quad \text{and} \quad K_{X_1} + \tfrac{1}{m}A_1 \sim_{\mathbb{Q}} \tfrac{m-1}{m}K_{X_1};$$

hence,

$$E_2 - \tfrac{1}{m}F_2 \sim_{\mathbb{Q}} p_1^*\bigl(\tfrac{m-1}{m}K_{X_1}\bigr) + E_1 - \tfrac{1}{m}F_1. \tag{44.3}$$

If $p_2 \circ p_i^{-1}$ is an isomorphism at the generic points of the D_i, then linear equivalence is preserved; hence, $m = 1$. Otherwise, D_1 is the image of an irreducible component D_1^Y of $E_2 - \frac{1}{m}F_2$, whose coefficient is $d_1 \geq -1+\varepsilon$ by assumption. Thus,

$$d_1 D_1 = (p_1)_*\bigl(E_2 - \tfrac{1}{m}F_2\bigr) \sim_{\mathbb{Q}} \tfrac{m-1}{m}K_{X_1} \sim_{\mathbb{Q}} -\tfrac{m-1}{m}D_1.$$

Hence, $-\frac{m-1}{m} = d_1 \geq -1+\varepsilon$, and so $m\varepsilon \leq 1$. ■

COROLLARY 45. *Let* $A \subset \overline{S}_{a,b}$ *be a smooth hyperplane section with only* 1 *place at infinity, and* $\psi : S_{a,b} \cong S_{a',b'}$ *an isomorphism. Then there is a* $\phi \in \langle \sigma_x, \sigma_y \rangle$ *such that under the composite*

$$\phi \circ \psi : S_{a,b} \cong S_{a',b'} \cong S_{a'',b''}, \tag{45.1}$$

the image $(\phi \circ \psi)_*(A)$ *lies in* $|-K|$ *or* $|-2K|$ *on* $S_{a'',b''}$.

Proof. Apply Proposition 36 and Corollary 43 to $\psi_*(A)$. We get a $\phi \in \langle \sigma_x, \sigma_y \rangle$ such that

$$\left(S_{a'',b''}, \tfrac{1}{m}(\phi \circ \psi)_*(A)\right)$$

has discrepancy $\geq -\frac{1}{2}$, where $m = \frac{1}{3}\deg((\phi \circ \psi)_*(A))$.

As we noted in [19, Paragraph 16], the class group of S is $\mathrm{Cl}(\overline{S})/\mathbb{Z}[-K_{\overline{S}}]$. Thus, the class of A is trivial in $\mathrm{Cl}(S_{a,b})$, so the class of $(\phi \circ \psi)_*(A)$ is trivial in $\mathrm{Cl}(S_{a'',b''})$. Therefore, $(\phi \circ \psi)_*(A) \in |-mK|$ for some m of $S_{a'',b''}$.

We can now apply Theorem 44 to get that $m \leq 2$. ∎

Paragraph 46 (End of the proof of Theorem 28). Using Corollary 45, we get $\phi \in \langle \sigma_x, \sigma_y \rangle$ such that $A^m := (\phi \circ \psi)_*(A)$ lies in $|-K|$ or $|-2K|$ on $S_{a'',b''}$.

The possibility $A^m \in |-2K|$ is excluded by Lemma 38 (3). Thus, $A^m \in |-K|$ and so $\phi \circ \psi$ is a linear isomorphism. ∎

8. Integral points

We do not have a good understanding of all integral points on cubic surfaces, but Zariski density is known in many cases.

Paragraph 47 (Cubics containing a line). Let K be a number field with ring of integers R. Let S be an affine, cubic surface that contains a K-line L. Projecting S from L gives a map $\pi : S \setminus L \dashrightarrow \mathbb{P}^1$ whose general fibers are $\mathbb{G}_m$-torsors. This is not very helpful if $\overline{S}$ has a singular point on $\overline{L}$. However, if $\overline{S}$ is smooth along $\overline{L}$, then π was used to prove the Zariski density of R-points in increasing generality in the papers [2, 3, 16, 23, 29]; see also [15] for related results. The surface S_0 is treated already in [29].

As we noted in Paragraph 18, there is such a K-line on $S_{a,b}$ iff both $a(x)$ and $b(y)$ have a root in K. Next, we see how to use this approach to get integral points.

Paragraph 48. Let $S_{a,b}$ be as in Claim 15. As in Paragraph 17, we think of it as the blow-up of $\mathbb{A}^2$ at 6 base points, corresponding to the roots of $a(x)$ and $b(y)$.

Let α be a root of $a(x)$ and β a root of $b(y)$. Write $a(x) = (\alpha^{-1}x - 1)p(x)$ and $b(y) = (\beta^{-1}y - 1)q(y)$. Then

$$Q_{\lambda,\mu} := \lambda\big(p(x) + q(y) - c\big) + \mu xy$$

is a pencil of quadratic polynomials, vanishing at 4 of the 6 base points. Thus, on $S_{a,b}$ the birational transforms $C_{\lambda,\mu}$ of the curves $(Q_{\lambda,\mu} = 0)$ form a pencil of conics residual to the line $L_{\alpha,\beta}$, which is the birational transform of $(\alpha^{-1}x + \beta^{-1}y = 1)$.

Assume now that α, β are rational and let (x_0, y_0, z_0) be an integral point on $S_{a,b}$. There is a unique (up to scalar) Q_{λ_0,μ_0} that vanishes at (x_0, y_0).

If we are lucky, then $Q_{\lambda_0,\mu_0}(x,y)=0$ has infinitely many integer solutions. This is not automatic, and we used [1] to check cases.

If there are infinitely many integer solutions, then they are described by starting values and a linear recursion. This implies that

(1) the number of solutions of $Q_{\lambda_0,\mu_0}(x,y)=0$ grows like $\varepsilon\log N$, and

(2) the solutions are periodic modulo any integer.

We need to understand when the corresponding z values are integers. The $C_{\lambda,\mu}$ have 2 points at infinity; hence, the restriction of z to $C_{\lambda,\mu}$ is a linear function in x, y, with rational coefficients. Thus, Paragraph 48 (2) implies that z is periodic modulo 1. So, if there is one value (x_0, y_0) for which z_0 is an integer, then there are infinitely many. Also, in these cases, the $Q_{\lambda_0,\mu_0}=0$ are hyperbolas; hence, one of the branches lies in the fundamental domain for $x \gg 1$.

The automorphism group then moves these solutions to get new ones, but these seem to grow double exponentially. We have thus proved the following.

Corollary 48 (3). *If there is such a* $Q_{\lambda_0,\mu_0}(x,y)$ *and* $\operatorname{Aut}(S_{a,b})$ *is infinite, then the number of integral points on* $S_{a,b}$ *grows at least like* $\varepsilon\log N\log\log N$. ■

Now consider $S_0=(xyz=x^3+y^3+1)$. The rational roots are $\alpha=\beta=-1$, and

$$Q_{\lambda,\mu}=\lambda(x^2+y^2-x-y+1)+\mu xy.$$

For the first 3 solutions $(1,1,3)$, $(1,2,5)$, $(2,3,6)$, the resulting degree 2 equation $Q_{\lambda_0,\mu_0}=0$ has only finitely many integer solutions. For the next case $(2,9,41)$, the degree 2 equation is

$$6x^2-25xy+6y^2-6x-6y+6=0 \tag{48.4}$$

which has infinitely many integer solutions; see [1]. Thus, Paragraph 48 (1) applies and we get the following.

Corollary 48 (5). *The number of integral points on* $xyz=x^3+y^3+1$ *satisfying* $1\le x, y\le N$ *grows at least like* $\varepsilon\log N\log\log N$ *for some* $\varepsilon>0$. ■

Remark 48 (6). The recursion for the equation (48.4) involves 13 digit coefficients. The next 2 cases $(3,14,66)$, $(5,9,19)$ also give infinitely many solutions; for these the recursion involves 3 digit coefficients.

Paragraph 49 (Which power of log?). In order to develop a heuristic, we work with the divisibility conditions $x \mid b(y)$ and $y \mid a(x)$. The likelihood that x divides a random integer is $\frac{1}{x}$. Thus, if $b(y)$ and $a(x)$ are random, then we expect that the number of solutions satisfying $1\le x,y\le N$ is $\int_1^N\int_1^N \frac{1}{xy}dx\,dy=\log^2 N$.

Taking into account that we also need $x \le b(y)$ and $y \le a(x)$ refines this guess to $(1 - \frac{n+m}{2nm}) \log^2 N$, where $m = \deg a, n = \deg b$.

By contrast, the extension of the Batyrev–Manin conjectures to K3 surfaces (see, for example, [14, 38]) suggests that the exponent of $\log N$ should be the Picard number (over $\mathbb{Q}$). Over a field k, the Picard number of $S_{a,b}$ equals

$$\#\bigl(\text{irreducible factors of } a(x)\bigr) + \#\bigl(\text{irreducible factors of } b(y)\bigr) - 2.$$

For various choices of $a(x), b(y)$, we searched for all positive integral solutions up to 10^5. The results suggest a Batyrev–Manin–type dependence.

A detailed theoretical and numerical study of integral points on cubic surfaces is given in [4]. For the surfaces $S_{a,b}$, [4, Conjecture 1.1] predicts that the exponent of $\log N$ should be the Picard number plus 2. This seems larger than what we found. Note, however, that in many examples, the number of points fluctuates wildly for $N < 10^5$ but stabilizes for $N \gg 10^5$; see especially [4, Figure 2].

Further numerical work would be needed to get a clearer picture.

PARAGRAPH 50 (Density of orbits). Using the notation of Definition 16 for cubic surfaces $S_{a,b}$, by direct computation, we see that σ_y increases the multiplicity of a curve germ. Thus, a 1-dimensional, closed subscheme of $S_{a,b}$ cannot be invariant under $\mathrm{Aut}(S_{a,b})$. Therefore, every infinite orbit of $\mathrm{Aut}(S_{a,b})$ is Zariski dense, and there are no non-trivial $\mathrm{Aut}(S_{a,b})$-equivariant morphisms from $S_{a,b}$ to another variety. So $\sigma_{a,b}$ is loxodromic.

As an example, assume in addition that all coefficients of $a(x)$ and $b(y)$ are non-negative. Then $S_{a,b} \cap (\mathbb{R}^+)^3$ is $\mathrm{Aut}(S_{a,b})$-invariant, and $x \ge y > 0$ implies that $a(x)/y > x$. Thus, every $\mathrm{Aut}(S_{a,b})$ orbit on $S_{a,b} \cap (\mathbb{R}^+)^3$ is infinite.

If $a(t), b(t) \in \mathbb{Z}[t]$, then $S_{a,b}$ has a positive, integral point by [34, Theorem 4], so the positive, integral points are Zariski dense on $S_{a,b}$.

Any $\mathbb{Z}$-morphism $\mathbb{A}^1 \to S_{a,b}$ gives infinitely many integral points of $S_{a,b}$; these are called *affine lines*. The existence of affine lines depends subtly on $a(x), b(y)$. The surface S_0 seems rather special.

EXAMPLE 51 (Affine lines on S_0). For $P_0 = \mathbb{P}^2 \setminus (xyz = x^3 + y^3)$, all $\mathbb{C}$-embeddings $\mathbb{A}^1 \hookrightarrow P_0$ were enumerated in [19, Corollary 8]. Since $S_0 \to P_0 = \mathbb{P}^2 \setminus (xyz = x^3 + y^3)$ is unramified, each of these lifts to 3 embeddings $\mathbb{A}^1 \hookrightarrow S_0$, permuted by μ (Paragraph 10). We discuss these and their integral structures.

(1) If $x = 0$, then $y^3 = -1$. The rational solution is $y = -1$, giving the line $\ell_1 : t \mapsto (0, -1, t)$. We get integral points for $t \in \mathbb{Z}$. This line and its images by $\mathrm{Aut}_{\mathbb{Z}}(S_0)$ are disjoint from $(\mathbb{R}^+)^3$. The same holds for $\ell_2 : t \mapsto (-1, 0, t)$.

(2) $(3x+3y+z=0)$. Then $xyz - x^3 - y^3 = -(x+y)^3$. So $(x+y)^3 = -1$, giving the parametrization $\ell_3 : t \mapsto (t, -1-t, 3)$. We get integral points for $t \in \mathbb{Z}$. As before, this line and its images by $\mathrm{Aut}_{\mathbb{Z}}(S_0)$ are disjoint from $(\mathbb{R}^+)^3$.

(3) $(21x^2 - 22xy + 21y^2 - 6xz - 6yz + z^2 = 0)$. As a conic in $\mathbb{P}^2$, an explicit birational parametrization is given by

$$\phi : (u{:}v) \mapsto (u^2 - uv + v^2, u^2 + uv + v^2, 2u^2 + 10v^2).$$

For $u, v \in \mathbb{R}$, the image $\phi(u,v)$ lies in $(\mathbb{R}^+)^3$, but we check in Corollary 53 that this curve does not even have $\mathbb{Z}_2$-points. Thus, we do not get any integral points in $(\mathbb{R}^+)^3$ from this or its $\mathrm{Aut}_{\mathbb{Z}}(S_0)$-images.

However, setting $v = 1$, we get

$$t \mapsto (t^2 - t + 1, t^2 + t + 1, 2t^2 + 10),$$

giving $\sqrt{2N}$ positive integer solutions satisfying $1 \le x, y \le N$, for the equation $xyz = x^3 + y^3 + 8$.

See also [37] for further examples and applications.

LEMMA 52. *Let R be a UFD such that either* 2 *is invertible in R or* $\deg(R/pR : \mathbb{F}_2)$ *is odd for every* $p \mid 2$. *Then all R-solutions of* $(21x^2 - 22xy + 21y^2 - 6xz - 6yz + z^2 = 0)$ *are*

$$\phi(u,v) := (u^2 - uv + v^2, u^2 + uv + v^2, 10u^2 + 2v^2) \quad \textit{where } u, v \in R.$$

PROOF. One checks that ϕ is an isomorphism over $\mathbb{Q}$.

Let p be a prime in R such that $p \mid \phi(u,v)$. Then $p \mid 2uv$. So if $p \nmid 2$, then either $p \mid u$ or $p \mid v$. Then $p \mid u^2 - uv + v^2$ gives that it divides the other too.

If $p \mid 2$, then the only condition is $p \mid u^2 - uv + v^2$. If $\deg(R/pR : \mathbb{F}_2)$ is odd, then $x^2 - x + 1$ has no roots in R/pR, and then p divides both u and v. ■

COROLLARY 53. $(21x^2 - 22xy + 21y^2 - 6xz - 6yz + z^2 = 0)$ *has no* $\mathbb{Z}_2$ *points on* S_0.

PROOF. Computing $x^3 + y^3 - xyz$ on $\phi(u,v)$, we get $8v^6$. This cannot be 1. ■

EXAMPLE 54 (More affine lines). Let S be a smooth cubic surface as in Paragraph 12. Arguing as in [36, Proposition 2.3], we see that there are infinitely many smooth, rational curves on $\bar{S}$ with only 1 place at infinity.

These can be constructed using morphisms $\rho : \bar{S} \to B_1\mathbb{P}^2$, the 1 point blow-up of $\mathbb{P}^2$. Moreover, we get such rational curves defined over a field k iff there is such a ρ defined over k.

For the cubics $S_{a,b}$, such a ρ exists iff all roots of $a(t), b(t)$ are in k. On $\bar{S}_0$, smooth rational curves defined over $\mathbb{Q}$ are either lines or conics.

For integral points, the relevant question is the existence of $\mathbb{Z}$-morphisms $\mathbb{A}^1 \to S_0$; being an embedding is not important. There are many other $\mathbb{C}$-morphisms $\mathbb{A}^1 \to S_0$ [5, 27], but we do not know any other $\mathbb{Z}$-morphisms $\mathbb{A}^1 \to S_0$. Our methods do not seem to be able to prove that there are none.

Example 55. In the pencil of elliptic curves [19, Example 46]

$$\lambda(xyz - x^3 - y^3) + \mu(-xy^2 - x^2z + yz^2) = 0,$$

there are 3 rational curves with 1 place on $(xyz = x^3 + y^3)$. One of them, for $(\lambda : \mu) = (-3 : 1)$, is defined over $\mathbb{Q}$. It can be parametrized as

$$t \mapsto (-t^2 + t - 1, t, t^3 - 2t^2 + 3t - 3),$$

giving a family of integer solutions, but no positive ones.

Note that, as a curve in $\mathbb{P}^2$, it has a node at (2:1:5). The corresponding t values are the non-trivial cube roots of 1. When we lift to S_0, the curve becomes smooth, and the point $(2, 1, 5)$ is not on it. This curve turns out to be the same as $\sigma_y \circ \ell_3$ (Example 51 (2)).

Paragraph 56 (Another form of $S_{a,b}$). It is easy to see that for any n, m, the pair of divisibility conditions

$$x \mid y^m + 1 \quad \text{and} \quad y \mid x^n + 1$$

is equivalent to the equation $xyz = x^n + y^m + 1$. More generally, set

$$S'_{a,b} := \big(yu - a(x) = xv - b(y) = 0\big) \subset \mathbb{A}^4_{xyuv}. \tag{56.1}$$

There is a natural morphism $S_{a,b} \to S'_{a,b}$ given by $u := xz - b^*(y)$ and $v := yz - a^*(x)$. Then

$$cz = uv - a^*(x)b^*(y);$$

hence, $S_{a,b} \cong S'_{a,b}$ over R if c is a unit.

If $a(x), b(y)$ are cubics, then the closure of $S'_{a,b}$ in $\mathbb{P}^4$ is a non-normal surface whose canonical class is ample, so, for geometric questions it is harder to work with. On the other hand, $S'_{a,b}$ exists even if $a(0) \neq b(0)$, giving different integral structures on the surfaces $S_{a,b}$.

Furthermore, for any polynomials $a_i(x), b_i(y)$, there is a natural morphism

$$S'_{a_1,b_1} \to S'_{a_1a_2,b_1b_2}. \tag{56.2}$$

Thus, if $a_1(0) = b_1(0) = a_2(0) = b_2(0) = 1$, then we get a natural morphism

$$S_{a_1,b_1} \to S_{a_1a_2,b_1b_2}. \tag{56.3}$$

Example 57. Applying (56.3) to $S_0 = (xyz = x^3 + y^3 + 1)$, we get 4 surfaces with $\mathbb{Z}$-morphisms to it. These are $(xyz = x^2 + y^2 - x - y + 1)$, $(xyz = x^2 - x + y + 1)$, $(xyz = y^2 - y + x + 1)$, and $(xyz = x + y + 1)$. It is easy to see that the integral points are not Zariski dense on the last 3 surfaces. For example, for $(xyz = x^2 - x + y + 1)$, write $xu = y + 1$. Then $xu - 1 \mid x^2 - x + 1$ and $xu - 1 \mid ux^2 - ux + u = x(xu - 1) - (xu - 1) + x + u - 1$. Thus, $xu - 1 \mid x + u - 1$.

The integral points on $(xyz = x^2 + y^2 - x - y + 1)$ are also not Zariski dense; see [30, pp. 299–300].

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