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On the Stability of Symmetric Flows in a Two-Dimensional Channel

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Abstract

We consider the stability of symmetric flows in a two-dimensional channel (including the Poiseuille flow). In 2015 Grenier, Guo, and Nguyen have established instability of these flows in a particular region of the parameter space, affirming formal asymptotics results from the 1940s. We prove that these flows are stable outside this region in parameter space. More precisely, we show that the Orr–Sommerfeld operator

$$\mathcal{B} = \Big(-\frac{d^2}{dx^2} + i\beta(U + i\lambda)\Big)\Big(\frac{d^2}{dx^2} - \alpha^2\Big) - i\beta U'',$$

which is defined on

$$D(\mathcal{B}) = \big\{u \in H^4(0,1), u'(0) = u^{(3)}(0) = 0 \text{ and } u(1) = u'(1) = 0\big\}$$

is bounded on the half-plane $\Re\lambda \geq 0$ for $\alpha \gg \beta^{-1/10}$ or $\alpha \ll \beta^{-1/6}$.

Keywords. hydrodynamic stability, Orr–Sommerfeld, Poiseuille, non-self-adjoint

Mathematics Subject Classification (2020). Primary 76E05; Secondary 35P15

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Chapter 1

Introduction

1.1 Main results

Consider the incompressible Navier–Stokes equations in the two-dimensional pipe $D = \mathbb{R} \times (-1, 1)$

$$\begin{cases} \partial_t \mathbf{v} - \epsilon \Delta \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p & \text{in } \mathbb{R}_+ \times D, \\ \mathbf{v} = v_b \hat{i}_1 & \text{on } \mathbb{R}_+ \times \partial D, \end{cases} \tag{1.1.1}$$

where $\hat{i}_1 = (1, 0)$, $\mathbf{v} = (v_1, v_2)$ is the fluid velocity, and p is the pressure.

The parameter

$$R := \frac{1}{\epsilon} \tag{1.1.2}$$

is the Reynolds number of the flow and

$$v_b : \partial D \to \mathbb{R}$$

is the boundary velocity.

Since the flow is incompressible we must have

$$\operatorname{div} \mathbf{v} = 0.$$

We linearize (1.1.1) near the laminar flow (cf. [3])

$$\mathbf{v} = U(x_2)\hat{i}_1,$$

to obtain the linearized equation

$$\mathbf{u}_t - \mathcal{T}_0(\mathbf{u}, q) = 0,$$

where $\mathbf{u} = (u_1, u_2)$ and q are defined on $\mathbb{R}_+ \times D$, and $\mathcal{T}_0$ is the map

$$(\mathbf{u}, q) \mapsto \mathcal{T}_0(\mathbf{u}, q) := -\epsilon\, \Delta \mathbf{u} + U\, \frac{\partial \mathbf{u}}{\partial x_1} + u_2\, U'\, \hat{i}_1 - \nabla q. \tag{1.1.3}$$

We proceed with a formal derivation of the Orr–Sommerfeld equation, intentionally skipping the definitions of $\mathbf{v}$, p, $\mathbf{u}$, and q. Interested readers can read the entire derivation in [3]. The associated resolvent equation for $\mathcal{T}_0$ assumes the form

$$\mathcal{T}_0(\mathbf{u}, q) - \Lambda \mathbf{u} = \mathbf{f}, \tag{1.1.4}$$

where $\operatorname{div} \mathbf{u} = 0$ and $\Lambda \in \mathbb{C}$ is the spectral parameter.

Hence, we may define a stream function

$$\mathbf{u} = \nabla_\perp \psi = (-\psi_{x_2}, \psi_{x_1}).$$

Substituting the above into (1.1.4) and then taking the curl of the ensuing equation for ψ yields

$$\Big(-\epsilon\Delta^2 + U\frac{\partial}{\partial x_1}\Delta - U''\frac{\partial}{\partial x_1} - \Lambda\,\Delta\Big)\psi = F, \tag{1.1.5}$$

where $F = \operatorname{curl} f$.

We consider $U \in C^2([-1,1])$ (we later restrict ourselves to $U \in C^4([-1,1])$) satisfying the following:

$$U(\pm 1) = 0, \tag{1.1.6a}$$

$$\max_{x\in[-1,1]} U''(x) < 0, \tag{1.1.6b}$$

$$U(-x) = U(x). \tag{1.1.6c}$$

We normalize U so that

$$U'(\pm 1) = \mp 1. \tag{1.1.6d}$$

Substituting $\psi(x_1, x_2) = \phi(x_2)\, e^{i\alpha x_1}$ into (1.1.5) yields for $\phi : (-1,1) \to \mathbb{C}$ the equation

$$\mathcal{B}^{\mathfrak{D}}_{\lambda,\alpha,\beta}\phi = f, \tag{1.1.7a}$$

where (setting $x_2 = x$)

$$\mathcal{B}^{\mathfrak{D}}_{\lambda,\alpha,\beta} = (\mathcal{L}_\beta - \beta\lambda)\Big(\frac{d^2}{dx^2} - \alpha^2\Big) - i\beta U'', \tag{1.1.7b}$$

where

$$\mathcal{L}_\beta = -\frac{d^2}{dx^2} + i\beta U. \tag{1.1.7c}$$

In the above

$$\beta = \alpha\epsilon^{-1} = \alpha R \tag{1.1.8}$$

(R being the Reynolds number introduced in (1.1.2)), and, for $\beta \neq 0$,

$$\lambda = \beta^{-1}\Big(\frac{\Lambda}{\epsilon} - \alpha^2\Big). \tag{1.1.9}$$

We refer to [3, Section 3] for the details of the derivation. We use the pair of parameters (α, β) instead of (α, R) since the asymptotic limit we consider in the sequel is $\beta \to \infty$.

We define $\mathcal{B}^{\mathfrak{D}}_{\lambda,\alpha,\beta}$ on

$$D(\mathcal{B}^{\mathfrak{D}}_{\lambda,\alpha,\beta}) = \big\{u \in H^4(-1,1), u(1) = u'(1) = u(-1) = u'(-1) = 0\big\}. \tag{1.1.10}$$

We focus our interest on the restriction $\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta}$ of the operator $\mathcal{B}^{\mathfrak{D}}_{\lambda,\alpha,\beta}$ to functions that are symmetric with respect to the reflection $x \mapsto -x$. Hence, we are led to consider the equivalent restricted operator $\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}$ on $(0,1)$ whose domain is

$$D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}) = \{u \in H^4(0,1), u'(0) = u^{(3)}(0) = 0 \text{ and } u(1) = u'(1) = 0\}. \quad (1.1.11)$$

We leave the discussion of anti-symmetric modes to future research. Note that since

$$\mathcal{B}^{U}_{\lambda,\alpha,\beta} = \overline{\mathcal{B}^{-U}_{\bar{\lambda},\alpha,\beta}}$$

(where $\bar{\mathcal{B}}$ denotes the complex conjugate of $\mathcal{B}$) the analysis in the sequel applies to the case where $\min_{x\in[-1,1]} U''(x) > 0$ as well. Clearly, the Poiseuille flow associated with $U(x) = (1-x^2)/2$ meets all the criteria in (1.1.6). Another example mentioned in [13] is given by $U(x) = (2/\pi)\cos \pi x/2$. Note that U is decreasing on $(0,1)$.

In [13, Theorem 1.1], it has been established by Grenier–Guo–Nguyen that for sufficiently large R and for each α satisfying

$$C_L\, R^{-1/7} \le \alpha \le C_R\, R^{-1/11},$$

or equivalently when β is large and

$$C_L^{7/6}\,\beta^{-1/6} \le \alpha \le C_R^{11/10}\,\beta^{-1/10},$$

there exists $\lambda \in \mathbb{C}$ with negative real part such that $\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}$ is not invertible. For the case of a Poiseuille flow, the positive constants C_L and C_R have been determined from well-known formal asymptotic calculations (cf. the book [12] by P. G. Drazin and W. H. Reid).

In the present contribution, we consider the converse problem, i.e., we attempt to show that, for any $\delta > 0$, $(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}$ is bounded for $\Re\lambda \le 0$ when

$$\alpha \ge \beta^{-1/10+\delta} \quad \text{or} \quad 0 \le \alpha \le \alpha_L\,\beta^{-1/6}.$$

Note that unlike [13] we do not provide the precise estimate derived by formal asymptotics, i.e., $\alpha_L < C_L$ and $\beta^{-1/10+\delta} > \beta^{1/10}$. The determination of the precise curves is left to future research.

Recall from equation (1.1.8) that $\beta = \alpha/\epsilon$. Our main results are the following two theorems.

Theorem 1.1.1. *Let $U \in C^4([-1,1])$ satisfy (1.1.6). Then there exist positive α_L, C, Υ, and $\beta_0 > 1$ such that for all $\beta > \beta_0$ it holds that*

$$\sup_{\substack{0\le\alpha\le\alpha_L\beta^{-1/6}\\ \Re\lambda<\Upsilon\beta^{-1/2}}} \left\|(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\right\| + \left\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\right\| \le C\,\beta^{-1/2}\log\beta. \quad (1.1.12)$$

The condition $\alpha \in [0, \alpha_L \beta^{-1/6}]$ can be rephrased in terms of the pair (α, R) as $\alpha \in [0, \alpha_L^{6/7} R^{-1/7}]$.

Theorem 1.1.2. *Let $U \in C^4([-1,1])$ satisfy (1.1.6). Let further $\hat{\mu}_m > 0$ be given by* [3, equation (6.57)]. *Then for any $\delta > 0$ and any $\hat{\Upsilon} > 0$, there exist positive C, Υ, and β_0 such that for all $\beta > \beta_0$ we have*

$$\sup_{\substack{\beta^{-1/10+\delta} \leq \alpha \\ \Re\lambda \leq \min(\Upsilon\beta^{-1/2}, \beta^{-1/3}[\hat{\mu}_m - \hat{\Upsilon} - \alpha^2\beta^{-2/3}/2])}} \big\|(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\big\| + \Big\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\Big\| \leq C\beta^{-1/2+\delta}. \tag{1.1.13}$$

The condition $\alpha \geq \beta^{-1/10+\delta}$ can be rephrased in terms of the pair (α, R) as $\alpha \geq R^{(-1+10\delta)/(11-10\delta)}$. Note that the condition $\Re\lambda \leq \beta^{-1/3}([\hat{\mu}_m - \hat{\Upsilon} - \alpha^2\beta^{-2/3}/2])$ guarantees, by (1.1.9) that $\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta}$ is invertible for $\Lambda \leq \alpha\beta^{-1/3}[\hat{\mu}_m + \alpha^2\beta^{-2/3}/2 - \hat{\Upsilon}]$ and hence the stability of the laminar flow even for $\alpha \gtrsim \beta^{1/3}$.

In the above

$$\big\|(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\big\| = \sup_{\substack{f \in L^2(0,1) \\ \|f\|_2 = 1}} \|(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1} f\|_2$$

and

$$\Big\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\Big\| = \sup_{\substack{f \in L^2(0,1) \\ \|f\|_2 = 1}} \Big\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1} f\Big\|_2,$$

where $\|\cdot\|_2$ denotes the standard $L^2(0,1)$ norm.

In recent years, hydrodynamic stability of shear flows has attracted significant attention. For the case of a Couette flow we mention only a partial list of rigorous analytical results [4–6, 8]. In [3], we have established similar estimates for the Orr–Sommerfeld operator, together with semigroup estimates for the linearized Navier–Stokes operator in the case where $|U'| > 0$ in $[-1,1]$ (see also the works of Chen, Wei, and Zhang [9] and of Jia [17] for recent generalizations). In contrast with the present case the Orr–Sommerfeld operator has, when $|U''| > 0$, a bounded resolvent in the half-plane $\Re\lambda \leq 0$.

The hydrodynamic stability of symmetric flows in a channel has been considered extensively in physics (cf. for instance [12,19–21]). These works, just like that of [13], all attempt to determine as function of β the region in the $(\alpha, \Im\lambda)$ plane where the Orr–Sommerfeld is unstable. In a recent work [11], the stability of Poiseuille flow has been established in the case of a Navier-slip boundary condition. This means that the boundary condition $u'(\pm 1) = 0$ in (1.1.11) is replaced by $u''(\pm 1) = 0$. The stability of a pipe Poiseuille flow has also been addressed in [10].

1.2 Proof strategy

In the following informal discussion, we present the main ingredients of the proofs of Theorems 1.1.1 and 1.1.2. Some of the definitions appearing in the discussion will remain slightly vague and will be reformulated more precisely in the next sections. The reader may be interested in reviewing the relevant part of this presentation before diving into the technical details of any part of the analysis in the sequel.

We begin with a rather heuristic discussion. Consider the equation

$$\mathcal{B}_{\lambda,\alpha,\beta}\phi = f, \tag{1.2.1}$$

where $(\phi, f) \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}) \times L^2(0,1)$ and $\mathcal{B}_{\lambda,\alpha,\beta} = \mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}$.

Our goal is to estimate the operator $\mathcal{B}^{-1}_{\lambda,\alpha,\beta}$ under specific conditions on the parameters $(\lambda, \alpha, \beta) \in \mathbb{C} \times \mathbb{R}_+ \times \mathbb{R}_+$. We may rewrite the above equation in the form

$$\mathcal{A}_{\lambda,\alpha}\phi = v, \tag{1.2.2a}$$

where

$$v = \beta^{-1}[f + \phi^{(4)} - \alpha^2\phi''] \tag{1.2.2b}$$

and

$$\mathcal{A}_{\lambda,\alpha} = (U + i\lambda)\Big(-\frac{d^2}{dx^2} + \alpha^2\Big) + U'' \tag{1.2.2c}$$

is the inviscid (Rayleigh) operator whose study will be the main object of Section 1.2. We define $\mathcal{A}_{\lambda,\alpha}$ for $\Re\lambda \neq 0$ or when $\Re\lambda = 0$ and $\Im\lambda \notin [0, U(0)]$, on

$$D(\mathcal{A}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha}) = \big\{\phi \in H^2(0,1) \mid \phi'(0) = 0 \text{ and } \phi(1) = 0\big\}. \tag{1.2.2d}$$

It intuitively appears that v should tend to 0 as $\beta \to \infty$, and thus, we adopt the following proof strategy.

(1) We prove that v becomes small as $\beta \to \infty$.

(2) We obtain a bound for $\|\mathcal{A}^{-1}_{\lambda,\alpha}v\|_{1,2}$ (where $\|\cdot\|_{1,2}$ denotes the standard $H^1(0,1)$ norm).

After successfully completing the above stages we expect to obtain an inequality of the form

$$\|\phi'\|_2 \leq \delta_1(\beta)\|f\|_2 + \delta_2(\beta)\|\phi'\|_2.$$

If for sufficiently large β it holds that $\delta_2(\beta) < 1/2$, we can conclude from here an estimate for $\|\mathcal{B}^{-1}_{\lambda,\alpha,\beta}\| + \|\frac{d}{dx}\mathcal{B}^{-1}_{\lambda,\alpha,\beta}\|$.

Estimation of $\mathcal{A}_{\lambda,\alpha}^{-1}$. We use in Chapter 2 a similar procedure to the one used in [3]. Let $\lambda = \mu + i\nu$. Given that $|U''| > 0$ in $[0,1]$ and since

$$\Im\Big\langle \phi, \frac{\mathcal{A}_{\lambda,\alpha}\phi}{U+i\lambda}\Big\rangle = -\mu\Big\|\frac{|U''|^{1/2}\phi}{U+i\lambda}\Big\|_2^2, \tag{1.2.3}$$

we easily obtain that

$$\Big\|\frac{\phi}{U+i\lambda}\Big\|_2 \leq \frac{C}{|\mu|}\|v\|_2. \tag{1.2.4}$$

From the above (accompanied by a rather straightforward integration by parts) it is not difficult to show that

$$\|\mathcal{A}_{\lambda,\alpha}^{-1}\| + \Big\|\frac{d}{dx}\mathcal{A}_{\lambda,\alpha}^{-1}\Big\| \leq \frac{C}{|\mu|}.$$

The above estimate is unsatisfactory in the limit $\mu \to 0$, and hence finer estimates need to be established. We use the fact that $\mathcal{A}_{i\nu,\alpha}$ is self-adjoint. Thus, for $\nu \notin (0, U(0)]$ (see Sections 2.7 and 2.10) we may write

$$\Big\langle \frac{\phi}{U-\nu}, \mathcal{A}_{i\nu,\alpha}\phi\Big\rangle = \Big\|(U-\nu)\Big(\frac{\phi}{U-\nu}\Big)'\Big\|_2^2 + \alpha^2\|\phi\|_2^2,$$

and obtain from it an estimate for $\|\phi'\|_2$ in the case where $|\mu|$ is small.

For $\nu \in (0, U(0)]$, we have to address the singularity where $U = \nu$. Given the fact that U is increasing in $[0,1]$ there exists a unique $x_\nu \in [0,1)$ where $U(x_\nu) = \nu$. Let $\chi \in C^\infty([0,1],[0,1])$ denote a cutoff function supported on $[0,(1+x_\nu/2)]$. Setting $\varphi = \phi - \phi(x_\nu)\chi$ we may write

$$\begin{aligned}\Big\|(U-\nu)\Big(\frac{\varphi}{U-\nu}\Big)'\Big\|_2^2 &+ \alpha^2\|\phi\|_2^2 \\ &= \Big\langle \frac{\varphi}{U-\nu}, \mathcal{A}_{i\nu,\alpha}\varphi\Big\rangle - \phi(x_\nu)\Big[\Big\langle \frac{\varphi}{U-\nu}, U''\chi\Big\rangle - \langle \varphi, \chi'' - \alpha^2\chi\rangle\Big].\end{aligned} \tag{1.2.5}$$

For the above balance to become useful for the purpose of obtaining estimates for $\|\phi'\|_2$, we need to obtain an estimate for $\phi(x_\nu)$. To this end we use (1.2.3) to obtain (for $\mu \neq 0$)

$$\mu|\phi(x_\nu)|^2\Big\|\frac{1}{U+i\lambda}\|_2^2 \leq \|\phi\|_\infty\Big\|\frac{v}{U+i\lambda}\Big\|_1 + C|\mu|\Big\|\frac{\phi-\phi(x_\nu)}{U-\nu}\Big\|_2^2.$$

Under the condition in [3] on U, which is assumed to be strictly monotone, the above estimates leads to

$$|\phi(x_\nu)| \leq \|\phi\|_\infty\Big\|\frac{v}{U+i\lambda}\Big\|_1 + C|\mu|\,\|\phi'\|_2.$$

Substituting the above into (1.2.5) (properly amended to account for small values of $|\mu|$) yields an estimate for $\|\phi'\|_2$.

To adapt the above method to the present context we need to overcome several difficulties.

(1) It holds that $\mathcal{A}_{0,0}U = 0$ and since $U \in D(\mathcal{A}_{0,0})$, $(\mathcal{A}_{\lambda,\alpha})^{-1}$ becomes strongly singular in the limit $(\lambda, \alpha) \to (0, 0)$.

(2) The boundary condition at $x = 0$ is a Neumann condition in contrast with the Dirichlet condition in [3]. Thus, we have to write (1.2.5) separately on the intervals $(0, x_\nu)$ and $(x_\nu, 1)$. On $(x_\nu, 1)$ we may use the same method used in [3]. However, on $(0, x_\nu)$, considering $\phi/(U - \nu)$, we obtain bounds of this quotient for small values of α that are significantly greater than those obtained for larger values of α.

(3) The quadratic behavior of $U - U(0)$ as opposed to the linear behavior considered in [3].

The first pair of difficulties is addressed by the same techniques.

- For small values of α we use the fact that we can consider $(\mathcal{A}_{0,\lambda})^{-1}$ as an integral operator to obtain satisfactory estimates for its norm (see Proposition 2.4.1).
- For larger values of α we use again (1.2.5) (see Section 2.5).

In Section 2.6, we present a different analysis, which is valid for any $\alpha \geq 0$ with stronger singularity in the limit $\lambda \to 0$. In all cases, we use the orthogonal decomposition $\phi = C_\parallel(U - \nu) + \phi_\perp$ to obtain separate estimates for $C_\parallel$ and $\phi_\perp$, estimates of the latter being significantly smaller than the former. To overcome the last difficulty we simply use (1.2.4) in Section 2.9. In Section 2.8, we consider the transition between the quadratic behavior of U near x_ν and the linear behavior considered in Section 2.6.

Estimate of $\mathcal{B}^{-1}_{\lambda,\alpha,\beta}$. To obtain an estimate of v (see (1.2.2b)) we set

$$v_{\mathfrak{D}} := \mathcal{A}_{\lambda,\alpha}\phi + (U + i\lambda)\phi''(1)\hat{\psi}, \tag{1.2.6}$$

where

$$\hat{\psi}(x) = \frac{\mathrm{Ai}\big(\beta^{1/3}e^{-i\pi/6}[(1-x) - i\lambda]\big)}{\mathrm{Ai}\big(e^{-i2\pi/3}\beta^{1/3}\lambda\big)}\eta(x).$$

Here, Ai is the Airy function and $\eta \in C^\infty([0,1],[0,1])$ is supported on $(1/4, 1]$, and satisfies $\eta \equiv 1$ on $[1/2, 1]$. Note that $v_{\mathfrak{D}}(1) = v'_{\mathfrak{D}}(0) = 0$ and that $\hat{\psi}$ is a good approximation for the $L^2(-\infty, 1)$ solution of

$$\begin{cases} \big(-\frac{d^2}{dx^2} + i\beta[(1-x) + i\lambda]\big)u = 0 & \text{in } (-\infty, 1), \\ u(1) = 1. \end{cases} \tag{1.2.7}$$

We can now rewrite (1.2.1) in the form

$$(\mathcal{L}_\beta - \beta\lambda)v_{\mathfrak{D}} = g_{\mathfrak{D}},$$

where

$$\mathcal{L}_\beta = -\frac{d^2}{dx^2} + i\beta U$$

is defined on

$$D(\mathcal{L}_\beta) = \{u \in H^2(0,1) \mid u(1) = u'(0) = 0\}.$$

While the precise form of $g_{\mathfrak{D}}$ need not concern us in this brief summary of the proof we still need to obtain an estimate of its $L^2(0,1)$ norm. Thus, we get an estimate of v by working through the following steps.

(1) Estimate of $\phi''(1)$.

(2) Estimate of $g_{\mathfrak{D}}$.

(3) Estimate of $v_{\mathfrak{D}}$.

(4) Estimate of $\mathcal{A}_{\lambda,\alpha}^{-1} v_{\mathfrak{D}}$ and of $\phi''(1)\mathcal{A}_{\lambda,\alpha}^{-1}(U + i\lambda)\hat{\psi}$.

We use two different methods for the estimation of $\phi''(1)$.

For α values that are not too small. We rewrite (1.2.1) in the form

$$\Big(-\frac{d^2}{dx^2} + i\beta[U + i\lambda]\Big)(\phi'' - \alpha^2\phi) = i\beta U''\phi + f.$$

Given $\phi(1) = \phi'(1) = \phi'(0) = 0$, we may conclude that for $\mathfrak{z}(x) = \cosh(\alpha x)/\cosh\alpha$ it holds that

$$\langle \mathfrak{z}, \phi'' - \alpha^2\phi\rangle = 0. \tag{1.2.8a}$$

Consequently, we define the Schrödinger operator $\mathcal{L}_\beta^{\mathfrak{z}}$ with the same differential operator as for $\mathcal{L}_\beta$ but with the following domain

$$D(\mathcal{L}_\beta^{\mathfrak{z}}) = \big\{u \in H^2(0,1) \mid u'(0) = 0, \langle \mathfrak{z}, u\rangle = 0\big\}. \tag{1.2.8b}$$

Let $(v, g) \in D(\mathcal{L}_\beta^{\mathfrak{z}}) \times L^2(0,1)$ satisfy

$$(\mathcal{L}_\beta^{\mathfrak{z}} - \beta\lambda)v = g. \tag{1.2.9}$$

In Chapter 4, we obtain estimates for $v(1)$ that we later use in Chapter 5 (except for Sections 5.8 and 5.7) to obtain an estimate for $\phi''(1)$. Again, we have to distinguish between the quadratic case (Section 4.3) and the linear case (Section 4.2).

For smaller values of α and $|\lambda|$. The estimate of $\phi''(1)$, obtained by the above technique becomes deficient, given the singularity of $\mathcal{A}_{0,0}$. We thus integrate (1.2.1) for $\alpha = 0$ to obtain

$$\phi^{(3)}(1) = -\int_0^1 f(x)\,dx. \tag{1.2.10}$$

Then we use the identity

$$\|(U'')^{-1/2}\phi^{(3)}\|_2^2 = -\Re\langle (U'')^{-1}\phi'', \mathcal{B}_{\lambda,0,\beta}\phi\rangle - \frac{1}{U''(1)}\Re(\bar{\phi}''(1)\phi^{(3)}(1))$$
$$- \Re\langle [(U'')^{-1}]'\phi'', \phi^{(3)}\rangle + \mu\beta\|(U'')^{-1/2}\phi''\|_2^2$$

to obtain a proper bound for $\|\phi^{(3)}\|_2$, which together with Sobolev embedding (skipping, of course, some of the details) leads to a satisfactory bound for $|\phi''(1)|$. Note that the effectiveness of this technique is lost when α is not small since (1.2.10) is no longer valid. We use it only in Section 5.8.

Finally, we note that for $\alpha \gtrsim \beta^{-1/3}$ $\mathfrak{z}$ undergoes significant changes through an $\mathcal{O}(\beta^{-1/3})$ boundary layer near $x = 1$. Hence, we can no longer make any good use of (1.2.7) as an estimate for the behavior near $x = 1$ of the solution of (1.2.9). Instead, we need to use the same method developed in [3] for this case. Note that, since $\mathfrak{z}$ is localized near $x = 1$ for large values of α, the effect of the different boundary conditions at $x = 0$ here and in [3] is exponentially small. Resolvent estimates for $\mathcal{L}_\beta^{\mathfrak{z}}$ in this case are brought in Section 4.7 whereas estimates for the inverse of $\mathcal{B}_{\lambda,\alpha,\beta}$ are given in Section 5.7.

Remark. We note an error in the derivation of [3, equation (8.96)] where the term $\Theta_+'\psi_+'/\psi_+(1)$ (Θ_+ is analogous to χ in the present contribution) was overlooked. This error does not affect at all the validity of [3, equation (8.96)] given that the error generated by the missing term can be estimated using (4.2.17), and is much smaller than the right-hand side of [3, equation (8.96)] (which is greater or equal than $C\beta^{-1/4}$ for some positive C).

We skip the rather technical stage of estimating $g_{\mathfrak{D}}$. Once it is done, we need to estimate $(\mathcal{L}_\beta - \beta\lambda)^{-1}$ in $\mathcal{L}(L^2, L^p)$ for $1 \leq p \leq \infty$ in order to obtain an appropriate estimate for $v_{\mathfrak{D}}$. These estimates are obtained in Chapter 3 for various ranges of λ values. Again we need to distinguish between the linear behavior of $U - U(x_\nu)$ near $x = x_\nu$ (Section 3.1) and the quadratic behavior near $x = 0$ for $\nu = U(0)$ (Section 3.2). Special attention is also devoted to $\mathcal{L}(L^2, L^1)$ and $\mathcal{L}(H^1, L^1)$ estimates (Section 3.3).

Next, we estimate $\mathcal{A}_{\lambda,\alpha}^{-1} v_{\mathfrak{D}}$ using the aforementioned techniques of Chapter 2. For $\mathcal{A}_{\lambda,\alpha}^{-1}(U + i\lambda)\widehat{\psi}$ we use the exponential decay of $\widehat{\psi}$ away from $x = 1$ to obtain the desired estimate in a rather straightforward manner, except in the case $(\lambda, \alpha) \to (0, 0)$. These estimates are addressed in Section 5.2.

Finally, in Chapter 6 we summarize the results of Chapter 5 and prove the main theorems.

Chapter 2

The inviscid operator

2.1 Preliminaries

We begin by presenting the notation, frequently used in the sequel

$$\langle f, g\rangle_{L^2(a,b)} = \int_a^b \bar{f}(x)g(x)\,dx.$$

Note that when $(a, b) = (0, 1)$ the abbreviated notation $\langle f, g\rangle$ is used instead of $\langle f, g\rangle_{L^2(0,1)}$.

Next, recall that the differential expression of the inviscid operator (also called the Rayleigh operator) is given by

$$\mathcal{A}_{\lambda,\alpha} \stackrel{\text{def}}{=} (U + i\lambda)\Big(-\frac{d^2}{dx^2} + \alpha^2\Big) + U'', \tag{2.1.1}$$

where $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{R}$.

Note that, for any $\phi \in D(\mathcal{B}^{\mathfrak{D}}_{\lambda,\alpha,\beta})$, we have

$$\mathcal{A}_{\lambda,\alpha}\phi = \lim_{\beta\to\infty} \beta^{-1}\mathcal{B}^{\mathfrak{D}}_{\lambda,\alpha,\beta}\phi.$$

We consider here only spaces of even functions in $(-1, 1)$, hence, as explained in the introduction, we restrict the operator to $(0, 1)$ and consider the Neumann condition at 0 and the Dirichlet condition at 1. We thus define $\mathcal{A}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha}$ as follows:

- for $\Re\lambda \neq 0$ or when $\Re\lambda = 0$ and $\Im\lambda \notin [0, U(0)]$, on

$$D(\mathcal{A}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha}) = \big\{\phi \in H^2(0, 1) \mid \phi'(0) = 0 \text{ and } \phi(1) = 0\big\}, \tag{2.1.2a}$$

- for $\Re\lambda = 0$ and $\Im\lambda \in [0, U(0)]$

$$D(\mathcal{A}^{\mathfrak{N},\mathfrak{D}}_{0,\alpha}) = \big\{\phi \in H^2((0, 1), (U - \Im\lambda)^2dx) \mid \phi'(0) = 0 \text{ and } \phi(1) = 0\big\}, \tag{2.1.2b}$$

  which is equipped with the norm

$$\|u\|_{D(\mathcal{A}^{\mathfrak{N},\mathfrak{D}}_{0,\alpha})} = \int_0^1 [|u''|^2] + [|u'|^2 + |u|^2](U - \Im\lambda)^2\,dx.$$

Hence, in the following, we restrict attention to the interval [0, 1] assuming in this section that $U \in C^3([0,1])$ and satisfies the condition:

$$U(1) = 0, \tag{2.1.3a}$$

$$\max_{x\in[0,1]} U''(x) < 0, \tag{2.1.3b}$$

$$U'(0) = 0. \tag{2.1.3c}$$

We normalize U so that

$$U'(1) = -1. \tag{2.1.3d}$$

For convenience of notation we omit the superscripts $\mathfrak{N}$ and $\mathfrak{D}$ in the sequel whenever there is not any fear of ambiguity. Since

$$\mathcal{A}^U_{\lambda,\alpha} = \overline{\mathcal{A}^{-U}_{\bar{\lambda},\alpha}},$$

the analysis in this section applies to the case where $\min_{x\in[-1,1]} U''(x) > 0$ replaces equation (2.1.3b) as well.

In this section, we obtain a variety of estimates for $\mathcal{A}^{-1}_{\lambda,\alpha}$ that are necessary in order to obtain bounds in the same parameter regime for $\mathcal{B}^{-1}_{\lambda,\alpha}$. We note that since

$$\mathcal{A}_{\lambda,\alpha} = \overline{\mathcal{A}_{-\bar{\lambda},\alpha}}$$

the results in this section do not depend on the sign of $\Re\lambda$.

Let $\lambda = \mu + i\nu$. We begin by summarizing the results of this section in Figure 2.1. We map in this figure the regions in the $(|\mu|, \nu)$ plane where the various estimates of $\mathcal{A}^{-1}_{\lambda,\alpha}$ can be found. We refer the reader to Section 1.2 where the consideration that led us to split the λ plane to these regimes are briefly explained. We note that the results of Sections 2.6–2.11 are valid for any $\alpha \geq 0$. Section 2.3 addresses the case $\alpha = \lambda = 0$, Section 2.4 addresses small α values, and Section 2.5 relatively large values of α. The constants determining the boundaries of the various regimes satisfy $0 < \nu_1 < U(0)$, $\nu_0 < 0$, and κ_0 must be sufficiently large while μ_0 and λ_0 must be sufficiently small.

2.2 Some more preliminaries

We begin by defining the following (formally) self-adjoint unbounded operator on

$$H^0_U(0,1) := L^2((0,1), U^2dx)$$

by

$$\mathcal{M}_U = -U^{-2}\frac{d}{dx}U^2\frac{d}{dx}, \tag{2.2.1a}$$

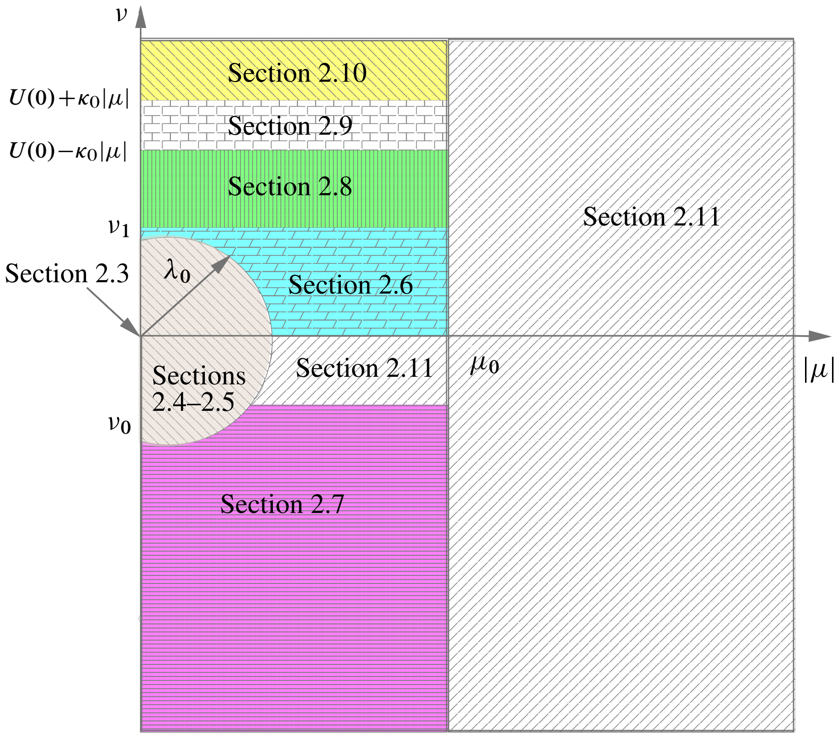

Figure 2.1. Summary of the results in Chapter 2. For each zone in $\mathbb{C}$ for the parameter λ, we indicate the section where the basic inequality is proved. Recall that $\mu = \Re\lambda$, and $\nu = \Im\lambda$, μ_0, ν_0, ν_1, λ_0, κ_0 are positive constants which are introduced in the various sections.

which is naturally associated via the Lax–Migram lemma with the quadratic form $\mathcal{Q}_U$, defined on

$$H^1_U(0,1) = \left\{u \in L^2_{\text{loc}}(0,1) \mid Uu \in L^2(0,1) \text{ and } Uu' \in L^2(0,1)\right\} \tag{2.2.1b}$$

by

$$\mathcal{Q}_U(w) = \|Uw'\|_2^2. \tag{2.2.1c}$$

Recall that U satisfies (2.1.3).

Remark 2.2.1. When U is replaced by $e^{-\phi}$, we arrive at a well-known problem considered in Statistical Mechanics and Morse theory (Witten Laplacians), see, for example, [14] and references therein. As observed in [7], we arrive at a singular case of this theory because U is vanishing at $x = 1$.

We can now state the following lemma.

Lemma 2.2.2. *Suppose that $\mathcal{M}_U$ is a self-adjoint operator with compact resolvent on $L^2((0,1), U^2\,dx)$. If $\{\kappa_n\}_{n=1}^{\infty}$ denotes the non-decreasing sequence of which the* *spectrum $\sigma(\mathcal{M}_U)$ is consisted, then*

$$\kappa_1 = 0$$

and

$$\kappa_2 \geq \lambda_2^N |U(0)|^2, \tag{2.2.2}$$

where λ_2^N denotes the second eigenvalue of the radially symmetric Neumann–Laplacian (i.e., the Laplacian reduced to the radially symmetric functions satisfying the Neumann condition.) in the unit ball in $\mathbb{R}^3$.

Proof. The proof that $\kappa_1 = 0$ is trivial (the associated eigenfunction is the non-vanishing constant function). To prove the lower bound for κ_2 we first observe that, by the variational max-min characterization of the second eigenvalue,

$$\kappa_2 = \sup_{\psi \in H_U^1(0,1)} \inf_{\substack{w \in H_U^1(0,1) \\ \langle w, U^2\psi\rangle = 0}} \frac{\|Uw'\|_2^2}{\|Uw\|_2^2} = \sup_{\phi \in H_U^1(0,1)} \inf_{\substack{w \in H_U^1(0,1) \\ \langle w, (1-x)^2\phi\rangle = 0}} \frac{\|Uw'\|_2^2}{\|Uw\|_2^2}. \tag{2.2.3}$$

The second equality can be proved by writing $\psi = \frac{(1-x)^2}{U^2}\phi$. Then, we use the fact that by the concavity of U

$$U(0)(1-x) \leq U(x) \leq (1-x) \quad \text{(see (2.1.3b))} \tag{2.2.4}$$

to obtain

$$|U(0)|^{-2}\frac{\|(1-x)w'\|_2^2}{\|(1-x)w\|_2^2} \geq \frac{\|Uw'\|_2^2}{\|Uw\|_2^2} \geq |U(0)|^2\frac{\|(1-x)w'\|_2^2}{\|(1-x)w\|_2^2}. \tag{2.2.5}$$

Let $U_0(x) = 1-x$ and κ_n^0 denote the nth eigenvalue of $\mathcal{M}_{U_0}$. By (2.2.5) and [15, Theorem 11.12] we have that

$$|U(0)|^2\kappa_2^0 \leq \kappa_2 \leq |U(0)|^{-2}\kappa_2^0. \tag{2.2.6}$$

Setting $\rho(x) = 1-x$ we obtain that

$$\mathcal{M}_{U_0} = -\rho^{-2}\frac{d}{d\rho}\rho^2\frac{d}{d\rho},$$

which is defined on (the Neumann condition is the natural boundary condition associated with Q_{U_0})

$$D(\mathcal{M}_{U_0}) = \{u \in H^2([0,1];\rho^2 d\rho) \mid u'(1) = 0\}.$$

Hence, $\mathcal{M}_{U_0}$ is the radially symmetric Neumann Laplacian, and we may conclude that

$$\kappa_2^0 = \lambda_2^N.$$

The above, together with (2.2.6) yields (2.2.2). ∎

We next recall Hardy's inequality on finite intervals (see, for example, [18, equation (1.25)]).

Lemma 2.2.3. *Let $w \in H^1(a,b)$ satisfy $w(b) = 0$. Then, we have*

$$\|([x-a]w)'\|_2^2 = \|[x-a]w'\|_2^2 \geq \frac{1}{4}\|w\|_2^2. \tag{2.2.7}$$

Proof. Let $\tilde{w} \in H^1(a,\infty)$ be given by

$$\tilde{w}(x) = \begin{cases} w(x), & x \in (a,b), \\ 0, & x \geq b. \end{cases}$$

Then Hardy's inequality on $\mathbb{R}_+$ applied to $\tilde{w}$ implies that

$$\|([x-a]w)'\|_{L^2(a,b)}^2 = \|([x-a]\tilde{w})'\|_{L^2(a,\infty)}^2 \geq \frac{1}{4}\|\tilde{w}\|_{L^2(a,\infty)}^2 = \frac{1}{4}\|w\|_{L^2(a,b)}^2.$$

To complete the proof we first write

$$\|([x-a]w)'\|_2^2 = \|[x-a]w'\|_2^2 + \|w\|_2^2 + 2\Re\langle w, [x-a]w'\rangle.$$

An integration by parts then yields

$$2\Re\langle w, [x-a]w'\rangle = \Re\langle [x-a], (|w|^2)'\rangle = -\|w\|_2^2.$$

Hence,

$$\|([x-a]w)'\|_2^2 = \|[x-a]w'\|_2^2,$$

which completes the proof of the lemma. ∎

If we drop the requirement that $w(b) = 0$ we can state the following lemma.

Lemma 2.2.4. *Let $w \in H^1(a,b)$. Then, we have*

$$\|([x-a]w)'\|_2^2 \geq \frac{1}{4}\|w\|_2^2. \tag{2.2.8}$$

Proof. We use Hardy's inequality in $\mathbb{R}_+$ for the extension

$$\tilde{w}(x) = \begin{cases} w(x), & x \in (a,b), \\ w(b)\frac{b-a}{x-a}, & x \geq b. \end{cases}$$ ∎

2.3 Estimates in the case $\alpha = \lambda = 0$

We begin with the simplest possible case, for which $\alpha = \lambda = 0$. We recall that

$$\mathcal{A}_{0,0} := -U\frac{d^2}{dx^2} + U''$$

is defined on

$$D(\mathcal{A}_{0,0}) = \{u \in H^2((0,1), U^2dx) \mid u(1) = u'(0) = 0\},$$

corresponding to a Dirichlet–Neumann problem on $(0,1)$.

Next, let $W^{1,p}(0,1)$ denote the normed space

$$W^{1,p}(0,1) := \{u \in L^p(0,1) \mid u' \in L^p(0,1)\},$$

with its natural norm denoted by $\| \cdot \|_{1,p}$.

Observing that U belongs to the kernel of $\mathcal{A}_{0,0}$, we set

$$\phi = c_\parallel U + \phi_\perp, \tag{2.3.1a}$$

where

$$c_\parallel = \frac{\langle U, \phi \rangle}{\|U\|_2^2}, \tag{2.3.1b}$$

in which $\langle \cdot, \cdot \rangle$ denotes the natural $L^2(0,1)$ inner product.

Lemma 2.3.1. *Let $U \in C^2([0,1])$ satisfy (2.1.3). There exists $C > 0$ such that for any $(\phi, v) \in D(\mathcal{A}_{0,0}) \times W^{1,2}(0,1)$ satisfying*

$$\mathcal{A}_{0,0}\,\phi = v, \tag{2.3.2a}$$

$$\langle 1, v \rangle = 0 \quad \textit{and} \quad v(1) = 0 \tag{2.3.2b}$$

and

$$\lim_{\substack{x\to 1\\ x<1}} \phi'(x) = 0, \tag{2.3.2c}$$

we have

$$\|\phi_\perp\|_{1,2} \leq C\,\|v\|_2 \tag{2.3.3a}$$

and

$$|c_\parallel| \leq C\,\|v\|_2^{1/2}\|v\|_{1,2}^{1/2}. \tag{2.3.3b}$$

Proof. Let $w = \phi/U$. Clearly, $w = c_\parallel + w_\perp$ with $w_\perp = \phi_\perp/U$, and

$$\mathcal{A}_{0,0}\,\phi = U^2\mathcal{M}_U\,w = U^2\mathcal{M}_U\,w_\perp = v. \tag{2.3.4}$$

Step 1. Estimate of $\|\phi_\perp'\|_2$. Taking the inner product with $w_\perp$ yields

$$\|Uw_\perp'\|_2^2 = \langle w_\perp, v \rangle. \tag{2.3.5}$$

Since $\langle w_\perp, U^2 \rangle = \langle \phi_\perp, U \rangle = 0$, we now use (2.2.2), (2.3.5), and Hardy's inequality (2.2.7) to obtain that

$$\|\phi_\perp\|_2^2 = \|Uw_\perp\|_2^2 \leq \kappa_2\|Uw_\perp'\|_2^2 \leq \kappa_2\left\|\frac{\phi_\perp}{U}\right\|_2\|v\|_2 \leq C\,\|\phi_\perp'\|_2\|v\|_2. \tag{2.3.6}$$

Note that by (2.1.3) and (2.2.7) it holds that

$$\begin{aligned}\|w_\perp\|_2 &\leq \Big\|\frac{\phi_\perp}{U(0)(1-x)}\Big\|_2 \leq \Big\|\frac{\tilde{\phi}_\perp}{U(0)(1-x)}\Big\|_{L^2(-\infty,1)} \\ &\leq \frac{2}{U(0)}\|\tilde{\phi}_\perp'\|_{L^2(-\infty,1)} = \frac{2}{U(0)}\|\phi_\perp'\|_2,\end{aligned} \tag{2.3.7}$$

where $\tilde{\phi}_\perp \in H^1_{\mathrm{loc}}((-\infty,1])$ is given by

$$\tilde{\phi}_\perp(x) = \begin{cases}\phi_\perp(x), & x \in [0,1],\\ \phi_\perp(0), & x < 0.\end{cases}$$

Integration by parts yields

$$\|\phi_\perp'\|_2^2 = \|(Uw_\perp)'\|_2^2 = \|Uw_\perp'\|_2^2 - \langle Uw_\perp, U''w_\perp\rangle \leq \|Uw_\perp'\|_2^2 + C\|\phi_\perp\|_2\|w_\perp\|_2. \tag{2.3.8}$$

Using (2.3.5) and (2.3.7), we obtain

$$\|\phi_\perp'\|_2 \leq C(\|v\|_2 + \|\phi_\perp\|_2).$$

By (2.3.6) we then obtain

$$\|\phi_\perp'\|_2 \leq C\|v\|_2. \tag{2.3.9}$$

Note that to obtain (2.3.9) we have used the mere fact that $v \in L^2(0,1)$.

Step 2. Estimate of $c_\parallel$. By (2.3.2a) and the fact that $\mathcal{A}_{0,0}\phi_\perp = v$, it holds that

$$\|\phi_\perp''\|_2 \leq C\Big(\Big\|\frac{\phi_\perp}{U}\Big\|_2 + \Big\|\frac{v}{U}\Big\|_2\Big).$$

By Hardy's inequality (2.2.8) we then obtain that

$$\|\phi_\perp''\|_2 \leq C(\|\phi_\perp'\|_2 + \|v'\|_2).$$

Using (2.3.9) we may conclude that

$$\|\phi_\perp''\|_2 \leq C\|v\|_{1,2}. \tag{2.3.10}$$

Using Sobolev embedding, (2.1.3d) and (2.3.2c) yields

$$|c_\parallel| = |\phi_\perp'(1)| \leq \|\phi_\perp'\|_\infty \leq \|\phi_\perp''\|_2^{1/2}\|\phi_\perp'\|_2^{1/2}.$$

We can now conclude (2.3.3b) from (2.3.9) and (2.3.10). ■

2.4 Estimate of $(\mathcal{A}_{\lambda,\alpha})^{-1}$ for $\Re\lambda \neq 0$ and $\alpha \ll |\lambda|^{1/2}$

We continue with the following estimate of $(\mathcal{A}_{\lambda,0})^{-1}$ when $\Re\lambda \neq 0$. From (2.1.1), we recall that

$$\mathcal{A}_{\lambda,0} \stackrel{\text{def}}{=} -(U+i\lambda)\frac{d^2}{dx^2} + U'',$$

and that its domain is defined in (2.1.2a):

$$D(\mathcal{A}_{\lambda,0}) = \{u \in H^2(0,1) \mid u(1) = u'(0) = 0\}.$$

We shall then consider $\mathcal{A}_{\lambda,\alpha}^{-1}$ for α small enough.

Proposition 2.4.1. *Let $p > 1$ and $U \in C^3([0,1])$ satisfy (2.1.3). There exists $C > 0$ such that, for $\lambda \in \mathbb{C}$ satisfying $\Re\lambda \neq 0$ and $|\lambda| < U(3/4)$, it holds for any $(\phi, v) \in D(\mathcal{A}_{\lambda,0}) \times L^p(0,1)$ satisfying $\mathcal{A}_{\lambda,0}\,\phi = v$ that*

$$\|\phi\|_{1,2} \leq C\Big(\frac{1}{|\lambda|}\Big|\int_0^1 v\,dx\Big| + \Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|v\|_p\Big). \tag{2.4.1}$$

Proof. Step 1. We prove that

$$\|\phi\|_\infty \leq C\Big(\frac{1}{|\lambda|}\Big|\int_0^1 v\,dx\Big| + \Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|v\|_p\Big). \tag{2.4.2}$$

Step 1.1. The estimate on $[1/2, 1]$. As, for any $\phi \in D(\mathcal{A}_{\lambda,0})$,

$$\mathcal{A}_{\lambda,0}\phi = -[(U+i\lambda)\phi' - U'\phi]',$$

we may conclude that

$$\frac{\phi(x)}{U(x)+i\lambda} = -\int_x^1 [K_2(x,\lambda) - K_2(t,\lambda)]v(t)dt + K_2(x,\lambda)\int_0^1 v(t)\,dt, \tag{2.4.3}$$

where

$$K_2(x,\lambda) = \int_x^1 \frac{ds}{(U+i\lambda)^2(s)}.$$

A first integration by parts yields

$$K_2(x,\lambda) = \frac{1}{U'(U+i\lambda)} + \frac{1}{i\lambda} - \int_x^1 \frac{U''ds}{(U')^2(U+i\lambda)}.$$

An additional integration by parts further gives

$$\begin{aligned}-\int_x^1 &\frac{U''ds}{(U')^2(U+i\lambda)} \\ &= \frac{U''}{(U')^3}\log(U+i\lambda) - U''(1)\log(i\lambda) + \int_x^1 \Big(\frac{U''}{(U')^3}\Big)'\log(U+i\lambda)\,ds.\end{aligned}$$

For $|\lambda| < U(3/4)$ (a bounded set in $\mathbb{C}$), there exists $C > 0$ such that for all $x \in [1/2, 1]$ (where $U'(x) \neq 0$)

$$\Big| \int_x^1 \Big(\frac{U''}{(U')^3}\Big)' \log(U + i\lambda)\, ds \Big| \leq C. \tag{2.4.4}$$

To prove (2.4.4), we introduce, for $\nu > 0$, the real value x_ν, which is defined by

$$U(x_\nu) = \nu \text{ for } 0 < \nu < U(0),\ x_\nu = 1 \text{ if } \nu \leq 0 \text{ and } x_\nu = 0 \text{ if } \nu > U(0). \tag{2.4.5}$$

Note that, for $0 < \nu \leq U(0)$, $x_\nu \in [0, 1)$, is indeed uniquely defined by the assumed monotonicity of U (see (2.1.3)).

Then we use the fact that for $|\lambda| < U(3/4)$ (implying $|\nu| < U(3/4)$, $|U'(x_\nu)| > 0$), there exists $C > 0$ such that

$$|\log(U(x) + i\lambda)| \leq C\,[1 + \log|x - x_\nu|^{-1}].$$

Hence,

$$\log(U + i\lambda) \text{ is uniformly bounded in } L^q(0,1) \quad \forall\, 1 \leq q < +\infty, \tag{2.4.6}$$

readily verifying (2.4.4).

Consequently, it holds that

$$\Big| K_2(x,\lambda) - \frac{1}{U'(U + i\lambda)} - \frac{1}{i\lambda} - \frac{U''}{(U')^3} \log(U + i\lambda) + U''(1)\log(i\lambda) \Big| \leq C. \tag{2.4.7}$$

Let

$$\hat{K}_2(x,\lambda) = K_2(x,\lambda) - \frac{1}{i\lambda} + U''(1)\log(i\lambda). \tag{2.4.8}$$

By (2.4.7), we get the existence of $C > 0$ such that, for any $x \in [1/2, 1]$ and $|\lambda| < U(3/4)$ we have

$$\big|(U + i\lambda)(x)\hat{K}_2(x,\lambda)\big| \leq C. \tag{2.4.9}$$

We now write, for any $x \in [1/2, 1]$,

$$\Big| \int_x^1 \frac{v}{U'(U + i\lambda)} dt \Big| \leq C \Big\| \frac{v}{U + i\lambda} \Big\|_1,$$

and hence, by (2.4.6) and (2.4.7), for all $x \in [1/2, 1]$ and $|\lambda| \leq U(3/4)$ we have, for (p, q) satisfying $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} \Big| \int_x^1 \hat{K}_2(t,\lambda) v(t) dt \Big| &\leq C\Big(\Big\| \frac{v}{U + i\lambda} \Big\|_1 + \|\log(U + i\lambda)\|_q \|v\|_p + \|v\|_1 \Big) \\ &\leq \hat{C}\Big(\Big\| \frac{v}{U + i\lambda} \Big\|_1 + \|v\|_p \Big). \end{aligned} \tag{2.4.10}$$

Substituting (2.4.8), (2.4.9), and (2.4.10) into (2.4.3) given that $K_2(x,\lambda)-K_2(t,\lambda)=\widehat{K}_2(x,\lambda)-\widehat{K}_2(t,\lambda)$ yields

$$\|\phi\|_{L^\infty(1/2,1)} \leq C\Big(\frac{1}{|\lambda|}\Big|\int_0^1 v\,dx\Big| + \Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|v\|_p\Big), \tag{2.4.11}$$

which completes the estimate of ϕ in $[\frac{1}{2},1]$.

Step 1.2. Estimate of $\|\phi\|_{L^\infty(0,1/2)}$. Next, we consider the case where $x\in[0,1/2)$. In this interval we need to address the fact that $U'(0)=0$, as is assumed in (2.1.3a). Here, we use the assumption $|\lambda|<U(3/4)$ to obtain that for $x\in(0,1/2)$,

$$\Big|K_2(x,\lambda)-K_2\Big(\frac{1}{2},\lambda\Big)\Big| = \Big|\widehat{K}_2(x,\lambda)-\widehat{K}_2\Big(\frac{1}{2},\lambda\Big)\Big| \leq C. \tag{2.4.12}$$

Since by (2.4.9)

$$\big|(U(1/2)+i\lambda)\widehat{K}_2(1/2,\lambda)\big| \leq C,$$

we obtain that

$$\Big|\widehat{K}_2(1/2,\lambda)\Big| \leq C. \tag{2.4.13}$$

Combining (2.4.13) with (2.4.12) and (2.4.8) yields

$$\|K_2(\cdot,\lambda)\|_{L^\infty(0,1/2)} \leq \frac{C}{|\lambda|}.$$

Hence, for all $x\in[0,1/2]$ it holds that

$$|K_2(x,\lambda)(U+i\lambda)(x)| \leq \frac{C}{|\lambda|}. \tag{2.4.14}$$

Furthermore, by (2.4.8), (2.4.12), and (2.4.13) we have that

$$\begin{aligned}
&\Big|\int_x^1 (K_2(t,\lambda)-K_2(x,\lambda))v(t)dt\Big| \\
&\quad \leq \Big|\int_x^{1/2} (\widehat{K}_2(t,\lambda)-\widehat{K}_2(x,\lambda))v(t)dt\Big| + \Big|\int_{1/2}^1 (\widehat{K}_2(t,\lambda)-\widehat{K}_2(x,\lambda))v(t)dt\Big| \\
&\quad \leq \Big|\int_{1/2}^1 (\widehat{K}_2(t,\lambda)-\widehat{K}_2(x,\lambda))v(t)dt\Big| + C\|v\|_1 \\
&\quad \leq \widehat{C}\Big(\frac{1}{|\lambda|}\Big|\int_0^1 v\,dx\Big| + \Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|v\|_p\Big),
\end{aligned}$$

where, to obtain the last inequality, we used (2.4.10).

We can then conclude that

$$\Big|(U+i\lambda)(x)\int_x^1 \Big(K_2(t,\lambda)-K_2(x,\lambda)\Big)v(t)dt\Big| \\ \leq C\Big(\frac{1}{|\lambda|}\Big|\int_0^1 v\,dx\Big| + \Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|v\|_p\Big).$$

Substituting the above, together with (2.4.14) into (2.4.3) yields

$$\|\phi\|_{L^\infty(0,1/2)} \leq C\Big(\frac{1}{|\lambda|}\Big|\int_0^1 v\,dx\Big| + \Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|v\|_p\Big).$$

Combined with (2.4.11) the above readily yields (2.4.2).

Step 2. We prove (2.4.1). We begin by rewriting $\mathcal{A}_{\lambda,0}\,\phi = v$ in the form

$$-\phi'' = -\frac{U''}{U+i\lambda}\phi + \frac{v}{U+i\lambda}.$$

Taking the inner product with ϕ in $L^2(0,1)$, integration by parts yields, as $\phi'(0) = \phi(1) = 0$,

$$\|\phi'\|_2^2 = \Re\Big\langle\phi, \frac{U''\phi}{U+i\lambda}\Big\rangle + \Re\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle. \tag{2.4.15}$$

Let $\widehat{\chi} \in C^\infty(\mathbb{R},[0,1])$ satisfy

$$\widehat{\chi}(x) = \begin{cases} 0 & |x| < \frac{1}{4}, \\ 1 & |x| > \frac{1}{2}. \end{cases} \tag{2.4.16}$$

Let further $\tilde{\chi} = 1 - \widehat{\chi}$. The first term on the right-hand side of (2.4.12) can be rewritten after an integration by part as

$$-\Re\Big\langle\phi, \frac{U''\phi}{U+i\lambda}\Big\rangle = \Re\Big\langle\Big(\frac{U''|\phi|^2\widehat{\chi}}{U'}\Big)', \log(U+i\lambda)\Big\rangle - \Re\Big\langle\tilde{\chi}\phi, \frac{U''\phi}{U+i\lambda}\Big\rangle. \tag{2.4.17}$$

For the first term on the right-hand side of (2.4.17), we write

$$\Big|\Re\Big\langle\Big(\frac{U''|\phi|^2\widehat{\chi}}{U'}\Big)', \log(U+i\lambda)\Big\rangle\Big| \leq C\|\phi\|_\infty\|[|\phi|+|\phi'|]\log(U+i\lambda)\|_1.$$

Using (2.4.6) together with Poincaré's inequality leads to

$$\Big|\Re\Big\langle\Big(\frac{U''|\phi|^2\widehat{\chi}}{U'}\Big)', \log(U+i\lambda)\Big\rangle\Big| \leq \widehat{C}\|\phi\|_\infty\|\phi'\|_2. \tag{2.4.18}$$

For the second term on the right-hand side of (2.4.17) we have, observing that $U(x) + i\lambda$ does not vanish for x in the support of $\tilde{\chi}$ and $|\lambda| < U(3/4)$,

$$\Big|\Re\Big\langle\tilde{\chi}\phi, \frac{U''\phi}{U+i\lambda}\Big\rangle\Big| \leq C\|\phi\|_2^2. \tag{2.4.19}$$

Substituting (2.4.18) and (2.4.19) into (2.4.17) yields, again with the aid of Poincaré's inequality,

$$\Big|\Re\Big\langle\phi, \frac{U''\phi}{U+i\lambda}\Big\rangle\Big| \leq C\|\phi'\|_2\|\phi\|_\infty. \tag{2.4.20}$$

For the second term on the right-hand side of (2.4.15) we use Poincaré's inequality and Sobolev's embeddings to obtain

$$\Big|\Re\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big| \leq \|\phi\|_\infty\Big\|\frac{v}{U+i\lambda}\Big\|_1 \leq \|\phi'\|_2\Big\|\frac{v}{U+i\lambda}\Big\|_1. \tag{2.4.21}$$

Substituting (2.4.21) together with (2.4.20) into (2.4.15) yields

$$\|\phi'\|_2 \leq C\left(\Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|\phi\|_\infty\right).$$

We can now conclude (2.4.1) from (2.4.2). ∎

We can now state as a corollary, an estimate for $\mathcal{A}_{\lambda,\alpha}^{-1}$ when α is relatively small.

Corollary 2.4.2. *Under the assumptions of Proposition 2.4.1, there exist $C > 0$ and $\delta_0 > 0$ such that, for all $\lambda \in \mathbb{C}$ satisfying $\Re\lambda \neq 0$ and $|\lambda| < U(3/4)$, for all α such that*

$$|\alpha| \leq \delta_0 \min(|\lambda|^{1/2}, \log^{-1/2}|\mu|^{-1}),$$

and for any pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times L^p(0,1)$ satisfying

$$\mathcal{A}_{\lambda,\alpha}\phi = v, \tag{2.4.22}$$

it holds that

$$\|\phi\|_{1,2} \leq C\Big(\frac{1}{|\lambda|}\Big|\int_0^1 v\,dx\Big| + \Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|v\|_p\Big).$$

The proof is obtained by rewriting (2.4.22) in the manner

$$\mathcal{A}_{\lambda,0}\phi = v + \alpha^2\phi,$$

or

$$\phi = \mathcal{A}_{\lambda,0}^{-1}v + \alpha^2\mathcal{A}_{\lambda,0}^{-1}\phi.$$

We now use (2.4.1) to obtain

$$\begin{aligned}\|\phi\|_{1,2} \leq C\Big(\frac{1}{|\lambda|}\Big|\int_0^1 v\,dx\Big| + \Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|v\|_p\Big) \\ + C\alpha^2\Big(\frac{1}{|\lambda|}\|\phi\|_1 + \Big\|\frac{\phi}{U+i\lambda}\Big\|_1 + \|\phi\|_p\Big).\end{aligned}$$

Since $|\nu| \leq U(3/4) < 1$,

$$\left\| \frac{1}{U + i\lambda} \right\|_1 \leq C \log|\mu|^{-1}, \tag{2.4.23}$$

to obtain that

$$\left\| \frac{\phi}{U + i\lambda} \right\|_1 \leq \left\| \frac{1}{U + i\lambda} \right\|_1 \|\phi\|_\infty \leq C \log|\mu|^{-1} \|\phi\|_\infty .$$

Sobolev embeddings, given that $|\alpha| < \delta_0 \min(|\lambda|^{1/2}, \log^{-1/2}|\mu|^{-1})$, complete the proof of the corollary when $p < \infty$ for sufficiently small δ_0.

2.5 Small $|\lambda|$ and $\alpha > \|U\|_2^{-1}(\Im\lambda)_+^{1/2}$

Set for any $\lambda \in \mathbb{C}$ and $\delta > 0$

$$\alpha_{\lambda,\delta} = \|U\|_2^{-1}((\Im\lambda)_+(1 + 2\delta))^{1/2}. \tag{2.5.1}$$

In this section, we attempt to prove the invertibility of $\mathcal{A}_{\lambda,\alpha}$ as defined in (2.1.2) for sufficiently small $|\lambda|$ and $\alpha \geq \alpha_{\lambda,\delta}$.

To be able to state the result of this section we define, for $p > 1$ and $\Re\lambda \neq 0$, on $W^{1,p}(0,1)$ the maps

$$v \mapsto N_0(v,\lambda) := \min\left(\left\| (1-x)^{1/2} \frac{v}{U + i\lambda} \right\|_1, \|v\|_{1,p} \right) \tag{2.5.2a}$$

and

$$N_1(v,\lambda) = \left| \left\langle 1, \frac{v}{U + i\lambda} \right\rangle \right|. \tag{2.5.2b}$$

We can now state and prove the following proposition.

Proposition 2.5.1. *Let $r > 1$, $p > 1$, and $\delta > 0$ and $U \in C^3([0,1])$ satisfy (2.1.3). There exist $\lambda_0 > 0$ and $C > 0$ such that for $0 < |\lambda| < \lambda_0$, $\alpha \geq \alpha_{\lambda,\delta}$ and $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ satisfying $\mathcal{A}_{\lambda,\alpha}\phi = v$, we have, with $c_\parallel = \langle U, \phi\rangle / \|U\|_2^2$ and $\phi_\perp = \phi - c_\parallel U$,*

$$|c_\parallel| \leq \frac{1 + C|\lambda|^2 \log|\lambda|^{-1}}{|\alpha^2 \|U\|_2^2 + i\lambda|} \big[\|v\|_1 + C|\lambda| N_1(v,\lambda) \big] \tag{2.5.3a}$$

and

$$\|\phi_\perp\|_{1,2} \leq C\Big[N_0(v,\lambda) + \frac{|\lambda|}{|\alpha^2\|U\|_2^2 + i\lambda|} (\|v\|_1 + |\lambda| N_1(v,\lambda)) \Big]. \tag{2.5.3b}$$

Proof. Step 1. We prove the existence of $\lambda_0 > 0$ and $C > 0$ such that, for $|\lambda| \leq \lambda_0$ and $\alpha \geq \alpha_{\lambda,\delta}$ it holds that

$$|c_\parallel| \leq \frac{1 + C|\lambda|^2 \log|\lambda|^{-1}}{|\alpha^2 \|U\|_2^2 + i\lambda|} \big(\|v\|_1 + C|\lambda| \|\phi_\perp\|_{1,2} + |\lambda| N_1(v,\lambda)\big). \tag{2.5.4}$$

As

$$U(-\phi'' + \alpha^2\phi) - U''\phi = v - i\lambda(-\phi'' + \alpha^2\phi) \tag{2.5.5}$$

or equivalently, by (2.3.2) and (2.3.4)

$$U^2(\mathcal{M}_U + \alpha^2)w = \frac{Uv}{U + i\lambda} + i\lambda \frac{U''\phi}{U + i\lambda}, \tag{2.5.6}$$

where $\mathcal{M}_U$ is given by (2.2.1) and $w = U^{-1}\phi$.

Taking the inner product with 1 and integrating by parts then yields

$$\alpha^2 \|U\|_2^2 c_\parallel = \Big\langle 1, \frac{Uv}{U + i\lambda} \Big\rangle + i\lambda \Big\langle 1, \frac{U''\phi}{U + i\lambda} \Big\rangle. \tag{2.5.7}$$

We now write

$$\Big\langle 1, \frac{U''\phi}{U + i\lambda} \Big\rangle = c_\parallel \Big\langle 1, \frac{U''U}{U + i\lambda} \Big\rangle + \Big\langle 1, \frac{U''\phi_\perp}{U + i\lambda} \Big\rangle. \tag{2.5.8}$$

For the first term on the right-hand side we write, as $U'(0) = 0$ and $U'(1) = -1$,

$$c_\parallel \Big\langle 1, \frac{U''U}{U + i\lambda} \Big\rangle = -c_\parallel \Big(1 + i\lambda \Big\langle 1, \frac{U''}{U + i\lambda} \Big\rangle\Big). \tag{2.5.9}$$

Since, for $\Im\lambda < U(1/2)$,

$$\begin{aligned} \Big\langle 1, \frac{U''}{U + i\lambda} \Big\rangle_{L^2(1/2,1)} &= -U''(1)\log(i\lambda) - \frac{U''(1/2)}{U'(1/2)} \log(U(1/2) + i\lambda) \\ &\quad - \Big\langle \Big(\frac{U''}{U'}\Big)', \log(U + i\lambda) \Big\rangle_{L^2(1/2,1)}, \end{aligned}$$

we may conclude the existence of $C > 0$ and $0 < \lambda_0 < U(1/2)$, such that, for $|\lambda| \leq \lambda_0$,

$$\Big| \Big\langle 1, \frac{U''}{U + i\lambda} \Big\rangle_{L^2(1/2,1)} \Big| \leq C \log|\lambda|^{-1}.$$

As

$$\Big| \Big\langle 1, \frac{U''}{U + i\lambda} \Big\rangle_{L^2(0,1/2)} \Big| \leq C,$$

we obtain

$$\Big| \Big\langle 1, \frac{U''}{U + i\lambda} \Big\rangle \Big| \leq C \log|\lambda|^{-1}. \tag{2.5.10}$$

Next, we consider the second term on the right-hand side of (2.5.8) (note that $\phi_\perp(1) = 0$)

$$\begin{aligned}
&\Big|\Big\langle 1, \frac{U''\phi_\perp}{U + i\lambda}\Big\rangle_{L^2(1/2,1)}\Big| \\
&\quad = \Big| - \frac{U''\phi_\perp}{U'} \log((U(1/2) + i\lambda))_{|x=1/2} + \Big\langle\Big(\frac{U''\phi_\perp}{U'}\Big)', \log(U + i\lambda)\Big\rangle\Big| \\
&\quad \leq C\|\phi_\perp\|_{1,2}.
\end{aligned}$$

For $\Im\lambda < U(1/2)$, we can write

$$\Big|\Big\langle 1, \frac{U''\phi_\perp}{U + i\lambda}\Big\rangle_{L^2(0,1/2)}\Big| \leq C\|\phi_\perp\|_1,$$

and hence we may conclude that

$$\Big|\Big\langle 1, \frac{U''\phi_\perp}{U + i\lambda}\Big\rangle\Big| \leq C\|\phi_\perp\|_{1,2}. \tag{2.5.11}$$

Substituting the above, together with (2.5.9) and (2.5.10) into (2.5.8) yields

$$\Big|\Big\langle 1, \frac{U''\phi}{U + i\lambda}\Big\rangle + c_\parallel\Big| \leq C|\lambda|\log|\lambda|^{-1}c_\parallel + C\|\phi_\perp\|_{1,2}. \tag{2.5.12}$$

We next rewrite (2.5.7) in the form

$$(\alpha^2\|U\|_2^2 + i\lambda)\,c_\parallel = \Big\langle 1, \frac{Uv}{U + i\lambda}\Big\rangle + i\lambda\Big(\Big\langle 1, \frac{U''\phi}{U + i\lambda}\Big\rangle + c_\parallel\Big), \tag{2.5.13}$$

and then observe that

$$\Big\langle 1, \frac{Uv}{U + i\lambda}\Big\rangle = \Big|\langle 1, v\rangle - i\lambda\Big\langle 1, \frac{v}{U + i\lambda}\Big\rangle\Big| \leq \|v\|_1 + |\lambda|N_1(v, \lambda).$$

Substituting the above together with (2.5.12) into (2.5.13) yields

$$|c_\parallel| \leq \frac{1}{|\alpha^2\|U\|_2^2 + i\lambda|}\big(\|v\|_1 + C|\lambda|^2\log|\lambda|^{-1}c_\parallel + C|\lambda|\|\phi_\perp\|_{1,2} + |\lambda|N_1(v, \lambda)\big).$$

Given the fact that when $\Im\lambda > 0$, $\alpha \geq \alpha_{\lambda,\delta}$, we have

$$\frac{1}{|\alpha^2\|U\|_2^2 + i\lambda|} \leq \frac{1}{\delta|\lambda|},$$

and that when $\Im\lambda \leq 0$ we have

$$\frac{1}{|\alpha^2\|U\|_2^2 + i\lambda|} \leq \frac{1}{|\lambda|},$$

we obtain (2.5.4) for sufficiently small $\lambda_0 > 0$.

Step 2. We prove (2.5.3).

Step 2.1. For $w_\perp = U^{-1}\phi_\perp$, we prove that

$$\|Uw_\perp'\|_2^2 + \alpha^2\|\phi_\perp\|_2^2 \leq C\|\phi_\perp'\|_2(N_0(v,\lambda) + |\lambda|\,|c_\||) + \Big|i\lambda\Big\langle\frac{\phi_\perp}{U}, \frac{U''\phi_\perp}{U+i\lambda}\Big\rangle\Big|. \tag{2.5.14}$$

Taking the inner product in (2.5.5) with $w_\perp$ yields (note that $w' = w_\perp'$)

$$\|Uw_\perp'\|_2^2 + \alpha^2\|\phi_\perp\|_2^2 = \Big\langle\phi_\perp, \frac{v}{U+i\lambda}\Big\rangle + i\lambda\Big\langle w_\perp, \frac{U''\phi}{U+i\lambda}\Big\rangle. \tag{2.5.15}$$

We now turn to estimate the right-hand side of (2.5.15). For the first term we have, using the fact that $|\phi_\perp(x)| \leq (1-x)^{1/2}\|\phi_\perp'\|_2$,

$$\Big|\Big\langle\phi_\perp, \frac{v}{U+i\lambda}\Big\rangle\Big| \leq \|\phi_\perp'\|_2\Big\|(1-x)^{1/2}\frac{v}{U+i\lambda}\Big\|_1. \tag{2.5.16}$$

Furthermore, splitting the domain of integration $(0,1)$ into two intervals: $(0,1/4)$ and $(1/4,1)$ and then integrating by parts on $(1/4,1)$ yield, for $0<|\lambda|<\lambda_0\leq U(1/2)$,

$$\begin{aligned}\Big\langle\phi_\perp, \frac{v}{U+i\lambda}\Big\rangle &= -\overline{\phi_\perp}\,v\log(U+i\lambda)\big|_{x=\frac14}\\ &\quad - \int_{1/4}^1 \log(U+i\lambda)\Big(\frac{\overline{\phi_\perp}\,v}{U'}\Big)'\,dx + \int_0^{1/4}\overline{\phi_\perp}\frac{v}{U+i\lambda}\,dx.\end{aligned} \tag{2.5.17}$$

Using a Sobolev embedding and Poincaré's inequality yields for $|\lambda|\leq U(1/2)$

$$\big|\overline{\phi_\perp}\,v\log(U+i\lambda)\big|_{x=\frac14}\big| \leq C\|\phi_\perp\|_\infty\|v\|_\infty \leq C\|\phi_\perp'\|_2\|v\|_{1,p}. \tag{2.5.18}$$

Furthermore, it holds that

$$\Big|\int_0^{1/4}\overline{\phi_\perp}\frac{v}{U+i\lambda}\,dx\Big| \leq C\,\|v\|_2\|\phi_\perp\|_2, \tag{2.5.19}$$

and, as in the proof of (2.4.20),

$$\Big|\int_{1/4}^1 \log(U+i\lambda)\Big(\frac{\overline{\phi_\perp}v}{U'}\Big)'\,dx\Big| \leq C(\|v\|_{1,p}\|\phi_\perp\|_\infty + \|v\|_\infty\|\phi_\perp'\|_2).$$

Substituting the above, together with (2.5.19) and (2.5.18) into (2.5.17) we can conclude, with the aid of Poincaré's inequality and Sobolev's embeddings, that

$$\Big|\Big\langle\phi_\perp, \frac{v}{U+i\lambda}\Big\rangle\Big| \leq C\|\phi_\perp'\|_2\|v\|_{1,p}. \tag{2.5.20}$$

Combining (2.5.20) with (2.5.16) and (2.5.2) yields

$$\Big|\Big\langle\phi_\perp, \frac{v}{U+i\lambda}\Big\rangle\Big| \leq C\|\phi_\perp'\|_2 N_0(v,\lambda). \tag{2.5.21}$$

For the second term we have, using the decomposition (2.3.1)

$$i\lambda\Big\langle w_\perp, \frac{U''\phi}{U+i\lambda}\Big\rangle = ic_\parallel\lambda\Big\langle \phi_\perp, \frac{U''}{U+i\lambda}\Big\rangle + i\lambda\Big\langle w_\perp, \frac{U''\phi_\perp}{U+i\lambda}\Big\rangle. \tag{2.5.22}$$

To estimate the first term in (2.5.22) we use (2.5.11) to obtain

$$\Big|c_\parallel\lambda\Big\langle \phi_\perp, \frac{U''}{U+i\lambda}\Big\rangle\Big| \leq C|\lambda|\,|c_\parallel|\|\phi_\perp\|_{1,2}. \tag{2.5.23}$$

Substituting the above together with (2.5.22) and (2.5.21) into (2.5.15) yields (2.5.14). The estimate of the last term in (2.5.14) is the object of the next step.

Step 2.2. We prove that for every $\epsilon > 0$ there exists $\lambda_0 > 0$ such that for all $|\lambda| \leq \lambda_0$ it holds that

$$\Big|i\lambda\Big\langle \frac{\phi_\perp}{U}, \frac{U''\phi_\perp}{U+i\lambda}\Big\rangle\Big| \leq \epsilon\|\phi_\perp'\|_2^2. \tag{2.5.24}$$

Clearly,

$$i\lambda\Big\langle \frac{\phi_\perp}{U}, \frac{U''\phi_\perp}{U+i\lambda}\Big\rangle = i\lambda\int_0^1 \frac{U''|\phi_\perp|^2}{U(U+i\lambda)}\,dx. \tag{2.5.25}$$

Recall the definition of x_ν in (2.4.5), and let $1-x_\nu \leq d < 1/2$.

The integral over $(1-2d, 1)$. We attempt, see Step 2 of the proof of [3, Proposition 4.14], to prove that for $d < \frac{1}{2}$ there exists $\widehat{C} > 0$ and λ_0 such that for $|\lambda| \leq \lambda_0$

$$\Big|i\lambda\int_{1-2d}^1 \frac{U''|\phi_\perp|^2}{U(U+i\lambda)}\,dx\Big| \leq \widehat{C}\,d\log|d|^{-1}\|\phi_\perp'\|_2^2. \tag{2.5.26}$$

To estimate this integral we use the identity

$$\frac{1}{U(U+i\lambda)} = \frac{1}{i\lambda}\Big[\frac{1}{U} - \frac{1}{U+i\lambda}\Big],$$

to obtain that

$$i\lambda\int_{1-2d}^1 \frac{U''|\phi_\perp|^2}{U(U+i\lambda)}\,dx = \int_{1-2d}^1 \frac{U''|\phi_\perp|^2}{U}\,dx - \int_{1-2d}^1 \frac{U''|\phi_\perp|^2}{U+i\lambda}\,dx. \tag{2.5.27}$$

Integration by parts yields

$$\begin{aligned}\int_{1-2d}^1 \frac{U''|\phi_\perp|^2}{U+i\lambda}\,dx = &\Big(\frac{U''}{U'}|\phi_\perp|^2|\log(U+i\lambda)|\Big)\Big|_{x=1-2d} \\ &- \int_{1-2d}^1 \Big(\frac{U''}{U'}|\phi_\perp|^2\Big)'|\log(U+i\lambda)|\,dx.\end{aligned} \tag{2.5.28}$$

Next, we observe that

$$|\phi_\perp(x)|^2 \leq 2d\|\phi_\perp'\|_2^2, \quad 0 \leq U(x) \leq 2d,\ \forall x \in [1-2d, 1]. \tag{2.5.29}$$

We also note that, for λ_0 small enough and $|\lambda| \le \lambda_0$,

$$|\log(U(1-2d)+i\lambda)| \le C \log|d|^{-1}. \tag{2.5.30}$$

Hence, the first term on the right-hand side of (2.5.28) can be estimated as follows:

$$\Big|\Big(\frac{U''}{U'}|\phi_\perp|^2|\log(U+i\lambda)|\Big)\Big|_{x=1-2d} \le C\, d \log|d|^{-1}\, \|\phi_\perp'\|_2^2. \tag{2.5.31}$$

For the second term, we write

$$\begin{aligned}
&\Big|\int_{1-2d}^1 \Big(\frac{U''}{U'}|\phi_\perp|^2\Big)'|\log(U+i\lambda)|\,dx\Big| \\
&\quad \le \Big\|\frac{U''}{U'}\Big\|_{W^{1,\infty}(1-2d,1)} \int_{1-2d}^1 (|\phi_\perp|^2 + 2|\phi_\perp|\,|\phi_\perp'|)|\log(U+i\lambda)|\,dx \\
&\quad \le C \int_{1-2d}^1 (|\phi_\perp|^2 + |\phi_\perp|\,|\phi_\perp'|)\log(U+i\lambda)|\,dx.
\end{aligned} \tag{2.5.32}$$

As

$$\|\log(U+i\lambda)\|_{L^p(1-2d,1)} \le C d^{1/p}\log|d|^{-1} \quad \text{for } p\in\{1,2\},$$

we obtain, using (2.5.29), from (2.5.32) that

$$\Big|\int_{1-2d}^1 \Big(\frac{U''}{U'}|\phi_\perp|^2\Big)'|\log(U+i\lambda)|\,dx\Big| \le C d\log|d|^{-1}\|\phi_\perp'\|_2^2.$$

Substituting the above, together with (2.5.29) into (2.5.28) then yields

$$\Big|\int_{1-2d}^1 \frac{U''|\phi_\perp|^2}{U+i\lambda}\,dx\Big| \le C d\log|d|^{-1}\|\phi_\perp'\|_2^2. \tag{2.5.33}$$

We now estimate the first term on the right-hand side of (2.5.27). Employing (2.3.7) and Poincaré's inequality yields

$$\Big|\int_{1-2d}^1 \frac{U''|\phi_\perp|^2}{U}\,dx\Big| \le C\|\phi_\perp\|_{L^2(1-2d,1)}\|\phi_\perp/U\|_{L^2(1-2d,1)} \le \widetilde{C}\, d\|\phi_\perp'\|_2^2.$$

Substituting the above together with (2.5.33) into (2.5.27) yields (2.5.26).

The integral over $[0,1-2d]$. By (2.1.3) there exists $C>0$ such that for all $1-x_\nu \le d < 1/2$,

$$\Big\|\frac{1}{U+i\lambda}\Big\|_{L^\infty(0,1-2d)} \le \frac{1}{U(1-2d)-\nu} \le \frac{C}{d}.$$

Hence, given that $U'(1)<0$,

$$\Big|\lambda\int_0^{1-2d} \frac{U''|\phi_\perp|^2}{U(U+i\lambda)}\,dx\Big| \le C\frac{|\lambda|}{d^2}\|\phi_\perp\|_2^2.$$

Combining the above with (2.5.26) yields that for all $1 - x_\nu \leq d$, with the aid of Poincaré's inequality

$$\left|\lambda \int_0^1 \frac{U''|\phi_\perp|^2}{U(U+i\lambda)}\,dx\right| \leq C\Big(\frac{|\lambda|}{d^2} + d\log|d|^{-1}\Big)\|\phi_\perp'\|_2^2. \tag{2.5.34}$$

Let $\epsilon > 0$. Clearly, there exists $d(\epsilon) > 0$ such that for $d \in (0, d(\epsilon))$

$$d\log|d|^{-1} \leq \frac{\epsilon}{2C}.$$

Furthermore, for (ϵ, d) as above we add the condition

$$\lambda_0 \leq \min\Big(\frac{\epsilon d^2}{2C}, d\,U(0)\Big).$$

As

$$U(0)(1 - x_\nu) \leq |\nu| < |\lambda| < \lambda_0 \leq d\,U(0),$$

we obtain that $d \geq 1 - x_\nu$, therefore, (2.5.24) can now be verified with the aid of (2.5.34).

Step 2.3. We complete the proof of (2.5.3). Substituting (2.5.24) into (2.5.14) yields for any $\epsilon > 0$, the existence of $C > 0$ and $\lambda_\epsilon > 0$, such that for $|\lambda| < \lambda_\epsilon$, it holds that

$$\|Uw_\perp'\|_2^2 + \alpha^2\|\phi_\perp\|_2^2 \leq C\|\phi_\perp'\|_2(N_0(v,\lambda) + |\lambda||c_\parallel|) + \epsilon\|\phi_\perp'\|_2^2. \tag{2.5.35}$$

We now attempt to bound $\|\phi_\perp'\|_2$. As

$$\phi_\perp' = Uw_\perp' + U'w_\perp,$$

and since $\|U'\|_\infty \leq 1$ we may use (2.3.7) to obtain

$$\|\phi_\perp'\|_2^2 \leq 2\|Uw_\perp'\|_2^2 + C\|\phi_\perp\|_2^2. \tag{2.5.36}$$

On the other hand, by (2.2.2) and (2.2.3) we have that

$$\|\phi_\perp\|_2^2 \leq C\|Uw_\perp'\|_2^2,$$

and hence, combining with (2.5.36), we obtain

$$\|\phi_\perp'\|_2 \leq C\|Uw_\perp'\|_2.$$

Substituting the above into (2.5.35) yields, with the aid of Poincaré's inequality and a suitable choice of ϵ, the existence of λ_0 and $C > 0$ such that for $|\lambda| \leq \lambda_0$,

$$\|\phi_\perp'\|_2 \leq C(N_0(v,\lambda) + |\lambda|\,|c_\parallel|). \tag{2.5.37}$$

We now combine (2.5.37) with (2.5.4) to obtain (2.5.3a) and (2.5.3b). ∎

While a direct use of (2.5.3) will be made in the proof of (5.2.2), we shall also need to transform $N_1(v,\lambda)$ into a more conventional bound, which is precisely what we achieve in the next lemma.

Lemma 2.5.2. *Let* $U \in C^2([0,1])$ *satisfy* (2.1.3), $p>1$, $0<\nu_0<U(1/2)$, *and* $\mu_0>0$. *There exist* $C>0$ *such that for all* $|\mu|<\mu_0$, *and* $|\nu|\leq\nu_0$ *it holds that*

$$N_1(v,\lambda) \leq |v(1)|\log|\lambda| + C\|v\|_{1,p}, \tag{2.5.38}$$

where N_1 *is introduced in* (2.5.2b).

Proof. Clearly,

$$\Big\langle 1, \frac{v}{U+i\lambda}\Big\rangle = \Big\langle 1, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,1/2)} + \Big\langle 1, \frac{v}{U+i\lambda}\Big\rangle_{L^2(1/2,1)}.$$

Integration by parts yields

$$\begin{aligned}\Big\langle 1, \frac{v}{U+i\lambda}\Big\rangle_{L^2(1/2,1)} &= v(1)\log(i\lambda) - \frac{v(1/2)}{U'(1/2)}\log(U(1/2)+i\lambda) \\ &\quad - \int_{1/2}^1 \Big(\frac{v}{U'}\Big)' \log(U+i\lambda)\,dx.\end{aligned}$$

Furthermore, as $\nu<U(1/2)$ it holds that

$$\Big|\Big\langle 1, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,1/2)}\Big| \leq \frac{\|v\|_1}{|U(1/2)+i\lambda|}.$$

Consequently, by (2.5.2b), we can conclude (2.5.38) for any $p>1$. ∎

2.6 The case $0<\Im\lambda<U(0)$

Lemma 2.3.1 and Propositions 2.4.1, 2.5.1 address some of the cases where $|\lambda|$ is small. We now consider the case $0<\nu<\nu_1$ for some $\nu_1<U(0)$, (recall that $\nu=\Im\lambda$). For later reference (see Lemma 5.2.1 and Proposition 5.4.1), we also consider the case where ν is small using a different approach than that of the previous section. We set, for $p>1$ and $v\in W^{1,p}(0,1)$,

$$\begin{aligned}N(v,\lambda) := \min\Big(&\Big\|[(1-x)^{1/2}+\nu^{-1/2}(1-x)]\frac{v}{U+i\lambda}\Big\|_1, \\ &|\nu|^{1/2}(\|v'\|_p + |v(1)|\log|\nu|^{-1}) + \|v\|_2 + \nu^{-1/2}\|v\|_1\Big).\end{aligned} \tag{2.6.1}$$

For small values of $|\lambda|$, since $\mathcal{A}_{i\nu,0}(U-\nu)=0$ and since $U(1)-\nu=\nu\ll 1$, we expect $\mathcal{A}_{\lambda,0}$ to be almost singular, and that the norm of $\phi=\mathcal{A}_{\lambda,0}^{-1}v$ would be much greater in the space spanned by $U-\nu$. We thus use the decomposition

$$\phi = c_\|^\nu(U-\nu) + [\phi - c_\|^\nu(U-\nu)],$$

where $c_\|^\nu$ is defined by (2.6.3).

The proof of the following proposition is somewhat similar to the proof of [3, Proposition 4.15]. In addition to the above-mentioned difference, resulting from the non-invertibility of $\mathcal{A}_{0,0}$, we need to address the Neumann condition at $x = 0$ here, which complicates the estimate of $\|\phi'\|_{L^2(0,x_\nu)}$, where x_ν is given by (2.4.5). This estimate, which is addressed in step 3 of the proof, significantly contributes to its length and complexity.

Proposition 2.6.1. *Let $p > 1$, $U \in C^3([0,1])$ satisfy (2.1.3), and $0 < \nu_1 < U(0)$. There exist $\mu_0 > 0$ and $C > 0$ such that for all $\lambda = \mu + i\nu$ with $0 < \nu < \nu_1$, and $0 < |\mu| \leq \mu_0$ and $\alpha \geq 0$, we have, for all $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ satisfying $\mathcal{A}_{\lambda,\alpha}\phi = v$ (where $\mathcal{A}_{\lambda,\alpha}$ is defined in (2.1.1)),*

$$\|\phi - c_\parallel^\nu (U - \nu)\|_{1,2} + \nu^{1/2}|c_\parallel^\nu| \leq \frac{C}{\nu} N(v, \lambda), \tag{2.6.2}$$

where

$$c_\parallel^\nu = \frac{\langle \phi - \phi(x_\nu), U - \nu \rangle_{L^2(0,x_\nu)}}{\|U - \nu\|^2_{L^2(0,x_\nu)}}, \tag{2.6.3}$$

in which x_ν is defined by (2.4.5).

Proof. Step 1. We prove that there exists $C > 0$ such that, for all $0 < |\mu| \leq 1$ it holds that

$$|\phi(x_\nu)| \leq C \Big|\Big\langle \phi, \frac{v}{U + i\lambda} \Big\rangle\Big|^{1/2} \tag{2.6.4}$$

for all pairs $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ satisfying (2.4.22).

It can be easily verified (since $\mathcal{A}_{\lambda,\alpha}\phi = v$) that

$$\Im\Big\langle \phi, \frac{v}{U - \nu + i\mu} \Big\rangle = -\mu \Big\langle \frac{U''\phi}{(U - \nu)^2 + \mu^2}, \phi \Big\rangle. \tag{2.6.5}$$

As

$$|\phi(x)|^2 \geq \frac{1}{2}|\phi(x_\nu)|^2 - |\phi(x) - \phi(x_\nu)|^2,$$

we may use (2.6.5) to obtain, observing that $-U'' > 0$,

$$\Big|\Im\Big\langle \phi, \frac{v}{U - \nu + i\mu} \Big\rangle\Big| \geq |\mu| \Big\langle \frac{|U''|}{(U - \nu)^2 + \mu^2}, \frac{1}{2}|\phi(x_\nu)|^2 - |\phi(x) - \phi(x_\nu)|^2 \Big\rangle. \tag{2.6.6}$$

Hence,

$$\frac{|\mu|}{2}|\phi(x_\nu)|^2 \Big\| \frac{|U''|^{1/2}}{U + i\lambda} \Big\|_2^2 \leq |\mu| \sup|U''| \Big\| \frac{\phi - \phi(x_\nu)}{U + i\lambda} \Big\|_2^2 + \Big|\Big\langle \phi, \frac{v}{U - \nu + i\mu} \Big\rangle\Big|. \tag{2.6.7}$$

Since $|U(x)-\nu|\leq|x-x_\nu|$, and $|U''|>0$ we can conclude, that for some positive C

$$\Big\|\frac{|U''|^{1/2}}{U+i\lambda}\Big\|_2^2\geq\frac{1}{C}\int_0^1\frac{1}{(x-x_\nu)^2+\mu^2}dx.$$

As $|\mu|\leq 1$ and $x_\nu\in[x_{\nu_1},1]$, we obtain after the change of variable $y=(x_\nu-x)/|\mu|$

$$\int_0^1\frac{1}{(x-x_\nu)^2+\mu^2}dx=\frac{1}{|\mu|}\int_{\frac{x_\nu-1}{|\mu|}}^{\frac{x_\nu}{|\mu|}}\frac{1}{1+y^2}dy\geq\frac{1}{|\mu|}\int_0^{\frac{x_{\nu_1}}{|\mu|}}\frac{1}{1+y^2}dy.$$

Consequently, there exists $\widehat{C}>0$, such that for all $|\mu|\leq 1$ and $\nu\in(0,\nu_1)$,

$$\Big\|\frac{|U''|^{1/2}}{U+i\lambda}\Big\|_2^2\geq\frac{1}{\widehat{C}}|\mu|^{-1}. \tag{2.6.8}$$

A similar argument is employed in the proof of [3, Proposition 4.14] (see between equations (4.59) and (4.60) there). By (2.6.7) we then have

$$|\phi(x_\nu)|^2\leq C\Big[|\mu|\Big\|\frac{\phi-\phi(x_\nu)}{U+i\lambda}\Big\|_2^2+\Big|\Big\langle\phi,\frac{v}{U-\nu+i\mu}\Big\rangle\Big|\Big]. \tag{2.6.9}$$

To estimate the first term on the right-hand side of (2.6.9) we first observe that for some $C=C(\nu_1)>0$ we have, for all $\lambda=\mu+i\nu$ such that $0<\nu<\nu_1$

$$\Big|\frac{1}{U(x)+i\lambda}\Big|\leq\frac{C}{|x-x_\nu|}\quad\forall x\in(x_{\nu_1}/4,1),$$

where $x_{\nu_1}=x_\nu\big|_{\nu=\nu_1}$, and

$$\Big|\frac{1}{U(x)+i\lambda}\Big|\leq C\quad\forall x\in(0,x_{\nu_1}/4).$$

Consequently, we may write

$$\Big|\frac{1}{U(x)+i\lambda}\Big|\leq\frac{C}{|x-x_\nu|}\quad\forall x\in(0,1). \tag{2.6.10}$$

We now apply Hardy's inequality (2.2.8) to $w=(x-x_\nu)^{-1}(\phi-\phi(x_\nu))$ in $(x_\nu,1)$. It follows that

$$\Big\|\frac{\phi-\phi(x_\nu)}{x-x_\nu}\Big\|_{L^2(x_\nu,1)}^2\leq\frac{1}{4}\|\phi'\|_{L^2(x_\nu,1)}^2. \tag{2.6.11}$$

A similar bound can be the interval $(0,x_\nu)$:

$$\Big\|\frac{\phi-\phi(x_\nu)}{x-x_\nu}\Big\|_{L^2(0,x_\nu)}^2\leq\frac{1}{4}\|\phi'\|_{L^2(0,x_\nu)}^2. \tag{2.6.12}$$

Consequently,

$$\left\| \frac{\phi - \phi(x_\nu)}{x - x_\nu} \right\|_{L^2(0,1)}^2 \leq \frac{1}{4} \|\phi'\|_{L^2(0,1)}^2 \tag{2.6.13}$$

from which we easily conclude, in view of (2.6.10),

$$\left\| \frac{\phi - \phi(x_\nu)}{U + i\lambda} \right\|_2^2 \leq C \|\phi'\|_2^2. \tag{2.6.14}$$

Substituting (2.6.14) into (2.6.9) readily yields

$$|\phi(x_\nu)|^2 \leq C \left(\left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| + |\mu| \, \|\phi'\|_2^2 \right). \tag{2.6.15}$$

Using the positivity of $-U''$ on $[0, 1]$ and (2.6.5) for the second inequality we then obtain that

$$\left\| \frac{\phi}{U + i\lambda} \right\|_2^2 \leq C \int_0^1 \frac{(-U'')}{(U - \nu)^2 + \mu^2} |\phi|^2 dx \leq \frac{\widehat{C}}{|\mu|} \left| \left\langle \phi, \frac{v}{U - \nu + i\mu} \right\rangle \right|. \tag{2.6.16}$$

Since

$$\|\phi'\|_2^2 + \alpha^2 \|\phi\|_2^2 = \Re \left\langle \phi, \frac{\mathcal{A}_{\lambda,\alpha}\phi}{U + i\lambda} \right\rangle - \Re \left\langle U''\phi, \frac{\phi}{U + i\lambda} \right\rangle,$$

we may conclude from (2.6.16), Poincaré's inequality, and (2.4.22) that

$$\begin{aligned} \|\phi'\|_2^2 + \alpha^2 \|\phi\|_2^2 &\leq \left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| + C \left\| \frac{\phi}{U + i\lambda} \right\|_2 \|\phi\|_2 \\ &\leq \left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| + \frac{\widehat{C}}{|\mu|^{1/2}} \left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right|^{1/2} \|\phi'\|_2, \end{aligned} \tag{2.6.17}$$

from which we conclude, given that $|\mu| \leq 1$,

$$\|\phi'\|_2^2 \leq \frac{C}{|\mu|} \left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right|. \tag{2.6.18}$$

Substituting (2.6.18) into (2.6.15) yields (2.6.4).

Step 2. We prove that for any $A > 0$, there exists C and μ_A such that, for $\alpha^2 \leq A$ and λ such that $|\mu| \leq \mu_A$ and $\nu \in (0, \nu_1)$

$$\|\phi\|_{H^1(x_\nu,1)} \leq C \left[\nu^{-1/2} \left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right|^{1/2} + N(v, \lambda) \right] \tag{2.6.19}$$

holds for any pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0, 1)$ satisfying (2.4.22).

Let $\chi \in C^\infty(\mathbb{R}, [0, 1])$ be given by

$$\chi(x) = \begin{cases} 1 & x < 1/2, \\ 0 & x > 3/4. \end{cases} \tag{2.6.20}$$

With $d = 1 - x_\nu$, let $\chi_d(x) = \chi((x - x_\nu)/d)$ and set

$$\phi = \varphi + \phi(x_\nu)\chi_d. \tag{2.6.21}$$

Note that by the choice of d, φ satisfies also the boundary conditions at $x \in \{0, 1\}$.

It can be easily verified that

$$\mathcal{A}_{\lambda,\alpha}\varphi = v + \phi(x_\nu)\big((U + i\lambda)(\chi_d'' - \alpha^2\chi_d) - U''\chi_d\big).$$

By (2.6.21) we have that

$$w := (U - \nu)^{-1}\varphi \in H^2(0, 1), \tag{2.6.22}$$

and hence we can rewrite the above equality (using (2.4.22) twice) in the form

$$\begin{aligned}
&-\Big((U-\nu)^2\Big(\frac{\varphi}{U-\nu}\Big)'\Big)' + \alpha^2(U-\nu)\varphi \\
&\quad = v + \phi(x_\nu)\big((U-\nu)(\chi_d'' - \alpha^2\chi_d) - U''\chi_d\big) + i\mu(\phi'' - \alpha^2\phi) \\
&\quad = \frac{(U-\nu)v}{U+i\lambda} + \phi(x_\nu)\big((U-\nu)(\chi_d'' - \alpha^2\chi_d) - U''\chi_d\big) + i\mu\frac{U''\phi}{U+i\lambda}.
\end{aligned} \tag{2.6.23}$$

Taking the inner product with w and integrating by parts, exploiting the fact that $\varphi(x_\nu) = 0$, then yields

$$\begin{aligned}
&\|(U-\nu)w'\|^2_{L^2(x_\nu,1)} + \alpha^2\|\varphi\|^2_{L^2(x_\nu,1)} \\
&\quad = \Big\langle\varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)} - \langle w, \phi(x_\nu)U''\chi_d\rangle_{L^2(x_\nu,1)} \\
&\qquad + \phi(x_\nu)\langle\varphi, \chi_d'' - \alpha^2\chi_d\rangle_{L^2(x_\nu,1)} + i\mu\Big\langle w, \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}.
\end{aligned} \tag{2.6.24}$$

We now estimate the four terms appearing in the right-hand side of (2.6.24), using precisely the same procedure as in the proof of [3, Proposition 4.13]. For the first term on the right-hand side of (2.6.24) we obtain with the aid of (2.6.21)

$$\begin{aligned}
\Big|\Big\langle\varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| &\leq \Big|\Big\langle\phi - \phi(x_\nu), \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| \\
&\quad + |\phi(x_\nu)|\Big|\Big\langle 1 - \chi_d, \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big|.
\end{aligned} \tag{2.6.25}$$

Since the integration is carried over $(x_\nu, 1)$ we can estimate the first term on the right-hand side of (2.6.25) by using Hardy's inequality (2.2.8) and (2.6.10)

$$\begin{aligned}
\Big|\Big\langle\phi - \phi(x_\nu), \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| &\leq \Big\|\frac{\phi - \phi(x_\nu)}{x - x_\nu}\Big\|_{L^2(x_\nu,1)}\Big\|\frac{(x-x_\nu)v}{U+i\lambda}\Big\|_{L^2(x_\nu,1)} \\
&\leq C\|\phi'\|_{L^2(x_\nu,1)}\|v\|_2.
\end{aligned}$$

For the second term on the right-hand side of (2.6.25) we first note that since

$$\phi(x_\nu) = -\int_{x_\nu}^1 \phi'(x)\,dx,$$

we may conclude that

$$|\phi(x_\nu)| \leq d^{1/2}\|\phi'\|_2. \tag{2.6.26}$$

By the definition of χ_d,

$$\Big\|\frac{1-\chi_d}{U+i\lambda}\Big\|_\infty \leq \frac{C}{d},$$

and hence,

$$\Big\|\frac{1-\chi_d}{U+i\lambda}\Big\|_{L^2(x_\nu,1)} = \Big\|\frac{1-\chi_d}{U+i\lambda}\Big\|_2 \leq \frac{C}{d^{1/2}}.$$

Hence, by the above and (2.6.26),

$$|\phi(x_\nu)|\Big|\Big\langle 1-\chi_d, \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| \leq d^{1/2}\|\phi'\|_2\Big\|\frac{1-\chi_d}{U+i\lambda}\Big\|_2\|v\|_2 \leq C\|\phi'\|_2\|v\|_2.$$

Hence,

$$\Big|\Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| \leq C\|\phi'\|_2\|v\|_2. \tag{2.6.27}$$

In addition, we may write, observing that $\varphi(1) = 0$,

$$\Big|\Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| \leq \|\varphi'\|_{L^2(x_\nu,1)}\Big\|(1-x)^{1/2}\frac{v}{U+i\lambda}\Big\|_1.$$

Using (2.6.26), yields

$$\|\varphi'\|_{L^2(x_\nu,1)} \leq \|\phi'\|_{L^2(x_\nu,1)} + |\phi(x_\nu)|\,\|\chi_d'\|_2 \leq C\|\phi'\|_2,$$

which leads to

$$\Big|\Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| \leq C\|\phi'\|_{L^2(x_\nu,1)}\Big\|(1-x)^{1/2}\frac{v}{U+i\lambda}\Big\|_1. \tag{2.6.28}$$

Combining (2.6.27) and (2.6.28) yields the existence of $C > 0$ such that

$$\Big|\Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| \leq C\|\phi'\|_{L^2(x_\nu,1)}N(v,\lambda). \tag{2.6.29}$$

To estimate the second term $\langle w, \phi(x_\nu)U''\chi_d\rangle_{L^2(x_\nu,1)}$ on the right-hand side of (2.6.24), we note that by Hardy's inequality (2.2.8) and (2.6.26), we have

$$\begin{aligned}\|w\|_{L^2(x_\nu,1)} \leq C\|\varphi'\|_{L^2(x_\nu,1)} &\leq \widehat{C}\Big(\|\phi'\|_{L^2(x_\nu,1)} + \frac{1}{d^{1/2}}|\phi(x_\nu)|\Big)\\ &\leq \widetilde{C}\|\phi'\|_{L^2(x_\nu,1)}.\end{aligned} \tag{2.6.30}$$

From (2.6.30) we then get

$$|\langle w, \phi(x_\nu)U''\chi_d\rangle_{L^2(x_\nu,1)}| \leq C\,|\phi(x_\nu)|\|\phi'\|_{L^2(x_\nu,1)}. \tag{2.6.31}$$

Next, we write for the third term $(\phi(x_\nu)\langle\varphi, \chi_d'' - \alpha^2\chi_d\rangle_{L^2(x_\nu,1)})$ on the right-hand side of (2.6.24), using integration by parts (note that $\chi_d'(x_\nu) = 0 = \chi_d'(1)$) and the fact that $\alpha^2 \leq A$

$$\begin{aligned} |\langle\varphi, \chi_d'' - \alpha^2\chi_d\rangle_{L^2(x_\nu,1)}| &\leq \|\varphi'\|_{L^2(x_\nu,1)}\|\chi_d'\|_2 + C_A\,\|\varphi\|_{L^2(x_\nu,1)} \\ &\leq \widehat{C}_A\Big(\frac{1}{d^{1/2}}\|\varphi'\|_{L^2(x_\nu,1)} + \|\varphi\|_{L^2(x_\nu,1)}\Big). \end{aligned}$$

(For convenience we drop the notation referring to the dependence on A in the sequel.) Consequently, by (2.6.21),

$$\begin{aligned} &|\phi(x_\nu)|\,|\langle\varphi, \chi_d'' - \alpha^2\chi_d\rangle_{L^2(x_\nu,1)}| \\ &\leq C\,|\phi(x_\nu)|\Big(\frac{1}{d^{1/2}}\Big(\|\phi'\|_{L^2(x_\nu,1)} + \frac{1}{d^{1/2}}|\phi(x_\nu)|\Big) + (\|\phi\|_{L^2(x_\nu,1)} + d^{1/2}|\phi(x_\nu)|)\Big). \end{aligned}$$

Hence, using Poincaré's inequality and (2.6.26), yields

$$|\phi(x_\nu)|\,|\langle\varphi, \chi_d'' - \alpha^2\chi_d\rangle_{L^2(x_\nu,1)}| \leq C\,\frac{|\phi(x_\nu)|}{d^{1/2}}\|\phi'\|_{L^2(x_\nu,1)}. \tag{2.6.32}$$

To estimate the last term $\big(i\mu\big\langle w, \frac{U''\phi}{U+i\lambda}\big\rangle_{(x_\nu,1)}\big)$ on the right-hand side of (2.6.24), we first write

$$\begin{aligned} &\Big|\Big\langle w, U''\frac{\phi}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| \\ &\qquad \leq \Big|\Big\langle w, U''\frac{\phi - \phi(x_\nu)}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| + \Big|\Big\langle w, U''\frac{\phi(x_\nu)}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big|. \end{aligned}$$

We then use (2.6.30), (2.6.26), and [3, equation (4.38)], which reads, for $\nu \in [0, \nu_1]$, with $\nu_1 < U(0)$,

$$\Big\|\frac{1}{U+i\lambda}\Big\|_2 \leq \check{C}_{\nu_1}|\mu|^{-1/2}, \tag{2.6.33}$$

to obtain that

$$\begin{aligned} \Big|\Big\langle w, U''\frac{\phi(x_\nu)}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| &\leq C\,\|w\|_{L^2(x_\nu,1)}|\phi(x_\nu)|\Big\|\frac{1}{U+i\lambda}\Big\|_2 \\ &\leq \widehat{C}\,|\mu|^{-1/2}d^{1/2}\|\phi'\|^2_{L^2(x_\nu,1)}. \end{aligned} \tag{2.6.34}$$

Furthermore, we have, by (2.6.14) and (2.6.30),

$$\Big|\Big\langle w, U''\frac{\phi - \phi(x_\nu)}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| \leq C\,\|\phi'\|^2_{L^2(x_\nu,1)}.$$

Substituting the above and (2.6.34) together with (2.6.29), (2.6.31), and (2.6.32) into (2.6.24) yields that there exists $C > 0$ such that

$$\begin{aligned}&\|(U-\nu)w'\|^2_{L^2(x_\nu,1)}+\alpha^2\|\varphi\|^2_{L^2(x_\nu,1)}\\&\leq C\Big([|\mu|^{1/2}d^{1/2}+|\mu|]\|\phi'\|^2_{L^2(x_\nu,1)}+\Big[\frac{|\phi(x_\nu)|}{d^{1/2}}+N(v,\lambda)\Big]\|\phi'\|_{L^2(x_\nu,1)}\Big).\end{aligned} \tag{2.6.35}$$

As $|U(x)-\nu| \geq \frac{1}{C}|x-x_\nu|$ for all $x \in (x_\nu,1)$, we can apply Hardy's inequality (2.2.7) to $(U-\nu)w'$ on $(x_\nu,1)$ to obtain

$$\begin{aligned}&\|w\|^2_{L^2(x_\nu,1)} \leq \widehat{C}\|(U-\nu)w'\|^2_{L^2(x_\nu,1)}\\&\leq \widetilde{C}\Big(\big[|\mu|^{1/2}d^{1/2}+|\mu|\big]\|\phi'\|^2_{L^2(x_\nu,1)}+\Big[\frac{|\phi(x_\nu)|}{d^{1/2}}+N(v,\lambda)\Big]\|\phi'\|_{L^2(x_\nu,1)}\Big).\end{aligned} \tag{2.6.36}$$

Continuing as in Step 2 of the proof of [3, Proposition 4.14] we write, using the definition of w and φ,

$$\|\phi'\|_{L^2(x_\nu,1)} \leq \|(U-\nu)w'\|_{L^2(x_\nu,1)}+\|U'w\|_{L^2(x_\nu,1)}+C\,d^{-1/2}|\phi(x_\nu)|, \tag{2.6.37}$$

from which we conclude with the aid of (2.6.35) and (2.6.36) that, for sufficiently small μ_A,

$$\|\phi'\|_{L^2(x_\nu,1)} \leq C\Big(\frac{|\phi(x_\nu)|}{d^{1/2}}+N(v,\lambda)\Big). \tag{2.6.38}$$

From (2.6.38) we can conclude (2.6.19) with the aid of Poincaré's inequality, the fact that $d \geq \frac{1}{C}\nu$, and (2.6.4).

Step 3. We prove that for any $A > 0$, there exist C and μ_A such that, for $\alpha^2 \leq A$, $|\mu| \leq \mu_A$, and $\nu \in [0,\nu_1)$

$$|c^\nu_\parallel|+\|\phi\|_{H^1(0,x_\nu)} \leq C\Big(\nu^{-1}\Big|\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2}+\nu^{-1/2}N(v,\lambda)\Big), \tag{2.6.39}$$

holds for any pair $(\phi,v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ satisfying (2.4.22).

Here, we need to obtain an estimate for $\|w\|_{L^2(0,x_\nu)}$, where we recall from (2.6.22) that $w := (U-\nu)^{-1}\varphi$.

To this end we need an estimate for $w(\hat{x}_0)$ for some $\hat{x}_0 \in (x_\nu/2, x_\nu)$, to be determined at later stage. Clearly, there exists $\hat{x}_1 \in ((1+x_\nu)/2, 1)$ such that

$$|\phi'(\hat{x}_1)| \leq \frac{\sqrt{2}}{d^{1/2}}\|\phi'\|_{L^2(x_\nu,1)}$$

and

$$|\phi(\hat{x}_1)| \leq d^{1/2}\|\phi'\|_{L^2(x_\nu,1)}.$$

Furthermore, it holds for all $x \in (x_\nu/2, x_\nu)$ that

$$|\phi'(x)| \leq |\phi'(\hat{x}_1)| + \Big|\int_x^{\hat{x}_1} \phi''(t)\,dt\Big|.$$

Consequently, as

$$|w(\hat{x}_0)| = \Big|\frac{\phi(\hat{x}_0)-\phi(x_\nu)}{U(\hat{x}_0)-\nu}\Big| \leq \frac{1}{|U(\hat{x}_0)-\nu|}\int_{\hat{x}_0}^{x_\nu} |\phi'(x)|\,dx,$$

we may conclude that

$$|w(\hat{x}_0)| \leq \frac{1}{|U(\hat{x}_0)-\nu|}\int_{\hat{x}_0}^{x_\nu} \Big[|\phi'(\hat{x}_1)| + \Big|\int_x^{\hat{x}_1} \phi''(t)\,dt\Big|\Big]\,dx.$$

With the aid of (2.4.22) we then have

$$|w(\hat{x}_0)| \leq \frac{1}{|U(\hat{x}_0)-\nu|}\int_{\hat{x}_0}^{x_\nu} \Big(|\phi'(\hat{x}_1)| + \Big|\int_x^{\hat{x}_1} \Big[\frac{U''\phi - v}{U+i\lambda} + \alpha^2\phi\Big]\,dt\Big|\Big)\,dx. \tag{2.6.40}$$

We now write

$$\int_x^{\hat{x}_1} \frac{U''\phi}{U+i\lambda}\,dt = \phi(x_\nu)\int_x^{\hat{x}_1} \frac{U''}{U+i\lambda}\,dt + \int_x^{\hat{x}_1} \frac{U''[\phi-\phi(x_\nu)]}{U+i\lambda}\,dt. \tag{2.6.41}$$

To facilitate the estimate of the integral appearing in the first term on the right-hand side of (2.6.41) we use an integration by parts to obtain

$$\int_x^{\hat{x}_1} \frac{U''}{U+i\lambda}\,dt = \Big(\frac{U''}{U'}\log(U+i\lambda)\Big)\Big|_{\hat{x}_1}^{x} - \int_x^{\hat{x}_1} \Big(\frac{U''}{U'}\Big)'\log(U+i\lambda)\,dt.$$

As $0<\nu<\nu_1$, U''/U' and $(\frac{U''}{U'})'$ are uniformly bounded in $(x_\nu/2, 1)$ and in view of the inequality

$$|\log(U+i\lambda)| \leq \log|x-x_\nu|^{-1} + C,$$

we observe that the L^1-norm of $\log(U(x)+i\lambda)$ is bounded and that

$$|\log(U+i\lambda)(\hat{x}_1)| \leq C(|\log(d^2+\mu^2)|+1). \tag{2.6.42}$$

Hence, we have

$$\Big|\phi(x_\nu)\int_x^{\hat{x}_1} \frac{U''(t)}{U(t)+i\lambda}\,dt\Big| \leq C|\phi(x_\nu)|[1+|\log(d^2+\mu^2)|+|\log(U+i\lambda)(\hat{x})|].$$

For the second term on the right-hand side of (2.6.41), we have by (2.6.14)

$$\begin{aligned}\Big|\int_{x_\nu}^{\hat{x}_1} \frac{U''(t)[\phi(t)-\phi(x_\nu)]}{U(t)+i\lambda}\,dt\Big| &\leq C|\hat{x}_1-x_\nu|^{1/2}\Big[\int_{x_\nu}^{\hat{x}_1}\Big|\frac{\phi(t)-\phi(x_\nu)}{U(t)+i\lambda}\Big|^2\,dt\Big]^{1/2} \\ &\leq \widetilde{C}\,d^{1/2}\|\phi'\|_{L^2(x_\nu,\hat{x}_1)}.\end{aligned}$$

In a similar manner we obtain that

$$\Big|\int_x^{x_\nu} \frac{U''[\phi(t)-\phi(x_\nu)]}{U(t)+i\lambda}\,dt\Big| \leq C(x_\nu - x)^{1/2}\|\phi'\|_{L^2(x,x_\nu)}.$$

Consequently,

$$\Big|\int_x^{\hat{x}_1} \frac{U''[\phi(t)-\phi(x_\nu)]}{U(t)+i\lambda}\,dt\Big| \leq C\big(\|\phi'\|_{L^2(x_\nu,\hat{x}_1)} + (x_\nu - x)^{1/2}\|\phi'\|_{L^2(x,x_\nu)}\big).$$

Hence,

$$\begin{aligned}\Big|\int_x^{\hat{x}_1} \frac{U''\phi}{U+i\lambda}\,dt\Big| &\\ \leq C\Big(&\|\phi'\|_{L^2(x_\nu,\hat{x}_1)} + (x_\nu - x)^{1/2}\|\phi'\|_{L^2(x,x_\nu)} \\ &+ |\phi(x_\nu)|[1+|\log(d^2+\mu^2)| + |\log(U+i\lambda)(x)|]\Big).\end{aligned} \tag{2.6.43}$$

Next, we write

$$\Big|\int_x^{\hat{x}_1} \phi(t)\,dt\Big| \leq C\,d^{1/2}\,\|\phi\|_{L^2(x_\nu,1)} + \Big|\int_x^{x_\nu} \phi(t)\,dt\Big|.$$

Then, with the aid of Poincaré's inequality we obtain

$$\Big|\int_x^{\hat{x}_1} \phi(t)\,dt\Big| \leq C\,d\,\|\phi'\|_{L^2(x_\nu,1)} + |\phi(x_\nu)|(x_\nu - x) + C(x_\nu - x)^{3/2}\|\phi'\|_{L^2(x,x_\nu)}.$$

Substituting the above, together with (2.6.43), into (2.6.40) yields

$$\begin{aligned}|w(\hat{x}_0)| \leq \frac{C(1+\alpha^2)}{|U(\hat{x}_0)-\nu|}\int_{\hat{x}_0}^{x_\nu} \big[&d^{-1/2}\|\phi'\|_{L^2(x_\nu,1)} + (x_\nu - x)^{1/2}\|\phi'\|_{L^2(x,x_\nu)} \\ &+ |\phi(x_\nu)|[1+|\log(U+i\lambda)(x)| + |\log(d^2+\mu^2)|]\big]\,dx.\end{aligned}$$

We now write, using (2.6.42),

$$\begin{aligned}\frac{1}{|U(\hat{x}_0)-\nu|}\int_{\hat{x}_0}^{x_\nu} |\log(U+i\lambda)(x)|\,dx &\leq \frac{C}{x_\nu - \hat{x}_0}\int_{\hat{x}_0}^{x_\nu} [1+|\log(x_\nu - x)|]\,dx \\ &\leq \widehat{C}[1+\log(|\hat{x}_0 - x_\nu|^{-1})].\end{aligned}$$

Consequently, since $|(\hat{x}_0 - x_\nu)(U(\hat{x})-\nu)^{-1}|$ is uniformly bounded for $\hat{x}_0 \in (x_\nu, x_\nu/2)$ and $\alpha^2 \leq A$,

$$\begin{aligned}|w(\hat{x}_0)| \leq C\big[&|\phi(x_\nu)|(1+\log|\hat{x}_0 - x_\nu|^{-1} + |\log(d^2+\mu^2)|) \\ &+ (x_\nu - \hat{x}_0)^{1/2}\|\phi'\|_{L^2(\hat{x}_0,x_\nu)} + d^{-1/2}\|\phi'\|_{L^2(x_\nu,1)}\big].\end{aligned} \tag{2.6.44}$$

We can now apply Hardy's inequality (2.2.7) to $w - w(\hat{x}_0)$ on the interval $(\hat{x}_0, x_\nu)$ to obtain

$$\begin{aligned}\|w - w(\hat{x}_0)\|^2_{L^2(\hat{x}_0,x_\nu)} &\leq 4\|([x - x_\nu][w - w(\hat{x}_0)])'\|^2_{L^2(\hat{x}_0,x_\nu)}\\ &\leq C(\nu_1)\|(U-\nu)w'\|^2_{L^2(\hat{x}_0,x_\nu)}.\end{aligned}$$

By (2.6.44) and (2.6.4) we then have

$$\begin{aligned}&\|w\|_{L^2(\hat{x}_0,x_\nu)}\\ &\leq C\big(\|(U-\nu)w'\|_{L^2(\hat{x}_0,x_\nu)} + d^{-1/2}\|\phi'\|_{L^2(x_\nu,1)}\\ &\quad + [\|\phi'\|_2^{1/2}N(v,\lambda)^{\frac{1}{2}} + |\mu|^{1/2}\|\phi'\|_2](1 + \log(|\hat{x}_0 - x_\nu|^{-1}) + |\log(d^2+\mu^2)|)\\ &\quad + (x_\nu - \hat{x}_0)^{1/2}\|\phi'\|_{L^2(\hat{x}_0,x_\nu)}\big).\end{aligned}$$

Note that C is independent of $\hat{x}_0 \in (x_\nu/2, x_\nu)$.

On the other hand by Poincaré's inequality, we have

$$\|w - w(\hat{x}_0)\|^2_{L^2(0,\hat{x}_0)} \leq C\|\, w'\|^2_{L^2(0,\hat{x}_0)}.$$

Observing that $\frac{1}{2}x_{\nu_1} \leq \hat{x}_0 \leq x_\nu \leq 1$ we obtain for all $x \in (0, \hat{x}_0)$

$$U(x) - \nu \geq U(\hat{x}_0) - U(x_\nu) \geq \Big|U'\Big(\frac{1}{2}x_{\nu_1}\Big)\Big|\,|\hat{x}_0 - x_\nu|.$$

Consequently,

$$\|w - w(\hat{x}_0)\|^2_{L^2(0,\hat{x}_0)} \leq C(x_\nu - \hat{x}_0)^{-2}\|(U-\nu)w'\|^2_{L^2(0,\hat{x}_0)}, \tag{2.6.45}$$

and hence, as $\hat{x}_0 \leq x_\nu$,

$$\begin{aligned}&\|w\|_{L^2(0,x_\nu)}\\ &\leq C\big((x_\nu - \hat{x}_0)^{-1}\|(U-\nu)w'\|_{L^2(0,x_\nu)} + d^{-1/2}\|\phi'\|_{L^2(x_\nu,1)}\\ &\quad + [\|\phi'\|_2^{1/2}N(v,\lambda)^{\frac{1}{2}} + |\mu|^{1/2}\|\phi'\|_2](1 + \log|\hat{x}_0 - x_\nu|^{-1} + |\log(d^2+\mu^2)|)\\ &\quad + (x_\nu - \hat{x}_0)^{1/2}\|\phi'\|_{L^2(\hat{x}_0,x_\nu)}\big).\end{aligned}$$

Continuing as in Step 2 of the proof of [3, Proposition 4.14] we establish that

$$\begin{aligned}&\|\phi'\|_{L^2(0,x_\nu)}\\ &\leq C\big((x_\nu - \hat{x}_0)^{-1}\|(U-\nu)w'\|_{L^2(0,x_\nu)} + d^{-1/2}\|\phi'\|_{L^2(x_\nu,1)}\\ &\quad + [\|\phi'\|_2^{1/2}N(v,\lambda)^{\frac{1}{2}} + |\mu|^{1/2}\|\phi'\|_2](1 + \log(|\hat{x}_0 - x_\nu|^{-1}) + |\log(d^2+\mu^2)|)\\ &\quad + (x_\nu - \hat{x}_0)^{1/2}\|\phi'\|_{L^2(\hat{x}_0,x_\nu)}\big).\end{aligned} \tag{2.6.46}$$

Taking the inner product in $L^2(0, x_\nu)$ of (2.6.23) with w yields, as in (2.6.24) (note that $\chi_d \equiv 1$ in $(0, x_\nu)$)

$$\begin{aligned}\|(U-\nu)w'\|^2_{L^2(0,x_\nu)} &+ \alpha^2\|\varphi\|^2_{L^2(0,x_\nu)} \\ &= \Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)} - \langle w, \phi(x_\nu)U''\rangle_{L^2(0,x_\nu)} \\ &\quad - \alpha^2\phi(x_\nu)\langle\varphi, 1\rangle_{L^2(0,x_\nu)} + i\mu\Big\langle w, \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}. \end{aligned} \tag{2.6.47}$$

As in the proof of (2.6.29) we obtain below for the first term on the right-hand side of (2.6.47)

$$\Big|\Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| \leq C\|\phi'\|_{L^2(0,x_\nu)}\, N(v,\lambda). \tag{2.6.48}$$

Indeed, as $\varphi' \equiv \phi'$ in $(0, x_\nu)$ and $\varphi(x_\nu) = 0$, we get for $x \in (0, x_\nu)$

$$|\varphi(x)| \leq \|\varphi'\|_{L^2(0,x_\nu)}(x_\nu - x)^{1/2} \leq \|\phi'\|_{L^2(0,x_\nu)}(1-x)^{1/2},$$

from which we conclude that

$$\Big|\Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| \leq \|\phi'\|_{L^2(0,x_\nu)}\, \Big\|(1-x)^{1/2}\frac{v}{U+i\lambda}\Big\|_1. \tag{2.6.49}$$

In addition, we can write

$$\begin{aligned}\Big|\Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| &= \Big|\Big\langle \phi - \phi(x_\nu), \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| \\ &= \Big|\Big\langle \frac{(\phi-\phi(x_\nu))}{(x-x_\nu)}, \frac{x-x_\nu}{U+i\lambda}v\Big\rangle_{L^2(0,x_\nu)}\Big|\end{aligned}$$

and then use (2.6.12) and (2.6.10) to obtain

$$\Big|\Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| \leq C\|\phi'\|_2\|v\|_2. \tag{2.6.50}$$

Using the definition of $N(v,\lambda)$ in (2.6.1) we can now conclude (2.6.48) from (2.6.49) and (2.6.50).

By (2.2.8) which reads in this case

$$\|w\|_{L^2(0,x_\nu)} \leq C\|\phi'\|_{L^2(0,x_\nu)}, \tag{2.6.51}$$

we have for the second term on the right-hand side of (2.6.47) that

$$|\langle w, \phi(x_\nu)U''\rangle_{L^2(0,x_\nu)}| \leq C\,|\phi(x_\nu)|\|\phi'\|_{L^2(0,x_\nu)}.$$

For the third term we have, using the fact that $\alpha^2 \leq A$, (2.6.21) and Poincaré's inequality,

$$|\alpha^2\phi(x_\nu)\langle\varphi,1\rangle_{L^2(0,x_\nu)}| \leq C\,|\phi(x_\nu)|\big(\|\phi\|_{L^2(0,x_\nu)} + |\phi(x_\nu)|\big) \leq \widehat{C}\,|\phi(x_\nu)|\,\|\phi'\|_{L^2(0,x_\nu)}.$$

Finally, we obtain for the last term of (2.6.47) using (2.6.10), (2.6.12) (as in the proof of (2.6.14) but on the interval $(0, x_\nu)$), and (2.6.51)

$$\left|\mu\left\langle w, \frac{U''(\phi - \phi(x_\nu))}{U + i\lambda}\right\rangle_{L^2(0,x_\nu)}\right| \leq C|\mu|\|\phi'\|^2_{L^2(0,x_\nu)}.$$

Hence, with the aid of (2.6.33) we conclude, as in the proof of (2.6.34),

$$\begin{aligned}\left|\mu\left\langle w, \frac{U''\phi(x_\nu)}{U + i\lambda}\right\rangle_{L^2(0,x_\nu)}\right| &\leq C|\mu|\,\|(U + i\lambda)^{-1}\|_{L^2(0,x_\nu)}\,|\phi(x_\nu)|\,\|\phi'\|_{L^2(0,x_\nu)} \\ &\leq \hat{C}\,|\mu|^{\frac{1}{2}}\,|\phi(x_\nu)|\|\phi'\|_{L^2(0,x_\nu)} \\ &\leq \hat{C}\left(|\mu|\|\phi'\|^2_{L^2(0,x_\nu)} + |\phi(x_\nu)|^2\right) \\ &\leq \tilde{C}\left(|\mu|\|\phi'\|_{L^2(0,x_\nu)} + |\phi(x_\nu)|\right)\|\phi'\|_{L^2(0,x_\nu)}.\end{aligned}$$

Combining the above starting from (2.6.47) yields

$$\begin{aligned}&\|(U-\nu)w'\|^2_{L^2(0,x_\nu)} + \alpha^2\|\varphi\|^2_{L^2(0,x_\nu)} \\ &\quad \leq C\left(|\mu|\,\|\phi'\|^2_{L^2(0,x_\nu)} + [|\phi(x_\nu)| + N(v,\lambda)]\|\phi'\|_{L^2(0,x_\nu)}\right).\end{aligned} \tag{2.6.52}$$

Hence, by (2.6.46) and (2.6.4), there exists $C > 0$ such that for any $\hat{x}_0 \in (x_\nu/2, x_\nu)$ we have, with $\varepsilon = x_\nu - \hat{x}_0$,

$$\begin{aligned}\|\phi'\|_2 &\leq C(N(v,\lambda) + [1 + |\mu|^{1/2} + d^{-1/2}]\|\phi'\|_{L^2(x_\nu,1)}) \\ &\quad + C[|\mu|^{1/2}(\varepsilon^{-1} + |\log(d^2+\mu^2)|) + \varepsilon^{1/2}]\|\phi'\|_2.\end{aligned}$$

We can now choose $\hat{x}_0$ such that $\varepsilon = \inf((\frac{1}{4C})^2, x_{\nu_1}/4)$. Then under the condition $|\mu|^{1/2}(\varepsilon^{-1} + |\log(d^2+\mu^2)|) \leq \frac{1}{4C}$, which is valid for μ_A which is small enough, we obtain

$$\|\phi'\|_2 \leq 3C\left(N(v,\lambda) + d^{-1/2}\|\phi'\|_{L^2(x_\nu,1)}\right) \leq \hat{C}\left(N(v,\lambda) + \nu^{-1/2}\|\phi'\|_{L^2(x_\nu,1)}\right). \tag{2.6.53}$$

Combining (2.6.53) with (2.6.38) yields,

$$\|\phi'\|_2 \leq C\left(\frac{|\phi(x_\nu)|}{d} + d^{-1/2}N(v,\lambda)\right), \tag{2.6.54}$$

from which we can conclude a bound on ϕ in $H^1(0, x_\nu)$ as stated in (2.6.39), upon use of Poincaré's inequality and (2.6.4).

To complete the proof of (2.6.39), we need to estimate $c^\nu_\parallel$, which is defined in (2.6.3) by

$$c^\nu_\parallel = \frac{\langle \phi - \phi(x_\nu), U - \nu\rangle_{L^2(0,x_\nu)}}{\|U-\nu\|^2_{L^2(0,x_\nu)}}.$$

We note that for $0 < \nu < \nu_1$ the denominator in the definition of $c_\parallel^\nu$ satisfies

$$\|(U-\nu)\|^2_{L^2(0,x_\nu)} \geq \|(U-\nu)\|^2_{L^2(x_{\nu_1/4},x_{\nu_1/2})} \geq \frac{1}{C}. \tag{2.6.55}$$

Hence,

$$|c_\parallel^\nu| \leq C \left\| \frac{\phi - \phi(x_\nu)}{U-\nu} \right\|_{L^2(0,x_\nu)} \leq \hat{C}\, \|\phi'\|_{L^2(0,x_\nu)}, \tag{2.6.56}$$

which together with (2.6.54) yields (2.6.39).

Step 4. We prove that for any $A > 0$, there exists C and μ_A such that, for $\alpha^2 \leq A$, $|\mu| \leq \mu_A$, and $\nu \in (0, \nu_1)$ such that

$$\|\phi - c_\parallel^\nu (U-\nu)\|_{1,2} \leq C\nu^{-1/2}\Big(\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} + N(v,\lambda)\Big) \tag{2.6.57}$$

holds for any pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ satisfying (2.4.22).

Since (2.2.2) remains valid if we replace U by $U-\nu$, $1-x$ by $x_\nu - x$, and $(0,1)$ by $(0, x_\nu)$ we may conclude from (2.2.2) that there exists $C > 0$ such that

$$\|(U-\nu)w'\|^2_{L^2(0,x_\nu)} \geq \frac{1}{C}\|\varphi_\perp^\nu\|^2_{L^2(0,x_\nu)}, \tag{2.6.58}$$

where, in the interval $(0, x_\nu)$, $\varphi_\perp^\nu$ is defined by

$$\varphi_\perp^\nu = \phi - \phi(x_\nu) - c_\parallel^\nu(U-\nu) = \varphi - c_\parallel^\nu (U-\nu) \tag{2.6.59}$$

and $c_\parallel^\nu$ is defined in (2.6.3). Note that by construction

$$\langle \varphi_\perp^\nu, U-\nu\rangle_{(0,x_\nu)} = 0. \tag{2.6.60}$$

Furthermore, by (2.6.60) and (2.6.55)

$$\frac{1}{C}|c_\parallel^\nu|^2 \leq |c_\parallel^\nu|^2 \|(U-\nu)\|^2_{L^2(0,x_\nu)} \leq \|\varphi\|^2_{L^2(0,x_\nu)}. \tag{2.6.61}$$

Substituting (2.6.61) and (2.6.58) into (2.6.52) (recall again that $\varphi' \equiv \phi'$ in $(0, x_\nu)$) yields, with the aid of (2.6.18) and (2.6.59), for a new constant C

$$\begin{aligned}&\|\varphi_\perp^\nu\|^2_{L^2(0,x_\nu)} + \alpha^2 |c_\parallel^\nu|^2 \\ &\leq C\Big(\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big| + \big[|\phi(x_\nu)| + N(v,\lambda)\big]\big(\|(\varphi_\perp^\nu)'\|_{L^2(0,x_\nu)} + |c_\parallel^\nu|\big)\Big).\end{aligned} \tag{2.6.62}$$

Let $w_\perp^\nu := (U-\nu)^{-1}\varphi_\perp^\nu$. As in (2.3.8) we obtain that

$$\|(\varphi_\perp^\nu)'\|^2_{L^2(0,x_\nu)} \leq C\big(\|(U-\nu)(w_\perp^\nu)'\|^2_{L^2(0,x_\nu)} + \|\varphi_\perp^\nu\|_{L^2(0,x_\nu)}\|w_\perp^\nu\|_{L^2(0,x_\nu)}\big). \tag{2.6.63}$$

By Hardy's inequality (2.2.8) applied to $w_\perp^\nu$, and since $(w_\perp^\nu)' = w'$, we may conclude from (2.6.63), with the aid of (2.6.58), that

$$\begin{aligned}\|(\varphi_\perp^\nu)'\|_{L^2(0,x_\nu)} &\leq C(\|(U-\nu)w'\|_{L^2(0,x_\nu)} + \|\varphi_\perp^\nu\|_{L^2(0,x_\nu)}) \\ &\leq \widehat{C}\|(U-\nu)w'\|_{L^2(0,x_\nu)}.\end{aligned} \tag{2.6.64}$$

We now rewrite (2.6.52) with the aid of (2.6.18) in the form

$$\|(U-\nu)w'\|_{L^2(0,x_\nu)}^2 \leq C\Big(\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big| + [|\phi(x_\nu)| + N(v,\lambda)]\|\phi'\|_{L^2(0,x_\nu)}\Big).$$

Combining the above with (2.6.64) and (2.6.59) yields

$$\begin{aligned}\|(\varphi_\perp^\nu)'\|_{L^2(0,x_\nu)}^2 \leq C\Big(&\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big| \\ &+ [|\phi(x_\nu)| + N(v,\lambda)](\|(\varphi_\perp^\nu)'\|_{L^2(0,x_\nu)} + |c_\parallel^\nu|)\Big).\end{aligned} \tag{2.6.65}$$

This yields by (2.6.39) and (2.6.4) that

$$\|(\varphi_\perp^\nu)'\|_{L^2(0,x_\nu)}^2 \leq C\,\nu^{-1}\Big(\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big| + N(v,\lambda)^2\Big). \tag{2.6.66}$$

Clearly, by Poincaré's inequality

$$\begin{aligned}&\|\phi - c_\parallel^\nu(U-\nu)\|_{L^2(0,x_\nu)}^2 \\ &\quad\leq 2(\|\varphi_\perp^\nu\|_{L^2(0,x_\nu)}^2 + |\phi(x_\nu)|^2) \leq C(\|(\varphi_\perp^\nu)'\|_{L^2(0,x_\nu)}^2 + |\phi(x_\nu)|^2),\end{aligned}$$

which implies

$$\|\phi - c_\parallel^\nu(U-\nu)\|_{H^1(0,x_\nu)}^2 \leq C(\|(\varphi_\perp^\nu)'\|_{L^2(0,x_\nu)}^2 + |\phi(x_\nu)|^2). \tag{2.6.67}$$

Combining (2.6.67) with (2.6.39), (2.6.4), and (2.6.66) then yields

$$\|\phi - c_\parallel^\nu(U-\nu)\|_{H^1(0,x_\nu)} \leq C\nu^{-1/2}\Big(\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} + N(v,\lambda)\Big). \tag{2.6.68}$$

Next, we write with the aid of (2.6.19) and (2.6.39) (the bound on $|c_\parallel^\nu|$),

$$\begin{aligned}&\|\phi - c_\parallel^\nu(U-\nu)\|_{H^1(x_\nu,1)} \\ &\leq \|\phi\|_{H^1(x_\nu,1)} + d^{1/2}|c_\parallel^\nu| \leq C\Big(\nu^{-1/2}\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} + N(v,\lambda)\Big).\end{aligned} \tag{2.6.69}$$

Combining (2.6.69) with (2.6.68) and (2.6.4) yields,

$$\|\phi - c_\parallel^\nu(U-\nu)\|_{1,2} \leq C\nu^{-1/2}\Big(\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} + N(v,\lambda)\Big),$$

verifying (2.6.57).

Step 5. We prove that for any $0 < \nu_1 < U(0)$ and $A > 0$, there exists $\mu_A > 0$ and $C_A > 0$ such that (2.6.2) holds for $\alpha^2 \leq A$ and $|\mu| \leq \mu_A$.

For $\nu \in (0, \nu_1)$, let $\zeta_\nu \in C^\infty(\mathbb{R}_+, [0,1])$ satisfy

$$\zeta_\nu(x) = \begin{cases} 0 & x < \frac{x_\nu}{4}, \\ 1 & x > \frac{x_\nu}{2} \end{cases} \tag{2.6.70}$$

and

$$|\zeta_\nu'(x)| \leq C(\nu_1) \quad \forall \nu \in (0, \nu_1),\ \forall x \in (0,1).$$

Let further

$$\tilde{\zeta}_\nu = 1 - \zeta_\nu. \tag{2.6.71}$$

We may now write, omitting the reference to ν for ζ_ν and $\tilde{\zeta}_\nu$,

$$\Big|\Big\langle \zeta\phi, \frac{v}{U+i\lambda}\Big\rangle\Big| \leq |\phi(x_\nu)|\Big|\Big\langle \zeta, \frac{v}{U+i\lambda}\Big\rangle\Big| + \Big|\Big\langle \zeta\big(\phi - \phi(x_\nu)\big), \frac{v}{U+i\lambda}\Big\rangle\Big|. \tag{2.6.72}$$

For the first term on the right-hand side of (2.6.72) we begin by writing that

$$\Big\langle \zeta, \frac{v}{U+i\lambda}\Big\rangle = -\Big\langle \Big(\frac{\zeta\bar{v}}{U'}\Big)', \log(U+i\lambda)\Big\rangle + \frac{\bar{v}(1)\log(U(1)+i\lambda)}{U'(1)}.$$

Then we observe that

$$|\log(U(1)+i\lambda)| = |\log(U(1)-U(x_\nu)+i\mu)| \leq C(\log(d^{-1}+1)),$$

and since $d \leq 1 - x_{\nu_1}$ we obtain

$$|\log(U(1)+i\lambda)| = |\log(U(1)-U(x_\nu)+i\mu)| \leq \widehat{C}(\log(d^{-1})).$$

We can then conclude that

$$\Big|\Big\langle \zeta, \frac{v}{U+i\lambda}\Big\rangle\Big| \leq \Big|\Big\langle \Big(\frac{\zeta\bar{v}}{U'}\Big)', \log(U+i\lambda)\Big\rangle\Big| + C|v(1)|\log|d|^{-1}.$$

In view of (2.4.6) we can use Hölder's inequality and the fact that

$$\|v\|_p \leq \|v\|_\infty \leq |v(1)| + \|v'\|_p$$

to obtain, with the aid of (2.4.6), that for any $p > 1$, we have

$$\Big|\Big\langle \Big(\frac{\zeta\bar{v}}{U'}\Big)', \log(U+i\lambda)\Big\rangle\Big| \leq C(|v(1)| + \|v'\|_p).$$

Consequently, there exist $C > 0$ such that

$$\Big|\Big\langle \zeta, \frac{v}{U+i\lambda}\Big\rangle\Big| \leq C\left(\|v'\|_p + |v(1)|\log|d|^{-1}\right). \tag{2.6.73}$$

For the second term on the right-hand side of (2.6.72) we have, using (2.6.59) and Hardy's inequality (2.6.13)

$$\begin{aligned}\Big|\Big\langle\zeta(\phi-\phi(x_\nu)),\frac{v}{U+i\lambda}\Big\rangle\Big| &\leq |c_\parallel^\nu|\Big|\Big\langle\zeta(U-\nu),\frac{v}{U+i\lambda}\Big\rangle\Big|+\Big|\Big\langle\zeta\varphi_\perp^\nu,\frac{v}{U+i\lambda}\Big\rangle\Big| \\ &\leq C|c_\parallel^\nu|\,\|v\|_1+\Big\|\frac{\varphi_\perp^\nu}{x-x_\nu}\Big\|_2\Big\|\frac{(x-x_\nu)\,v\,\zeta}{U+i\lambda}\Big\|_2 \\ &\leq \hat{C}\big(|c_\parallel^\nu|\,\|v\|_1+\|(\varphi_\perp^\nu)'\|_2\|v\|_2\big). \end{aligned} \tag{2.6.74}$$

Substituting (2.6.73) and (2.6.74) into (2.6.72) yields

$$\begin{aligned}&\Big|\Big\langle\zeta\phi,\frac{v}{U+i\lambda}\Big\rangle\Big| \\ &\leq C\Big[|\phi(x_\nu)|(\|v'\|_p+|v(1)|\log|d|^{-1})+|c_\parallel^\nu|\,\|v\|_1+\|(\varphi_\perp^\nu)'\|_2\|v\|_2\Big]. \end{aligned} \tag{2.6.75}$$

As $0<\nu<\nu_1<U(0)$ and $\operatorname{supp}\tilde{\zeta}\subset[0,x_\nu/4]$, it holds that $|\tilde{\zeta}(U+i\lambda)^{-1}|\leq C$ and hence,

$$\Big|\Big\langle\tilde{\zeta}\phi,\frac{v}{U+i\lambda}\Big\rangle\Big|\leq C\|\phi\|_\infty\|v\|_1\leq C\|\phi'\|_2\|v\|_1. \tag{2.6.76}$$

Combining (2.6.76) with (2.6.75) and (2.6.59) then yields

$$\Big|\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle\Big|\leq C\,\big[|\phi(x_\nu)|\big(\|v'\|_p+|v(1)|\log|d|^{-1}\big)+|c_\parallel^\nu|\,\|v\|_1+\|(\varphi_\perp^\nu)'\|_2\|v\|_2\big].$$

With the aid of (2.6.4) we then obtain that

$$\Big|\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle\Big|\leq C\,\big[\|v'\|_p^2+|v(1)|^2\log^2|d|^{-1}+|c_\parallel^\nu|\,\|v\|_1+\|(\varphi_\perp^\nu)'\|_2\|v\|_2\big]. \tag{2.6.77}$$

Substituting (2.6.77) into (2.6.39) leads to

$$|c_\parallel^\nu|\leq C\big(\nu^{-1}[\|v'\|_p+|v(1)|\log|d|^{-1}+\|v\|_1+\|(\varphi_\perp^\nu)'\|_2^{1/2}\|v\|_2^{1/2}]+\nu^{-1/2}N(v,\lambda)\big). \tag{2.6.78}$$

Next, we substitute (2.6.77) into (2.6.57) to obtain, in view of (2.6.59)

$$\begin{aligned}\|(\varphi_\perp^\nu)'\|_2\leq C\big(\nu^{-1/2}[\|v'\|_p&+|v(1)|\log\nu^{-1}+|c_\parallel^\nu|^{1/2}\,\|v\|_1^{1/2} \\ &+\nu^{-1/2}\|v\|_2]+\nu^{-1/2}N(v,\lambda)\big). \end{aligned} \tag{2.6.79}$$

Combining (2.6.79) with (2.6.78) yields

$$\nu^{1/2}\|(\varphi_\perp^\nu)'\|_2+\nu|c_\parallel^\nu|\leq C\big(\|v'\|_p+|v(1)|\log\nu^{-1}+\nu^{-1}\|v\|_1+\nu^{-1/2}\|v\|_2+N(v,\lambda)\big). \tag{2.6.80}$$

Substituting (2.6.80) and (2.6.4) into (2.6.77) leads to

$$\begin{aligned}|\phi(x_\nu)|&\leq\Big|\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} \\ &\leq C\big(\|v'\|_p+|v(1)|\log\nu^{-1}+\nu^{-1}\|v\|_1+\nu^{-1/2}\|v\|_2+N(v,\lambda)\big). \end{aligned} \tag{2.6.81}$$

On the other hand, given the fact that $\phi(1) = 0$ and in view of (2.6.59), it holds that

$$|\phi(x)| = \Big|\int_x^1 \phi'(t)\,dt\Big| \le |c_\parallel^\nu|(1-x) + \|(\varphi_\perp^\nu)'\|_2(1-x)^{1/2}.$$

Consequently,

$$\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big| \le |c_\parallel^\nu|\Big\|(1-x)\,\frac{v}{U+i\lambda}\Big\|_1 + \|(\varphi_\perp^\nu)'\|_2\Big\|(1-x)^{1/2}\,\frac{v}{U+i\lambda}\Big\|_1. \tag{2.6.82}$$

Substituting (2.6.82) into (2.6.39) yields

$$\begin{aligned}|c_\parallel^\nu| &\le \nu^{-2}\Big\|(1-x)\,\frac{v}{U+i\lambda}\Big\|_1 \\ &\quad + \nu^{-1}\|(\varphi_\perp^\nu)'\|_2^{1/2}\Big\|(1-x)^{1/2}\,\frac{v}{U+i\lambda}\Big\|_1^{1/2} + \nu^{-1/2}N(v,\lambda).\end{aligned} \tag{2.6.83}$$

Then, substituting (2.6.82) into (2.6.57) leads to

$$\begin{aligned}\|(\varphi_\perp^\nu)'\|_2 &\le \nu^{-1/2}|c_\parallel^\nu|^{1/2}\Big\|(1-x)\,\frac{v}{U+i\lambda}\Big\|_1^{1/2} \\ &\quad + \nu^{-1}\Big\|(1-x)^{1/2}\,\frac{v}{U+i\lambda}\Big\|_1 + \nu^{-1/2}N(v,\lambda).\end{aligned} \tag{2.6.84}$$

Combining (2.6.83) and (2.6.84) we may conclude

$$\begin{aligned}&\nu^{1/2}\|(\varphi_\perp^\nu)'\|_2 + \nu|c_\parallel^\nu| \\ &\quad \le C\Big(\nu^{-1}\Big\|(1-x)\,\frac{v}{U+i\lambda}\Big\|_1^{1/2} + \nu^{-1/2}\Big\|(1-x)^{1/2}\frac{v}{U+i\lambda}\Big\|_1 + N(v,\lambda)\Big).\end{aligned} \tag{2.6.85}$$

Combining (2.6.85) with (2.6.81) and (2.6.1) yields (2.6.2) for $\alpha^2 \le A$.

Step 6. There exists $A_0 \ge 0$ and $\widehat{C}$ such that if $\alpha^2 \ge A_0$, $|\mu| \le 1$, $\nu \in (0,\nu_1)$, then

$$\|\phi\|_{H^1(0,1)} \le \widehat{C}\,N(v,\lambda) \tag{2.6.86}$$

for any pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ satisfying (2.4.22).

We preliminarily observe that

$$\begin{aligned}\delta_2 &= \sup_{\nu\in(0,\nu_1)}\Big\|\zeta_\nu\frac{U''}{U'}\Big\|_{1,\infty} < +\infty, \\ \widehat{C}_0 &= \sup_{\substack{|\mu|\le 1\\ \nu\in(0,\nu_1)}}\|\log(U+i\lambda)\|_2 < +\infty\end{aligned}$$

and

$$\widehat{C}_1 = \sup_{\substack{|\mu|\le 1\\ \nu\in(0,\nu_1)}}\Big\|\tilde{\zeta}_\nu\frac{U''}{U+i\lambda}\Big\|_\infty < +\infty,$$

where ζ_ν and $\tilde{\zeta}_\nu$ are defined in (2.6.70)–(2.6.71).

Taking (as in (2.4.15) for the case when $\alpha = 0$) the inner product of (2.3.2) with ϕ yields for the real part

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 = \Re\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle - \Re\Big\langle \zeta U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle - \Re\Big\langle \tilde{\zeta} U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle. \tag{2.6.87}$$

For the first term on the right-hand side, we can use (5.2.16) to obtain

$$\Big|\Re\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big| \leq C\|\phi'\|_2 N(v,\lambda).$$

For the second term we apply Poincaré's inequality, the finiteness of $\widehat{C}_0$ and δ_2, and the Sobolev embedding

$$\|\phi\|_\infty^2 \leq 2\|\phi\|_2\|\phi'\|_2,$$

to conclude that for some new constant $\widehat{C}$

$$\begin{aligned}
&\Big|\Big\langle \zeta U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle\Big| \\
&\quad \leq \Big|\Big\langle \Big(\frac{U''}{U'}\zeta|\phi|^2\Big)', \log(U+i\lambda)\Big\rangle\Big| \\
&\quad \leq \|\log(U+i\lambda)\|_2\|\phi\|_\infty\Big(2\Big\|\zeta\frac{U''}{U'}\Big\|_\infty\|\phi'\|_2 + \Big\|\Big(\zeta\frac{U''}{U'}\Big)'\Big\|_\infty\|\phi\|_2\Big) \\
&\quad \leq \widehat{C}\,\|\phi'\|_2^{3/2}\|\phi\|_2^{1/2}.
\end{aligned}$$

For the last term on the right-hand side of (2.6.87), we have

$$\Big|\Big\langle \tilde{\zeta} U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle\Big| \leq \|\phi\|_2^2\Big\|\tilde{\zeta}\frac{U''}{U+i\lambda}\Big\|_\infty \leq \widehat{C}_1\,\|\phi\|_2^2.$$

Consequently,

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq \widehat{C}\big(\|\phi'\|_2^{3/2}\|\phi\|_2^{1/2} + \|\phi\|_2^2 + \|\phi'\|_2 N(v,\lambda)\big).$$

Using Young's inequality we obtain, for some $A_0 \geq 0$ and $\widehat{C} > 0$

$$\frac{1}{2}\|\phi'\|_2^2 \leq (A_0 - \alpha^2)\|\phi\|_2^2 + \widehat{C}\,N(v,\lambda)^2. \tag{2.6.88}$$

Hence, for $\alpha^2 \geq A_0$, (2.6.86) follows immediately from the above inequality in conjunction with Poincaré's inequality.

Conclusion. Observing that $\frac{1}{C}\nu \leq d \leq C\nu$, the proof of (2.6.2) follows from (2.6.39), (2.6.57), and (2.6.86). ∎

2.7 The case $\Im\lambda < 0$

We now consider the case where $\Im\lambda$ is negative. Due to the non-invertibility of $\mathcal{A}_{0,0}$ the estimates of $\mathcal{A}_{\lambda,\alpha}^{-1}$ become challenging in the limit $\Im\lambda \to 0$ and the bounds necessarily include negative powers of $\nu = \Im\lambda$.

Proposition 2.7.1. *Let $U \in C^3([0,1])$ satisfy (2.1.3). Then there exist $C > 0$ such that for any $\alpha \geq 0$, and $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$* (where $\mathcal{A}_{\lambda,\alpha}$ is defined in (2.1.1)) *satisfying (2.4.22), we have for all $\nu < 0$,*

$$\|\phi\|_{1,2} \leq C(1 + |\nu|^{-1}) \left\|(1-x)^{1/2} \frac{v}{U + i\lambda}\right\|_1, \tag{2.7.1a}$$

and, for $-1/2 < \nu < 0$

$$\|\phi'\|_{L^2(1-|\nu|^{1/2},1)} \leq C\,|\nu|^{-3/4} \left\|(1-x)^{1/2} \frac{v}{U + i\lambda}\right\|_1. \tag{2.7.1b}$$

Furthermore, it holds that for all $\nu < 0$

$$|\phi(x)| \leq C(1-x)^{1/2}[1 + |\nu|^{-1/2}(1-x)^{1/2}] \left|\left\langle \phi, \frac{v}{U + i\lambda}\right\rangle\right|^{1/2}. \tag{2.7.2}$$

Proof. We begin by rewriting $\mathcal{A}_{\lambda,\alpha}\phi = v$ in the form

$$-((U-\nu)^2 w')' + \alpha^2 (U-\nu)^2 w = v - i\mu \frac{v - U''\phi}{U + i\lambda} = \frac{(U-\nu)v + i\mu U''\phi}{U + i\lambda},$$

where

$$w = \phi/(U-\nu).$$

Taking the inner product with w on the left yields (see (2.6.24) and (2.6.47))

$$\|(U-\nu)w'\|_2^2 + \alpha^2\|\phi\|_2^2 = \left\langle \phi, \frac{v}{U + i\lambda}\right\rangle + i\mu\left\langle w, \frac{U''\phi}{U + i\lambda}\right\rangle.$$

For the last term on the right-hand side we have, since $U'' < 0$ and $U - \nu > 0$,

$$\Re\left(i\mu\left\langle w, \frac{U''\phi}{U + i\lambda}\right\rangle\right) = |\mu|^2\left\langle \frac{\phi}{U-\nu}, \frac{U''\phi}{|U + i\lambda|^2}\right\rangle < 0.$$

Hence,

$$\|(U-\nu)w'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq \Re\left\langle \phi, \frac{v}{U + i\lambda}\right\rangle. \tag{2.7.3}$$

We now write, using the fact that $U(x) - \nu \geq C^{-1}(1 - x - \nu)$,

$$\begin{aligned} |w(x)| &\leq \left|\int_x^1 w'(t)\,dt\right| \leq \left[\int_x^1 \frac{1}{(U-\nu)^2}\,dt\right]^{1/2} \|(U-\nu)w'\|_2 \\ &\leq C\frac{(1-x)^{1/2}}{|\nu|^{1/2}(1-x-\nu)^{1/2}}\,\|(U-\nu)w'\|_2. \end{aligned} \tag{2.7.4}$$

Using this time the bound $U(x) - \nu \leq 1 - x - \nu$ together with (2.7.3) and (2.7.4), we obtain (2.7.2). Integrating (2.7.4) squared over $[0, 1]$ yields

$$\|w\|_2^2 \leq C\,|\nu|^{-1}\,\|(U-\nu)w'\|_2^2. \tag{2.7.5}$$

By writing $\phi = (U-\nu)w$, we obtain that

$$\|\phi'\|_2^2 \leq 2\,\|(U-\nu)w'\|_2^2 + 2\,\|U'w\|_2^2,$$

which leads, together with (2.7.5), to

$$\|\phi'\|_2^2 \leq C(|\nu|^{-1}+1)\|(U-\nu)w'\|_2^2. \tag{2.7.6}$$

We can now establish (2.7.1a) by combining (2.7.6) with (2.7.3) and the fact that $|\phi(x)| \leq \|\phi'\|_2(1-x)^{1/2}$.

To obtain (2.7.1b) we write for $-1/2 < \nu < 0$,

$$\|\phi'\|_{L^2(1-|\nu|^{1/2},1)}^2 \leq 2\,\|(U-\nu)w'\|_{L^2(1-|\nu|^{1/2},1)}^2 + 2\,\|U'w\|_{L^2(1-|\nu|^{1/2},1)}^2. \tag{2.7.7}$$

By (2.7.4) and the fact that for all $x \in [0, 1]$

$$0 \leq \frac{1-x}{|\nu|(1-x-\nu)} \leq \frac{1}{|\nu|},$$

we obtain via integration over $(1-|\nu|^{1/2}, 1)$ that

$$\|U'w\|_{L^2(1-|\nu|^{1/2},1)}^2 \leq C\,\|w\|_{L^2(1-|\nu|^{1/2},1)}^2 \leq \hat{C}\,|\nu|^{-1/2}\|(U-\nu)w'\|_2^2.$$

Substituting the above into (2.7.7) yields

$$\|\phi'\|_{L^2(1-|\nu|^{1/2},1)}^2 \leq C\,|\nu|^{-1/2}\|(U-\nu)w'\|_2^2. \tag{2.7.8}$$

By (2.7.3) and (2.7.1a) we then obtain that for $-1/2 < \nu < 0$,

$$\begin{aligned}\|\phi'\|_{L^2(1-|\nu|^{1/2},1)}^2 &\leq C\,|\nu|^{-1/2}\|\phi'\|_2\left\|(1-x)^{1/2}\frac{v}{U+i\lambda}\right\|_1 \\ &\leq \hat{C}\,|\nu|^{-3/2}\left\|(1-x)^{1/2}\frac{v}{U+i\lambda}\right\|_1^2,\end{aligned}$$

readily verifying (2.7.1b). ∎

2.8 The case $\nu_1 \leq \Im\lambda < U(0) - \kappa_0|\Re\lambda|$, $|\Re\lambda|$ small

In the following, we establish estimates, similar to (2.5.3), for $\nu_1 \leq \nu < U(0) - \kappa_0|\mu|$, for sufficiently small $U(0) - \nu_1$ and $|\mu|$.

Proposition 2.8.1. *Let $U \in C^4([0,1])$ satisfy (2.1.3) and $U'''(0) = 0$. Let further $p \ge 2$. There exist $0 < \nu_1 < U(0)$, $\mu_0 > 0$, $\kappa_0 > 0$, and $C > 0$ such that for λ s.t. $0 < |\mu| \le \mu_0$ and $\nu_1 < \nu < U(0) - \kappa_0|\mu|$, for all $\alpha \ge 0$ we have, for all pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times L^\infty(0,1)$ satisfying (2.4.22),*

$$\|\phi\|_{1,2} \le \frac{C}{|\mu|^{\frac{1}{p}}\, x_\nu^{1/2-1/p}}\|v\|_p \tag{2.8.1a}$$

and

$$\|\phi\|_{1,2} \le C\frac{\log\frac{x_\nu}{|\mu|^{1/2}}}{x_\nu^{1/2}}\|v\|_\infty. \tag{2.8.1b}$$

Proof. Since $0 < \nu < U(0)$ and since by (2.4.5) $U(x_\nu) = \nu$, we must have that $x_\nu \in (0,1)$.

Step 1. We prove that there exist $C > 0$, $\mu_0 > 0$, and $\nu_1 < U(0)$, such that, for all $\lambda = \mu + i\nu$ such that $\nu_1 < \nu < U(0) - |\mu|$ and $0 < |\mu| \le \mu_0$ it holds that

$$|\phi(x_\nu)|^2 \le Cx_\nu\Big[\frac{\mu}{x_\nu^2}\|\phi'\|_2^2 + \Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|\Big] \tag{2.8.2}$$

for all pairs $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times L^\infty(0,1)$ satisfying (2.4.22).

As in (2.6.6) we write

$$\Big|\Im\Big\langle\phi, \frac{v}{U-\nu+i\mu}\Big\rangle\Big| \ge |\mu|\Big\langle\frac{|U''|}{(U-\nu)^2+\mu^2}, \frac{1}{2}|\phi(x_\nu)|^2 - |\phi(x)-\phi(x_\nu)|^2\Big\rangle. \tag{2.8.3}$$

We begin by observing that for some $C > 0$

$$\Big|U(x) - \nu - \frac{1}{2}U''(0)[2x_\nu(x-x_\nu) + (x-x_\nu)^2]\Big| \le C[(x-x_\nu)^4 + x_\nu^3|x-x_\nu|]. \tag{2.8.4}$$

By (2.8.4) we have

$$\begin{aligned}&\frac{1}{2}U''(0)|x^2-x_\nu^2| - C[(x-x_\nu)^4 + x_\nu^3|x-x_\nu|]\\ &\qquad\le |U-\nu| \le \frac{1}{2}U''(0)|x^2-x_\nu^2| + C[(x-x_\nu)^4 + x_\nu^3|x-x_\nu|].\end{aligned} \tag{2.8.5}$$

From the right inequality in (2.8.5), as $(x-x_\nu)^3 + x_\nu^3 \le 2(x+x_\nu)$, we conclude that there exists $C_1 > 0$ such that

$$|U-\nu| \le C_1|x^2-x_\nu^2|. \tag{2.8.6}$$

From the left inequality in (2.8.5), there exist $a_0 > 0$ and $\nu_1 < U(0)$ such that for all $\nu_1 < \nu < U(0)$ and $x + x_\nu \le a_0$ it holds that

$$\frac{1}{2}U''(0)|x^2-x_\nu^2| - C[(x-x_\nu)^4 + x_\nu^3|x-x_\nu|] \ge \frac{1}{4}U''(0)|x^2-x_\nu^2|, \tag{2.8.7}$$

from which we can conclude that whenever $x + x_\nu \le a_0$, we have

$$|U - \nu| \ge \frac{1}{C}|x^2 - x_\nu^2|. \tag{2.8.8}$$

On the other hand we have that, for $\nu_1 < \nu < U(0)$ such that $x_{\nu_1} < \frac{a_0}{2}$,

$$\inf_{x \ge a_0 - x_\nu} \frac{|U - \nu|}{x^2 - x_\nu^2} \ge \frac{1}{a_0} \inf_{x \ge a_0 - x_\nu} \frac{|U - \nu|}{x - x_\nu} \ge \frac{|U'(a_0 - x_{\nu_1})|}{a_0} > 0.$$

Combining the above with (2.8.8) and (2.8.6) yields the existence of $0 < \nu_1 < U(0)$ and $C > 0$ for which

$$\frac{1}{C}(x^2 - x_\nu^2)^2 \le (U(x) - \nu)^2 \le C(x^2 - x_\nu^2)^2 \tag{2.8.9}$$

for all $x \in [0, 1]$ and $\nu_1 < \nu < U(0)$.

From (2.8.9) we can get that

$$\int_0^1 \frac{|U''|}{(U - \nu)^2 + \mu^2}\, dx \ge \frac{1}{C} \int_0^1 \frac{1}{(x - x_\nu)^2 (x + x_\nu)^2 + \mu^2}\, dx.$$

As

$$\sup_{x \in [1, \infty)} \frac{x^4}{(x - x_\nu)^2 (x + x_\nu)^2 + \mu^2} \le \frac{1}{(1 - x_{\nu_1}^2)^2},$$

we obtain that

$$\begin{aligned} \int_0^1 & \frac{dx}{(x - x_\nu)^2 (x + x_\nu)^2 + \mu^2} \\ & \ge \int_0^\infty \frac{dx}{(x - x_\nu)^2 (x + x_\nu)^2 + \mu^2} - \frac{1}{(1 - x_{\nu_1}^2)^2} \int_1^\infty \frac{dx}{x^4} \\ & = \int_0^\infty \frac{dx}{(x - x_\nu)^2 (x + x_\nu)^2 + \mu^2} - C_2, \end{aligned}$$

where $C_2 := \frac{1}{3(1 - x_{\nu_1}^2)^2}$.

Using the substitution $\xi = x / x_\nu$ it can be easily verified that

$$\int_0^\infty \frac{dx}{(x - x_\nu)^2 (x + x_\nu)^2 + \mu^2} = \frac{1}{2x_\nu^3} \int_{-\infty}^\infty \frac{d\xi}{(\xi^2 - 1)^2 + \hat{a}^2},$$

where $\hat{a} = |\mu| / x_\nu^2$.

Hence, there exists $C > 0$ such that

$$\int_0^1 \frac{|U''|}{(U - \nu)^2 + \mu^2}\, dx \ge \frac{1}{C}\Big[\frac{1}{x_\nu^3} \int_{-\infty}^\infty \frac{dx}{(x^2 - 1)^2 + \hat{a}^2} - 2C_2\Big]. \tag{2.8.10}$$

Making use of the residue theorem yields

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2-1)^2+\hat{a}^2} = 2\pi i\Big[\text{Res}\Big(\frac{1}{(z^2-1)^2+\hat{a}^2}, \sqrt{1+i\hat{a}}\Big) + \text{Res}\Big(\frac{1}{(z^2-1)^2+\hat{a}^2}, -\sqrt{1-i\hat{a}}\Big)\Big] = \frac{\pi\Re(\sqrt{1+i\hat{a}})}{\hat{a}\sqrt{\hat{a}^2+1}}. \tag{2.8.11}$$

Consequently, given that, by (2.8.6) applied with $x = 0$, it holds, for sufficiently small $U(0) - \nu$, that

$$\mu < U(0) - \nu \le C_1 x_\nu^2,$$

hence $x_\nu^2 \ge c_0\mu$ with $c_0 = 1/C_1$ and finally that $\hat{a} \le \frac{1}{c_0}$.

Consequently, there exists $0 < \nu_1 < U(0)$ and $\mu_0 > 0$ such that for all $\nu_1 < \nu < U(0) - |\mu|$ and $0 < |\mu| \le \mu_0$ we have for some $\widehat{C} > 0$

$$\int_0^1 \frac{|U''|}{(U-\nu)^2+\mu^2}\,dx \ge \frac{1}{C}\Big[\frac{1}{x_\nu^3}\frac{\pi\Re(\sqrt{1+i\hat{a}})}{\hat{a}\sqrt{\hat{a}^2+1}} - 2C_2\Big] \ge \frac{\widehat{C}}{|\mu|x_\nu}. \tag{2.8.12}$$

In a similar manner we can also show the existence of a positive C such that

$$\int_0^1 \frac{|U''|}{(U-\nu)^2+\mu^2}\,dx \le \widetilde{C}\int_0^\infty \frac{dx}{(x-x_\nu)^2(x+x_\nu)^2+\mu^2}\,dx \le \frac{C}{|\mu|x_\nu}. \tag{2.8.13}$$

We proceed with the estimation of the right-hand side of (2.8.3) by observing that, in view of (2.8.9)

$$\Big\langle\frac{|U''|}{(U-\nu)^2+\mu^2}, |\phi(x)-\phi(x_\nu)|^2\Big\rangle \le C\Big\|\frac{\phi(x)-\phi(x_\nu)}{x^2-x_\nu^2}\Big\|_2^2. \tag{2.8.14}$$

Applying Hardy's inequality yields

$$\Big\|\frac{\phi(x)-\phi(x_\nu)}{x^2-x_\nu^2}\Big\|_2^2 \le C\Big\|\Big(\frac{\phi(x)-\phi(x_\nu)}{x+x_\nu}\Big)'\Big\|_2^2. \tag{2.8.15}$$

As

$$\Big\|\Big(\frac{\phi(x)-\phi(x_\nu)}{x+x_\nu}\Big)'\Big\|_2^2 \le 2\Big\|\frac{\phi'}{x+x_\nu}\Big\|_2^2 + 2\Big\|\frac{\phi-\phi(x_\nu)}{(x+x_\nu)^2}\Big\|_2^2,$$

and since by Hardy's inequality (2.2.8)

$$\Big\|\frac{\phi-\phi(x_\nu)}{(x+x_\nu)^2}\Big\|_2^2 \le \frac{1}{x_\nu^2}\Big\|\frac{\phi-\phi(x_\nu)}{x-x_\nu}\Big\|_2^2 \le \frac{C}{x_\nu^2}\|\phi'\|_2^2,$$

we obtain that

$$\Big\|\Big(\frac{\phi(x)-\phi(x_\nu)}{x+x_\nu}\Big)'\Big\|_2^2 \le \frac{C}{x_\nu^2}\|\phi'\|_2^2.$$

Substituting the above into (2.8.15) and then into (2.8.14) yields

$$\Big\langle\frac{|U''|}{(U-\nu)^2+\mu^2},|\phi(x)-\phi(x_\nu)|^2\Big\rangle\leq\frac{C}{x_\nu^2}\|\phi'\|_2^2 \tag{2.8.16}$$

which, when substituted into (2.8.3) together with (2.8.12) leads to (2.8.2).

Step 2. We prove that there exist $0<\nu_1<U(0)$, positive μ_0, C, and C_0, and $\kappa_0\geq 1$ such that, for all $\alpha\geq C_0/x_\nu$, $\nu_1<\nu<U(0)-\kappa_0|\mu|$, and $0<|\mu|\leq\mu_0$, the inequality

$$\|\phi\|_{1,2}\leq C\Big|\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2}, \tag{2.8.17}$$

holds for any pair $(\phi,v)\in D(\mathcal{A}_{\lambda,\alpha})\times L^\infty(0,1)$ satisfying (2.4.22).

Let

$$\tilde U=U(0)-U\quad\text{and}\quad \kappa=\sqrt{(i\lambda+U(0))}.$$

Note that $|\kappa|>|\mu|^{1/2}>0$. Clearly,

$$\frac{1}{U+i\lambda}=-\frac{1}{(\tilde U^{1/2}-\kappa)(\tilde U^{1/2}+\kappa)}=-\frac{1}{2\kappa}\Big[\frac{1}{\tilde U^{1/2}-\kappa}-\frac{1}{\tilde U^{1/2}+\kappa}\Big].$$

We now write

$$\int_0^1\frac{U''}{U+i\lambda}\,dx=-\frac{1}{4\kappa}\int_0^1U''\Big[\frac{1}{\tilde U^{1/2}-\kappa}-\frac{1}{\tilde U^{1/2}+\kappa}\Big]\,dx.$$

An integration by parts now yields

$$\begin{aligned}&\int_0^1U''\Big[\frac{1}{\tilde U^{1/2}-\kappa}-\frac{1}{\tilde U^{1/2}+\kappa}\Big]\,dx\\ &\quad=\frac{1}{2}\frac{U''}{\tilde U^{-1/2}U'}\log\frac{\tilde U^{1/2}-\kappa}{\tilde U^{1/2}+\kappa}\Big|_{x=0}^{x=1}+\frac{1}{2}\int_0^1\Big(\frac{U''}{\tilde U^{-1/2}U'}\Big)'\log\frac{\tilde U^{1/2}-\kappa}{\tilde U^{1/2}+\kappa}\,dx\end{aligned}$$

Since $\max_{x\in[0,1]}(\tilde U^{-1/2}U')(x)<0$ and since $\tilde U^{-1/2}U'\in C^2([0,1])$ we can conclude that there exist positive C_1,C_2,C such that

$$\Big|\int_0^1\frac{U''}{U+i\lambda}\,dx\Big|\leq C_1+\frac{C_2}{\kappa}\int_0^1[|\log\tilde U^{1/2}-\kappa|+\log\tilde U^{1/2}+\kappa|]\,dx\leq\frac{C}{\kappa}.$$

Since by (2.8.8) it holds that $\kappa\geq\sqrt{U(0)-\nu}\geq x_\nu/C$, we conclude from the above that there exists $\hat C>0$ such that

$$\Big|\int_0^1\frac{U''}{U+i\lambda}\,dx\Big|\leq\frac{\hat C}{x_\nu}. \tag{2.8.18}$$

Taking the inner product of (2.4.22) with $\phi/(U+i\lambda)$ yields for the real part

$$\Re\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle=\|\phi'\|_2^2+\alpha^2\|\phi\|_2^2+\Re\Big\langle U''\phi,\frac{\phi}{U+i\lambda}\Big\rangle. \tag{2.8.19}$$

We then write

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq \Re\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle + \Big[\int_0^1 \frac{|U''(x)|\,\big|\,|\phi(x)|^2 - |\phi(x_\nu)|^2\,\big|}{|U+i\lambda|}\,dx + |\phi(x_\nu)|^2\Big|\int_0^1 \frac{U''(x)}{U+i\lambda}\,dx\Big|\Big]. \tag{2.8.20}$$

By (2.8.18) it holds that

$$|\phi(x_\nu)|^2\Big|\int_0^1 \frac{U''(x)}{U+i\lambda}\,dx\Big| \leq \frac{\widehat{C}}{x_\nu}|\phi(x_\nu)|^2. \tag{2.8.21}$$

To estimate the first term on the right-hand side of (2.8.21) we use the inequality

$$\big|\,|\phi(x)|^2 - |\phi(x_\nu)|^2\,\big| \leq [|\phi(x)| + |\phi(x_\nu)|]\,|\phi(x) - \phi(x_\nu)|$$

together with Hardy's inequality and (2.8.9) to obtain that for some $C_0 > 0$

$$\begin{aligned}
&\int_0^1 \frac{|U''(x)|\big|\,|\phi(x)|^2 - |\phi(x_\nu)|^2\,\big|}{|U+i\lambda|}\,dx \\
&\quad\leq \Big\|\frac{x^2 - x_\nu^2}{U+i\lambda}\Big\|_\infty \Big\|\frac{|\phi| + |\phi(x_\nu)|}{x+x_\nu}\Big\|_2 \Big\|\frac{\phi - \phi(x_\nu)}{x - x_\nu}\Big\|_2 \\
&\quad\leq C_0[x_\nu^{-1}\|\phi\|_2 + x_\nu^{-1/2}|\phi(x_\nu)|]\,\|\phi'\|_2.
\end{aligned} \tag{2.8.22}$$

Note that to obtain the last inequality we need the estimate

$$\Big\|\frac{\phi(x_\nu)}{x+x_\nu}\Big\|_2 \leq C\,\frac{|\phi(x_\nu)|}{x_\nu^{1/2}}.$$

Substituting (2.8.22) together with (2.8.21) into (2.8.20) yields the existence of $\widehat{C}_0 > 0$ and $C > 0$ such that, for $\alpha \geq \widehat{C}_0/x_\nu$,

$$\|\phi'\|_2^2 \leq \frac{C}{x_\nu}|\phi(x_\nu)|^2 + 2\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|.$$

Substituting (2.8.2) into the above yields

$$\|\phi'\|_2^2 \leq \frac{|\mu|}{x_\nu^2}|\phi(x_\nu)|^2 + (2 + Cx_\nu)\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|.$$

For sufficiently large κ_0 (or equivalently for sufficiently small $|\mu|/x_\nu^2$) we obtain (2.8.17).

Step 3. With $N_2(v,\lambda)$ given by

$$N_2(v,\lambda) = \Big\|(1-x)^{1/2}\frac{v}{U+i\lambda}\Big\|_1, \tag{2.8.23}$$

we prove that, for any $\widehat{C}_0 > 0$, there exist $C > 0$, $\mu_0 > 0$, $\kappa_0 > 0$, and $\nu_1 > 0$ such that, for $\nu_1 < \nu < U(0) - \kappa_0|\mu|$, $|\alpha| \leq \widehat{C}_0/x_\nu$, and $|\mu| \leq \mu_0$, we have

$$\|\phi'\|_{L^2(0,x_\nu)} \leq C\Big(\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} + x_\nu^{1/2}N_2(v,\lambda)\Big), \tag{2.8.24}$$

holds for any pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ satisfying (2.4.22).

We seek an estimate for $\|\phi'\|_{L^2(0,x_\nu)}$, depending on $\|\phi\|_{L^2(0,x_\nu)}$. We begin, to this end, by obtaining an L^∞ estimate of ϕ'.

Estimate of $\|\phi'\|_{L^\infty}$. We separately consider the subintervals $(0, x_\nu/2)$ and $(x_\nu/2, x_\nu)$.

Estimate on $(0, x_\nu/2)$. To obtain an estimate for $\|\phi'\|_{L^\infty(0,x_\nu/2)}$ we integrate the relation $\mathcal{A}_{\lambda,\alpha}\phi = v$, to obtain for all $x \in (0, x_\nu)$,

$$|\phi'(x)| = \Big|\int_0^x \phi''(t)\,dt\Big| \leq \Big|\int_0^x \Big(\frac{U''\phi - v}{U+i\lambda} + \alpha^2\phi\Big)\,dt\Big|,$$

which leads to

$$|\phi'(x) \leq \Big|\int_0^x \Big(\frac{U''\phi}{U+i\lambda}\Big)\,dt\Big| + \Big\|\frac{v}{U+i\lambda}\Big\|_{L^1(0,x)} + \alpha^2\|\phi\|_{L^1(0,x)}. \tag{2.8.25}$$

We then use the following decomposition

$$\int_0^x \frac{U''\phi}{U+i\lambda}\,dt = \int_0^x \frac{U''[\phi-\phi(x_\nu)]}{U+i\lambda}\,dt + \phi(x_\nu)\int_0^x \frac{U''}{U+i\lambda}\,dt. \tag{2.8.26}$$

To estimate the first integral on the right-hand side of (2.8.26) we need the following bound which follows from (2.8.4):

$$\Big|\frac{U''}{U-\nu} - \frac{2}{x^2-x_\nu^2}\Big| \leq \Big|\frac{U''(0)}{U-\nu} - \frac{2}{x^2-x_\nu^2}\Big| + \Big|\frac{U''(x)-U''(0)}{U-\nu}\Big| \leq C\,\frac{x_\nu}{x_\nu^2-x^2}.$$

Consequently,

$$\begin{aligned}\Big|\int_0^x \frac{U''[\phi-\phi(x_\nu)]}{U+i\lambda}\,dt\Big| &\leq \int_0^x \frac{|U''|\,|\phi-\phi(x_\nu)|}{|U-\nu|}\,dt\\ &\leq (2+Cx_\nu)\int_0^x \frac{|\phi(t)-\phi(x_\nu)|}{x_\nu^2-t^2}\,dt,\end{aligned} \tag{2.8.27}$$

and hence, for sufficiently small $U(0) - \nu_1$,

$$\begin{aligned}\Big|\int_0^x \frac{U''[\phi-\phi(x_\nu)]}{U+i\lambda}\,dt\Big| &\leq (2+Cx_\nu)\log\frac{x_\nu+x}{x_\nu}\|\phi'\|_{L^\infty(0,x)}\\ &\leq (1+Cx_\nu)\log(9/4)\|\phi'\|_{L^\infty(0,x)}.\end{aligned} \tag{2.8.28}$$

To obtain the last inequality we have used the fact that $x \in (0, x_\nu/2)$. Note that $\log(9/4) < 1$. Hence, the coefficient of $|\phi'\|_{L^\infty(0,x)}$ is smaller than one for sufficiently small $U(0) - \nu_1$. Next, we write for the second integral in the right-hand side of equation (2.8.26)

$$\left|\phi(x_\nu)\int_0^x \frac{U''}{U+i\lambda}\,dt\right| \le |\phi(x_\nu)|\left|\int_0^x \frac{|U''|}{|U-\nu|}\,dt\right| \le (2+Cx_\nu)|\phi(x_\nu)|\int_0^x \frac{dt}{x_\nu^2 - t^2},$$

which leads to

$$\left|\phi(x_\nu)\int_0^x \frac{U''}{U+i\lambda}\,dt\right| \le \hat{C}\,\frac{|\phi(x_\nu)|}{x_\nu}\,\log\frac{x_\nu + x}{x_\nu - x}. \tag{2.8.29}$$

Substituting the above, together with (2.8.29) and (2.8.28) into (2.8.25) yields for all $t \in (0, x] \subset (0, x_\nu/2]$

$$|\phi'(t)| \le \log\frac{9}{4}(1 + Cx_\nu)\|\phi'\|_{L^\infty(0,t)} + C\,\frac{|\phi(x_\nu)|}{x_\nu}\log\frac{x_\nu + t}{x_\nu - t} + \left\|\frac{v}{U+i\lambda}\right\|_{L^1(0,x)} + \alpha^2\|\phi\|_{L^1(0,x)}.$$

Taking the supremum over $t \in (0, x]$ yields

$$\|\phi'(t)\|_{L^\infty(0,x)} \le \log\frac{9}{4}(1 + Cx_\nu)\|\phi'\|_{L^\infty(0,x)} + C\,\frac{|\phi(x_\nu)|}{x_\nu}\,\log\frac{x_\nu + x}{x_\nu - x} + \left\|\frac{v}{U+i\lambda}\right\|_{L^1(0,x)} + \alpha^2\|\phi\|_{L^1(0,x)}.$$

Hence, for sufficiently small $U(0) - \nu_1$, we obtain for all $\nu_1 < \nu < U(0)$ and all $x \in [0, x_\nu/2]$,

$$\|\phi'\|_{L^\infty(0,x)} \le C\Big(\frac{|\phi(x_\nu)|}{x_\nu}\log\frac{x + x_\nu}{x - x_\nu} + \alpha^2\|\phi\|_{L^1(0,x)} + \left\|\frac{v}{U+i\lambda}\right\|_{L^1(0,x)}\Big). \tag{2.8.30}$$

Estimate on $(x_\nu/2, x_\nu)$. To obtain a bound for $\|\phi'\|_{L^\infty(x_\nu/2,x)}$ for $x \in (x_\nu/2, x_\nu)$ we write

$$|\phi'(x)| \le |\phi'(x_\nu/2)| + \left|\int_{x_\nu/2}^x \phi''(t)\,dt\right|.$$

The first term can be estimated by using (2.8.32). To obtain a bound for the second term, we follow the same path as in (2.8.27). We get

$$\left|\int_{x_\nu/2}^x \frac{U''[\phi - \phi(x_\nu)]}{U+i\lambda}\,dt\right| \le (2 + Cx_\nu)\int_{x_\nu.}^x \frac{|\phi(t) - \phi(x_\nu)|}{x_\nu^2 - t^2}\,dt. \tag{2.8.31}$$

As in (2.8.28) we can conclude that

$$\begin{aligned}(2 + Cx_\nu)\int_{x_\nu/2}^x \frac{|\phi(t) - \phi(x_\nu)|}{x_\nu^2 - t^2}\,dt &\le (2 + Cx_\nu)\|\phi'\|_{L^\infty(x_\nu/2,x)}\log\frac{x + x_\nu}{3x_\nu/2}\\ &\le (2 + Cx_\nu)\log(4/3)\|\phi'\|_{L^\infty(x_\nu/2,x)}.\end{aligned}$$

Hence, the coefficient of $\|\phi'\|_{L^\infty(x_\nu/2,x)}$ is again smaller than 1 by a suitable choice of ν_1. Repeating the above other steps then yields, for $x \geq x_\nu/2$,

$$\|\phi'\|_{L^\infty(x_\nu/2,x)} \leq C\Big(\frac{1}{x_\nu}|\phi(x_\nu)|\log\frac{x_\nu+x}{x_\nu-x}+\alpha^2\|\phi\|_{L^1(0,x)}+\Big\|\frac{v}{U+i\lambda}\Big\|_{L^1(0,x)}+|\phi'(x_\nu/2)|\Big),$$

which, combined with (2.8.30) for $x = x_\nu/2$, finally gives

$$\|\phi'\|_{L^\infty(x_\nu/2,x)} \leq \widehat{C}\Big(\frac{1}{x_\nu}|\phi(x_\nu)|\log\frac{x_\nu+x}{x_\nu-x}+\alpha^2\|\phi\|_{L^1(0,x)}+\Big\|\frac{v}{U+i\lambda}\Big\|_{L^1(0,x)}\Big).$$

Combining the above and (2.8.30) lead to the existence of $C > 0$ such that

$$\|\phi'\|_{L^\infty(0,x)} \leq C\Big(\frac{1}{x_\nu}|\phi(x_\nu)|\log\frac{x_\nu+x}{x_\nu-x}+\alpha^2\|\phi\|_{L^1(0,x)}+\Big\|\frac{v}{U+i\lambda}\Big\|_{L^1(0,x)}\Big) \quad \forall x \in (0,x_\nu). \tag{2.8.32}$$

Estimate of $\|\phi'\|_{L^2(0,x_\nu)}$. Observing that

$$\int_0^{x_\nu}\log^2\frac{x_\nu+x}{x_\nu-x}\,dx \leq x_\nu\int_0^1\log^2\frac{1+t}{1-t}\,dt \leq Cx_\nu, \tag{2.8.33}$$

we may conclude from (2.8.32) by integrating over $(0, x_\nu)$, that

$$\|\phi'\|_{L^2(0,x_\nu)} \leq C\Big(x_\nu^{-1/2}|\phi(x_\nu)|+\alpha^2 x_\nu\|\phi\|_{L^2(0,x_\nu)}+x_\nu^{1/2}\Big\|\frac{v}{U+i\lambda}\Big\|_{L^1(0,x_\nu)}\Big).$$

Note that

$$\Big\|\frac{v}{U+i\lambda}\Big\|_{L^1(0,x_\nu)} \leq (1-x_\nu)^{-\frac{1}{2}}N_2(v,\lambda), \tag{2.8.34}$$

which leads to

$$\|\phi'\|_{L^2(0,x_\nu)} \leq C\big(x_\nu^{-1/2}|\phi(x_\nu)|+\alpha^2 x_\nu\|\phi\|_{L^2(0,x_\nu)}+x_\nu^{1/2}N_2(v,\lambda)\big). \tag{2.8.35}$$

Estimate of $\|\phi\|_{L^2(0,x_\nu)}$. Set

$$\phi = \varphi + \phi(x_\nu),$$

and recall from (2.6.23) that for all $x \in (0, x_\nu)$

$$-\Big((U-\nu)^2\Big(\frac{\varphi}{U-\nu}\Big)'\Big)'+\alpha^2(U-\nu)\varphi = \frac{(U-\nu)v}{U+i\lambda}-\alpha^2\phi(x_\nu)\big((U-\nu)-U''\big)+i\mu\frac{U''\phi}{U+i\lambda}.$$

Taking the inner product, in $L^2(0, x_\nu)$, with w defined as in equation (2.6.22) by $w := (U - \nu)^{-1}\varphi$, yields as in (2.6.24)

$$\begin{aligned}
&\|(U-\nu)w'\|^2_{L^2(0,x_\nu)} + \alpha^2\|\varphi\|^2_{L^2(0,x_\nu)} \\
&\quad = \Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)} - \langle w, \phi(x_\nu)U''\rangle_{L^2(0,x_\nu)} \\
&\qquad - \alpha^2\phi(x_\nu)\langle\varphi, 1\rangle_{L^2(0,x_\nu)} + i\mu\Big\langle w, \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}.
\end{aligned} \tag{2.8.36}$$

We now turn to estimate the various terms on the right-hand side of (2.8.36).

For the second term in (2.8.36) we use the fact that by Hardy's inequality and (2.8.9) it holds that

$$\|w\|_{L^1(0,x_\nu)} \leq C\Big\|\frac{\phi-\phi(x_\nu)}{x-x_\nu}\Big\|_{L^2(0,x_\nu)}\Big\|\frac{1}{x+x_\nu}\Big\|_{L^2(0,x_\nu)} \leq \frac{\widehat{C}}{x_\nu^{1/2}}\|\phi'\|_{L^2(0,x_\nu)}. \tag{2.8.37}$$

Consequently,

$$|\langle w, \phi(x_\nu)U''\rangle_{L^2(0,x_\nu)}| \leq C\frac{|\phi(x_\nu)|}{x_\nu^{1/2}}\|\phi'\|_{L^2(0,x_\nu)}. \tag{2.8.38}$$

For the third term in (2.8.36), it follows from that

$$\alpha^2|\phi(x_\nu)\langle\varphi, 1\rangle| \leq C\alpha^2 x_\nu^{1/2}(\|\phi\|_{L^2(0,x_\nu)} + x_\nu^{1/2}|\phi(x_\nu)|)|\phi(x_\nu)|. \tag{2.8.39}$$

Finally, for the last term in (2.8.36), proceeding as in the proof of (2.8.22), we obtain by using (2.8.37) and Hardy's inequality

$$\begin{aligned}
&\Big|\Big\langle w, \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| \\
&\leq |\phi(x_\nu)|\,\|w\|_{L^1(0,x_\nu)}\Big\|\frac{U''}{U+i\lambda}\Big\|_{L^\infty(0,x_\nu)} + \|w\|_{L^2(0,x_\nu)}\Big\|\frac{U''(\phi-\phi(x_\nu))}{U+i\lambda}\Big\|_{L^2(0,x_\nu)} \\
&\leq C\Big[\frac{|\phi(x_\nu)|}{|\mu|x_\nu^{1/2}}\|\phi'\|_{L^2(0,x_\nu)} + \frac{\|\phi'\|^2_{L^2(0,x_\nu)}}{x_\nu^2}\Big].
\end{aligned}$$

Substituting the above, together with (2.8.39) and (2.8.38) into (2.8.36) yields

$$\begin{aligned}
\alpha^2\|\varphi\|^2_{L^2(0,x_\nu)} \leq{}& C\Big[\frac{|\phi(x_\nu)|}{x_\nu^{1/2}}\|\phi'\|_{L^2(0,x_\nu)} + \frac{|\mu|}{x_\nu^2}\|\phi'\|^2_{L^2(0,x_\nu)}\Big] \\
&+ C\alpha^2(x_\nu^{1/2}\|\phi\|_{L^2(0,x_\nu)} + x_\nu|\phi(x_\nu)|)|\phi(x_\nu)| \\
&+ \Big|\Big\langle\varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big|,
\end{aligned}$$

from which we easily conclude that

$$\alpha^2\|\phi\|_{L^2(0,x_\nu)}^2 \leq C\Big[\frac{|\phi(x_\nu)|}{x_\nu^{1/2}}\|\phi'\|_{L^2(0,x_\nu)} + \frac{|\mu|}{x_\nu^2}\|\phi'\|_{L^2(0,x_\nu)}^2 + \alpha^2 x_\nu|\phi(x_\nu)|^2\Big] + \Big|\Big\langle\varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big|. \tag{2.8.40}$$

Substituting (2.8.40) into (2.8.35) we obtain

$$\|\phi'\|_{L^2(0,x_\nu)} \leq C\Big(\frac{|\phi(x_\nu)|}{x_\nu^{1/2}} + \alpha^2 x_\nu^{3/2}|\phi(x_\nu)| + \alpha|\mu|^{1/2}\|\phi'\|_{L^2(0,x_\nu)} + x_\nu^{1/2}N_2(v,\lambda) + \alpha x_\nu\Big|\Big\langle\varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big|^{1/2}\Big).$$

As $|\alpha| \leq C_0/x_\nu$, we obtain for sufficiently large κ_0 (implying that both $|\mu|^{1/2}/x_\nu$ and $\alpha|\mu|^{1/2}$ are sufficiently small)

$$\|\phi'\|_{L^2(0,x_\nu)} \leq \hat{C}\Big(\frac{|\phi(x_\nu)|}{x_\nu^{1/2}} + x_\nu^{1/2}N_2(v,\lambda) + \Big|\Big\langle\varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big|^{1/2}\Big). \tag{2.8.41}$$

Note that, by (2.8.34),

$$\Big|\Big\langle\varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| \leq \|\phi-\phi(x_\nu)\|_{L^\infty(0,x_\nu)}\Big\|\frac{v}{U+i\lambda}\Big\|_{L^1(0,x_\nu)} \leq 2x_\nu^{1/2}\|\phi'\|_{L^2(0,x_\nu)}N_2(v,\lambda).$$

Combining the above with (2.8.41) and (2.8.2) yields

$$\|\phi'\|_{L^2(0,x_\nu)} \leq C\Big[\frac{\mu^{1/2}}{x_\nu}\|\phi'\|_2 + x_\nu^{1/2}N_2(v,\lambda) + \Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2}\Big].$$

For sufficiently large κ_0 we easily obtain (2.8.24).

Step 4. We prove that, for any $C_0 > 0$, there exist $C > 0$, $\mu_0 > 0$, $\kappa_0 > 0$, and $\nu_1 > 0$ such that, for $\nu_1 < \nu < U(0) - \kappa_0|\mu|$, $|\alpha| \leq C_0/x_\nu$, and $|\mu| \leq \mu_0$, we have

$$\|\phi'\|_2 \leq C\Big(\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} + x_\nu^{1/2}N_2(v,\lambda)\Big), \tag{2.8.42}$$

for any pair (ϕ, v) satisfying (2.4.22).

Observing that $U''(U-\nu) > 0$ on $(x_\nu, 1)$ implies

$$\Re\Big\langle U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)} = \int_{x_\nu}^1 \frac{|\phi|^2U''(U-\nu)}{|U+i\lambda|^2}\,dx \geq 0,$$

and hence we may conclude from (2.8.19), that

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq -\Re\Big\langle U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)} + \Re\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle. \tag{2.8.43}$$

To estimate $\left|\Re\left\langle U''\phi, \frac{\phi}{U+i\lambda}\right\rangle_{L^2(0,x_\nu)}\right|$, we now proceed as in the proof of (2.8.21)–(2.8.22) to obtain

$$\begin{aligned}\left|\Re\left\langle U''\phi, \frac{\phi}{U+i\lambda}\right\rangle_{L^2(0,x_\nu)}\right| \\ \le \frac{C}{x_\nu}|\phi(x_\nu)|^2 + C\left(x_\nu^{-1}\|\phi\|_{L^2(0,x_\nu)} + x_\nu^{-1/2}|\phi(x_\nu)|\right)\|\phi'\|_{L^2(0,x_\nu)}. \end{aligned} \tag{2.8.44}$$

Using Poincaré's inequality applied in $(0, x_\nu)$ to $(\phi - \phi(x_\nu))$ yields

$$\begin{aligned}\left|\Re\left\langle U''\phi, \frac{\phi}{U+i\lambda}\right\rangle_{L^2(0,x_\nu)}\right| \\ \le \frac{C}{x_\nu}|\phi(x_\nu)|^2 + \widehat{C}\left(\|\phi'\|_{L^2(0,x_\nu)} + x_\nu^{-1/2}|\phi(x_\nu)|\right)\|\phi'\|_{L^2(0,x_\nu)}. \end{aligned} \tag{2.8.45}$$

Substituting (2.8.45) together with (2.8.24) and (2.8.2) into (2.8.43) yields

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \le C\left[\frac{|\mu|}{x_\nu^2}\|\phi'\|_2^2 + \left|\Re\left\langle\phi, \frac{v}{U+i\lambda}\right\rangle\right| + x_\nu N_2(v,\lambda)^2\right].$$

For sufficiently large κ_0 we readily obtain (2.8.42).

Step 5. We prove (2.8.1). We begin by deriving two conclusions of (2.8.42) and (2.8.17) under the assumptions of the proposition. Since by (2.6.82)

$$\left|\left\langle\phi, \frac{v}{U+i\lambda}\right\rangle\right| \le \|\phi'\|_2 N_2(v,\lambda),$$

where N_2 is given by (2.8.23), we obtain by (2.8.42) (for $|\alpha| \ge \frac{C_0}{x_\nu}$) and (2.8.17) (for $|\alpha| \le \frac{C_0}{x_\nu}$) that, under the assumption of the proposition,

$$\|\phi'\|_2 \le C N_2(v,\lambda), \tag{2.8.46}$$

which combined with (2.8.2) yields, for bounded $|\mu|/x_\nu^2$,

$$|\phi(x_\nu)| \le C x_\nu^{1/2} N_2(v,\lambda). \tag{2.8.47}$$

Proof of (2.8.1a). We begin by obtaining a bound on $\|(U+i\lambda)^{-1}\|_q$. By (2.8.9) we have for $q > 1$ and $\nu_1 < \nu < U(0)$

$$\left\|\frac{1}{U+i\lambda}\right\|_{L^q(0,1)}^q \le \frac{C}{|\mu|^{q-1/2}}\int_{\mathbb{R}_+}\frac{ds}{[(s^2-a^2)^2+1]^{q/2}},$$

where $a = x_\nu|\mu|^{-1/2}$.

We estimate the integral on the right-hand side in the following manner:

$$\begin{aligned}\int_{\mathbb{R}_+}\frac{ds}{[(s^2-a^2)^2+1]^{q/2}} &\le \int_{\mathbb{R}_+}\frac{ds}{[a^2(s-a)^2+1]^{q/2}} \\ &\le \int_{\mathbb{R}}\frac{d\tau}{[a^2\tau^2+1]^{q/2}} \le \frac{1}{a}\int_{\mathbb{R}}\frac{dt}{[t^2+1]^{q/2}} \le \frac{C}{a}. \end{aligned} \tag{2.8.48}$$

It follows that for $q > 1$ and $\nu_1 \leq \nu < U(0)$

$$\Big\| \frac{1}{U + i\lambda} \Big\|_{L^q(0,1)}^q \leq \frac{C}{x_\nu |\mu|^{q-1}}. \tag{2.8.49}$$

Since for $\nu < \nu_1$ we may establish (2.8.49), using (2.6.10), as in [3] we may conclude that (2.8.49) holds for any $0 \leq \nu < U(0)$.

We continue by estimating $\langle \phi, (U + i\lambda)^{-1} v \rangle$. To thus end we write

$$\Big| \Big\langle \phi, \frac{v}{U + i\lambda} \Big\rangle \Big| \leq |\phi(x_\nu)| \, \Big\| \frac{v}{U + i\lambda} \Big\|_1 + \Big\| \frac{\phi - \phi(x_\nu)}{x - x_\nu} \Big\|_2 \Big\| \frac{(x - x_\nu)\, v}{U + i\lambda} \Big\|_2. \tag{2.8.50}$$

Suppose first that $v \in L^p(0,1)$ for some $p \in [2, \infty)$. Then,

$$\Big\| \frac{v}{U + i\lambda} \Big\|_1 \leq \|v\|_p \Big\| \frac{1}{U + i\lambda} \Big\|_q,$$

where $q = p/(p-1)$.

Consequently, by (2.8.49)

$$\Big\| \frac{v}{U + i\lambda} \Big\|_1 \leq \frac{C}{|\mu|^{\frac{1}{p}} x_\nu^{1-\frac{1}{p}}} \|v\|_p. \tag{2.8.51}$$

Next, we estimate the second term on the right-hand side of (2.8.50). Consider first the case $p = 2$. Here, we write with the aid of (2.8.9)

$$\Big\| \frac{(x - x_\nu)v}{U + i\lambda} \Big\|_2 \leq \|v\|_2 \Big\| \frac{x - x_\nu}{U + i\lambda} \Big\|_\infty \leq \frac{C}{x_\nu} \|v\|_2. \tag{2.8.52}$$

For $p > 2$ we have

$$\Big\| \frac{(x - x_\nu)v}{U + i\lambda} \Big\|_2 \leq \|v\|_p \Big\| \frac{x - x_\nu}{U + i\lambda} \Big\|_{\tilde{q}},$$

where $\tilde{q} = 2p/(p-2)$.

As above we write

$$\Big\| \frac{x - x_\nu}{U + i\lambda} \Big\|_{\tilde{q}}^{\tilde{q}} \leq C \int_0^1 \frac{dx}{[x + x_\nu]^{\tilde{q}}} \leq \frac{\widehat{C}}{x_\nu^{\tilde{q}-1}}. \tag{2.8.53}$$

Consequently,

$$\Big\| \frac{(x - x_\nu)v}{U + i\lambda} \Big\|_2 \leq \frac{C}{x_\nu^{\frac{1}{2}+\frac{1}{p}}} \|v\|_p, \tag{2.8.54}$$

which is in accordance with (2.8.52) for $p = 2$.

Using (2.8.50) once again, we deduce from (2.8.51) and (2.8.54) that

$$\Big| \Big\langle \phi, \frac{v}{U + i\lambda} \Big\rangle \Big| \leq C \left(\frac{1}{|\mu|^{\frac{1}{p}} x_\nu^{1-\frac{1}{p}}} |\phi(x_\nu)| + \frac{1}{x_\nu^{\frac{1}{2}+\frac{1}{p}}} \Big\| \frac{\phi - \phi(x_\nu)}{x - x_\nu} \Big\|_2 \right) \|v\|_p.$$

By Hardy's inequality we then have

$$\left|\left\langle \phi, \frac{v}{U+i\lambda}\right\rangle\right| \le C\left(\frac{1}{|\mu|^{\frac{1}{p}} x_\nu^{1-\frac{1}{p}}}|\phi(x_\nu)| + \frac{1}{x_\nu^{\frac{1}{2}+\frac{1}{p}}}\|\phi'\|_2\right)\|v\|_p. \tag{2.8.55}$$

By (2.8.23) and (2.8.51), we have that

$$N_2(v,\lambda) \le \left\|\frac{v}{U+i\lambda}\right\|_1 \le \frac{C}{|\mu|^{\frac{1}{p}} x_\nu^{1-\frac{1}{p}}}\|v\|_p. \tag{2.8.56}$$

The above combined with (2.8.46) yields

$$\|\phi'\|_2 \le \frac{C}{|\mu|^{\frac{1}{p}} x_\nu^{1-\frac{1}{p}}}\|v\|_p, \tag{2.8.57}$$

which is weaker than (2.8.1a).

We obtain a better estimate in the following manner. By (2.8.55), (2.8.56), and (2.8.42) it holds that

$$\|\phi'\|_2 \le$$
$$C\left(\left[\frac{1}{|\mu|^{\frac{1}{2p}} x_\nu^{\frac{1}{2}-\frac{1}{2p}}}|\phi(x_\nu)|^{1/2} + \frac{1}{x_\nu^{\frac{1}{4}+\frac{1}{2p}}}\|\phi'\|_2^{1/2}\right]\|v\|_p^{1/2} + |\mu|^{-\frac{1}{p}} x_\nu^{-\frac{1}{2}+\frac{1}{p}}\|v\|_p\right),$$

from which we get

$$\begin{aligned}\|\phi'\|_2 &\le C\frac{1}{x_\nu^{\frac{1}{4}+\frac{1}{2p}}}\|\phi'\|_2^{\frac{1}{2}}\|v\|_p^{\frac{1}{2}} + C\left(x_\nu^{-1/2}|\phi(x_\nu)| + |\mu|^{-\frac{1}{p}} x_\nu^{-\frac{1}{2}+\frac{1}{p}}\|v\|_p\right)\\ &\le \hat{C}|\mu|^{\frac{1}{p}} x_\nu^{-\frac{2}{p}}\|\phi'\|_2 + \hat{C}\left(x_\nu^{-1/2}|\phi(x_\nu)| + |\mu|^{-\frac{1}{p}} x_\nu^{-\frac{1}{2}+\frac{1}{p}}\|v\|_p\right).\end{aligned}$$

Using the fact that $|\mu|^{1/2}x_\nu^{-1}$ can be assumed to be small for a suitable choice of κ_0, we can then conclude that

$$\|\phi'\|_2 \le C\left(x_\nu^{-1/2}|\phi(x_\nu)| + |\mu|^{-\frac{1}{p}} x_\nu^{-\frac{1}{2}+\frac{1}{p}}\|v\|_p\right). \tag{2.8.58}$$

By (2.8.2) and (2.8.55) it holds that

$$|\phi(x_\nu)|^2 \le Cx_\nu\left[\frac{\mu}{x_\nu^2}\|\phi'\|_2^2 + \left(|\mu|^{-\frac{1}{p}} x_\nu^{-1+\frac{1}{p}}|\phi(x_\nu)| + \frac{1}{x_\nu^{\frac{1}{2}+\frac{1}{p}}}\|\phi'\|_2\right)\|v\|_p\right],$$

and hence we obtain that for any $\delta > 0$ there exist $C_\delta > 0$ and $\kappa_0(\delta)$ such that, under the conditions of the proposition with $\kappa_0 = \kappa_0(\delta)$,

$$|\phi(x_\nu)| \le x_\nu^{1/2}\delta\|\phi'\|_2 + C_\delta\frac{x_\nu^{1/p}}{|\mu|^{1/p}}\|v\|_p. \tag{2.8.59}$$

Substituting (2.8.59) into (2.8.58) yields

$$\|\phi'\|_2 \le C|\mu|^{-\frac{1}{p}} x_\nu^{-\frac{1}{2}+\frac{1}{p}} \|v\|_p .$$

As $\phi(1) = 0$, we may use Poincaré's inequality to establish (2.8.1a).

Proof of (2.8.1b). Suppose now that $v \in L^\infty(0,1)$. Then,

$$\Big\|\frac{v}{U+i\lambda}\Big\|_1 \le \|v\|_\infty \Big\|\frac{1}{U+i\lambda}\Big\|_1 .$$

Then, we may use (2.8.9) to obtain

$$\Big\|\frac{1}{U+i\lambda}\Big\|_1 \le C\int_0^1 \frac{dx}{[(x^2-x_\nu^2)^2+\mu^2]^{1/2}} \le \frac{C}{|\mu|^{1/2}}\int_{\mathbb{R}_+}\frac{ds}{[(s^2-a^2)^2+1]^{1/2}}, \tag{2.8.60}$$

Then, we write, using the fact that for sufficiently large κ_0 we have $a \ge 2$,

$$\int_{\mathbb{R}_+}\frac{ds}{[(s^2-a^2)^2+1]^{1/2}} \le \int_0^{2a}\frac{ds}{[a^2(s-a)^2+1]^{1/2}} + \int_{2a}^\infty \frac{ds}{[(s-a)^4+1]^{1/2}} \le C\Big(\frac{\log a}{a}\Big). \tag{2.8.61}$$

Combining the above yields for $\nu_1 < \nu < U(0)$

$$\Big\|\frac{1}{U+i\lambda}\Big\|_1 \le C\frac{\log\frac{x_\nu}{|\mu|^{1/2}}}{x_\nu}. \tag{2.8.62}$$

Note that by (2.6.10) the above estimate holds for $\nu \le \nu_1$ as well (see [3]). From (2.8.62), we deduce immediately

$$x_\nu^{1/2}\Big\|\frac{v}{U+i\lambda}\Big\|_1 \le C\frac{\log\frac{x_\nu}{|\mu|^{1/2}}}{x_\nu^{1/2}}\|v\|_\infty. \tag{2.8.63}$$

Next, we estimate the second term on the right-hand side of (2.8.50) in the case $p=\infty$. Using (2.8.53) we obtain that

$$\Big\|\frac{(x-x_\nu)v}{U+i\lambda}\Big\|_2 \le \|v\|_\infty\Big\|\frac{x-x_\nu}{U+i\lambda}\Big\|_2 \le \frac{C}{x_\nu^{1/2}}\|v\|_\infty. \tag{2.8.64}$$

Combining (2.8.64) with (2.8.63), (2.8.17), (2.8.42), and (2.8.24) yields (2.8.1b). Note that by (2.8.2) and (2.8.50) we obtain that

$$|\phi(x_\nu)| \le C\log\Big(\frac{x_\nu}{|\mu|^{1/2}}\Big)\|v\|_\infty \tag{2.8.65}$$

This completes the proof of the proposition. ∎

2.9 The case $U(0) - \kappa_0|\Re\lambda| \leq \Im\lambda \leq U(0) + \kappa_0|\Re\lambda|$

In the following, we consider the case where ν is very close to $U(0)$. Here, we need to address the quadratic behavior of $U - U(0)$ near $x = 0$. This case deserves special attention whenever $|\nu - U(0)| \lesssim |\mu|$.

Proposition 2.9.1. *Let $p \in (2, +\infty]$, $\kappa_0 > 0$ and $U \in C^3([0, 1])$ satisfy (2.1.3). There exist $\mu_0 > 0$ and $C > 0$ such that, for any $\alpha \geq 0$ and any λ for which $0 < |\mu| \leq \mu_0$ and $U(0) - \kappa_0|\mu| \leq \nu \leq U(0) + \kappa_0|\mu|$, we have, for every pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times L^\infty(0, 1)$ satisfying (2.4.22)*

$$|\mu|^{\frac{1}{2p}+\frac{1}{4}}\|\phi\|_{1,2} \leq C\|v\|_p. \tag{2.9.1}$$

Proof. For $\nu \leq U(0)$ we choose $x_\nu \in [0, 1)$ so that $U(x_\nu) = \nu$. In the case $\nu > U(0)$ we set $x_\nu = 0$ and proceed in a similar manner. Obviously, the assumptions made on U and λ imply that there exists $C > 0$ such that

$$x_\nu < C|\mu|^{1/2} \quad \text{for all } 0 < |\mu| \leq 1. \tag{2.9.2}$$

Step 1. We prove that there exist $C > 0$ and $\mu_0 > 0$ such that, for all λ such that $0 < |\mu| \leq \mu_0$ and $U(0) - \kappa_0|\mu| \leq \nu \leq U(0) + \kappa_0|\mu|$ it holds that

$$|\phi(x_\nu)|^2 \leq C|\mu|^{1/2}\Big[\|\phi'\|_2^2 + \Big|\Big\langle\phi, \frac{v}{U + i\lambda}\Big\rangle\Big|\Big] \tag{2.9.3}$$

for all pairs $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times L^\infty(0, 1)$ satisfying (2.4.22).

We note that (2.8.3) can be rewritten in the form

$$\begin{aligned}\frac{|\mu|}{2}\,|\phi(x_\nu)|^2 &\int_0^1 \frac{|U''|}{(U-\nu)^2 + \mu^2}\,dx \\ &\leq \Big|\Im\Big\langle\phi, \frac{v}{U - \nu + i\mu}\Big\rangle\Big| + |\mu|\Big\langle\frac{|U''|}{(U-\nu)^2 + \mu^2}|\phi(x) - \phi(x_\nu)|^2\Big\rangle.\end{aligned} \tag{2.9.4}$$

Since

$$(U - \nu)^2 \leq C(x^2 + |\nu - U(0)|)^2 \leq C(x^2 + \kappa_0|\mu|)^2 \leq \hat{C}(x^4 + |\mu|^2),$$

we obtain that

$$\int_0^1 \frac{|U''|}{(U-\nu)^2 + \mu^2}\,dx \geq \frac{1}{\hat{C}}\int_0^1 \frac{dx}{x^4 + \mu^2}.$$

Using the substitution $x = |\mu|^{1/2}\xi$ yields

$$\int_0^1 \frac{dx}{x^4 + \mu^2} = |\mu|^{-3/2}\int_0^{|\mu|^{-1/2}} \frac{d\xi}{\xi^4 + 1} = |\mu|^{-3/2}\Big[\int_0^\infty \frac{d\xi}{\xi^4 + 1} - \int_{|\mu|^{-1/2}}^\infty \frac{d\xi}{\xi^4 + 1}\Big].$$

As

$$\int_{|\mu|^{-1/2}}^{\infty} \frac{d\xi}{\xi^4+1} \leq C|\mu|^{3/2},$$

we obtain the existence of $\mu_0 > 0$ and $\widehat{C}$ such that, under the conditions of this step

$$\int_0^1 \frac{|U''|}{(U-\nu)^2+\mu^2}\,dx \geq \frac{1}{\widehat{C}|\mu|^{3/2}}. \tag{2.9.5}$$

By (2.8.16) we have that

$$\Big\langle \frac{|U''|}{(U-\nu)^2+\mu^2}, |\phi(x)-\phi(x_\nu)|^2 \Big\rangle \leq \frac{C}{|\mu|}\Big\|\frac{\phi-\phi(x_\nu)}{|U-\nu|^{1/2}}\Big\|_2^2.$$

By (2.8.9) we have for all $\nu_1 < \nu < U(0)$ for some positive $\nu_1 > 0$ that (note for sufficiently small μ_0 we clearly have $\nu > U(0) - \kappa_0|\mu| > \nu_1$)

$$|U-\nu| \geq \frac{1}{C}(x^2-x_\nu^2) \geq \frac{1}{C}(x-x_\nu)^2, \tag{2.9.6}$$

which remains valid also for $\nu \geq U(0)$ given that

$$|U-\nu| \geq |U-U(0)| \geq \frac{1}{C}x^2 = \frac{1}{C}(x-x_\nu)^2. \tag{2.9.7}$$

Hence, by Hardy's inequality (2.6.13),

$$\Big\langle \frac{|U''|}{(U-\nu)^2+\mu^2}, |\phi(x)-\phi(x_\nu)| \Big\rangle \leq \frac{C}{|\mu|}\Big\|\frac{\phi(x)-\phi(x_\nu)}{x-x_\nu}\Big\|_2^2 \leq \frac{C}{|\mu|}\|\phi'\|_2^2.$$

Combining the above with (2.9.5) and (2.9.4) yields (2.9.3).

Step 2. We prove that for any $\kappa_1 > 0$ there exist positive C and μ_0 such that, for all $\nu > U(0) - \kappa_1|\mu|$ and $|\mu| \leq \mu_0$ it holds that

$$\|\phi'\|_2 \leq C\Big[\mu^{1/4}\Big\|\frac{v}{U+i\lambda}\Big\|_1 + \Big\|\frac{(x-x_\nu)v}{U+i\lambda}\Big\|_2\Big] \tag{2.9.8}$$

holds for any pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times L^\infty(0,1)$ satisfying (2.4.22).

We begin by restating (2.8.43):

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq -\Re\Big\langle U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)} + \Re\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle. \tag{2.9.9}$$

Then, we write

$$\Re\Big\langle \phi, \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)} = \Re\Big\langle \phi(x_\nu), \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)} + \Re\Big\langle \phi-\phi(x_\nu), \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}.$$

For the first term on the right-hand side we use (2.9.3) to obtain

$$\left|\left\langle \phi(x_\nu), \frac{U''\phi}{U+i\lambda}\right\rangle_{L^2(0,x_\nu)}\right| \leq C|\mu|^{1/4}x_\nu^{1/2}\left[\|\phi'\|_2 + \left|\left\langle \phi, \frac{v}{U+i\lambda}\right\rangle\right|^{1/2}\right]\left\|\frac{\phi}{U+i\lambda}\right\|_2.$$

Using (2.6.5) and the fact that $U'' < 0$, we obtain that

$$\left\|\frac{\phi}{U+i\lambda}\right\|_2^2 = \int_0^1 \frac{|\phi|^2}{(U-\nu)^2+\mu^2}dx \leq C\int_0^1 \frac{-U''|\phi|^2}{(U-\nu)^2+\mu^2}dx \leq \frac{C}{\mu}\left|\left\langle \phi, \frac{v}{U+i\lambda}\right\rangle\right|.$$

Hence, it holds that

$$\left\|\frac{\phi}{U+i\lambda}\right\|_2 \leq C|\mu|^{-1/2}\left|\left\langle \phi, \frac{v}{U+i\lambda}\right\rangle\right|^{1/2} \tag{2.9.10}$$

and we can conclude that

$$\left|\left\langle \phi(x_\nu), \frac{U''\phi}{U+i\lambda}\right\rangle_{L^2(0,x_\nu)}\right| \leq C\left[\|\phi'\|_2 + \left|\left\langle \phi, \frac{v}{U+i\lambda}\right\rangle\right|^{1/2}\right]\left|\left\langle \phi, \frac{v}{U+i\lambda}\right\rangle\right|^{1/2}. \tag{2.9.11}$$

For the second term on the right-hand side we use (2.9.10), (2.6.12), and (2.9.2) to obtain

$$\begin{aligned}\left|\left\langle \phi-\phi(x_\nu), \frac{U''\phi}{U+i\lambda}\right\rangle_{L^2(0,x_\nu)}\right| &= \left|\left\langle \frac{\phi-\phi(x_\nu)}{x-x_\nu}, \frac{(x-x_\nu)U''\phi}{U+i\lambda}\right\rangle_{L^2(0,x_\nu)}\right| \\ &\leq C\,x_\nu\|\phi'\|_2\left\|\frac{\phi}{U+i\lambda}\right\|_2 \\ &\leq \hat{C}\,\|\phi'\|_2\left|\left\langle \phi, \frac{v}{U+i\lambda}\right\rangle\right|^{1/2}.\end{aligned}$$

Hence,

$$\left|\left\langle \phi-\phi(x_\nu), \frac{U''\phi}{U+i\lambda}\right\rangle_{L^2(0,x_\nu)}\right| \leq \hat{C}\,\|\phi'\|_2\left|\left\langle \phi, \frac{v}{U+i\lambda}\right\rangle\right|^{1/2}. \tag{2.9.12}$$

Substituting (2.9.12) together with (2.9.11) into (2.9.9) yields

$$\|\phi'\|_2^2 \leq C\left|\left\langle \phi, \frac{v}{U+i\lambda}\right\rangle\right|. \tag{2.9.13}$$

Note that by combining (2.9.13) with (2.9.3) we can also conclude that

$$|\phi(x_\nu)|^2 \leq C|\mu|^{1/2}\left|\left\langle \phi, \frac{v}{U+i\lambda}\right\rangle\right|. \tag{2.9.14}$$

To prove (2.9.8) we now write, with the aid of Hardy's inequality

$$\begin{aligned}\left|\left\langle \phi, \frac{v}{U+i\lambda}\right\rangle\right| &\leq \left|\left\langle \phi(x_\nu), \frac{v}{U+i\lambda}\right\rangle + \left|\left\langle \phi-\phi(x_\nu), \frac{v}{U+i\lambda}\right\rangle\right|\right. \\ &\leq |\phi(x_\nu)|\left\|\frac{v}{U+i\lambda}\right\|_1 + \|\phi'\|_2\left\|\frac{(x-x_\nu)v}{U+i\lambda}\right\|_2.\end{aligned}$$

Combining the above with (2.9.14) gives

$$\left|\left\langle\phi,\frac{v}{U+i\lambda}\right\rangle\right|\leq C\Big(|\mu|^{\frac{1}{2}}\Big\|\frac{v}{U+i\lambda}\Big\|_1^2+\|\phi'\|_2\Big\|\frac{(x-x_\nu)v}{U+i\lambda}\Big\|_2\Big). \tag{2.9.15}$$

Then, (2.9.15) and (2.9.13) imply (2.9.8).

Step 3. We estimate $\|\frac{1}{U+i\lambda}\|_q$ for $q\geq 1$ and $\|\frac{x-x_\nu}{U+i\lambda}\|_q$ for $q\geq 2$ for $\nu\in(U(0)-\kappa_0|\mu|,U(0)+\kappa_0|\mu|)$.

We consider two separate cases by splitting $(U(0)-\kappa_0|\mu|,U(0)+\kappa_0|\mu|)$ into two subintervals.

The case $\nu\in(U(0)-\kappa_0|\mu|,U(0)-\delta|\mu|)$, $\delta\in(0,\kappa_0)$. In this case, observing that $x_\nu\geq C^{-1}|\delta\mu|^{\frac{1}{2}}$ we use (2.8.49) to establish that for any $\delta>0$ there exists C such that for all $\delta|\mu|<U(0)-\nu<\kappa_0|\mu|$ it holds for all $q>1$

$$\Big\|\frac{1}{U+i\lambda}\Big\|_q^q\leq\frac{C}{|\delta|^{\frac{q}{2}}\,|\mu|^{q-1/2}}. \tag{2.9.16}$$

In a similar manner we obtain from (2.8.60) and (2.8.61) (note that $a=x_\nu|\mu|^{-1/2}\geq C^{-1}\delta^{\frac{1}{2}}$ in the present regime of ν values) that there exists $C_\delta>0$ such that

$$\Big\|\frac{1}{U+i\lambda}\Big\|_1\leq\frac{C_\delta}{|\mu|^{1/2}}. \tag{2.9.17}$$

Finally, we use (2.8.53) and the fact that $x_\nu\geq C^{-1}|\delta\mu|^{\frac{1}{2}}$ to obtain for all $q\geq 2$

$$\Big\|\frac{x-x_\nu}{U+i\lambda}\Big\|_q^q\leq\frac{C_\delta}{|\mu|^{(q-1)/2}}. \tag{2.9.18}$$

The case $\nu\in(U(0)-\delta|\mu|,U(0)+\kappa_0|\mu|)$, $\delta\in(0,\kappa_0)$. In this case we use (2.8.9) to obtain that

$$(U-\nu)^2\geq\frac{1}{C}(x^2-x_\nu^2)^2\geq\frac{1}{C}\Big(\frac{x^4}{2}-x_\nu^4\Big)\geq\frac{1}{C_1}x^4-C_2\delta^2|\mu|^2.$$

The above inequality implies that there exist C and $\delta_0\leq\kappa_0$ such that, for $\delta\in(0,\delta_0)$ and $\nu\in(U(0)-\delta|\mu|,U(0)+\kappa_0|\mu|)$,

$$\frac{1}{(U(x)-\nu)^2+|\mu|^2}\leq\frac{C}{x^4+|\mu|^2} \tag{2.9.19}$$

for all $x\in[0,1]$.

Consequently, for all $q\geq 1$, using the substitution $x=|\mu|^{1/2}\xi$, we obtain

$$\Big\|\frac{1}{U+i\lambda}\Big\|_q^q\leq\int_0^1\frac{C}{[x^2+|\mu|]^q}\,dx\leq\frac{C}{|\mu|^{q-1/2}}\int_0^\infty\frac{d\xi}{[\xi^2+1]^q}\leq\frac{C}{|\mu|^{q-1/2}}. \tag{2.9.20}$$

Finally, we use (2.9.19) to obtain that

$$\Big\|\frac{x-x_\nu}{U+i\lambda}\Big\|_q^q \leq C\Big(|\mu|^{q/2}\Big\|\frac{1}{U+i\lambda}\Big\|_q^q + \Big\|\frac{x}{x^2+|\mu|}\Big\|_q^q\Big). \tag{2.9.21}$$

We now observe that for all $q>1$

$$\Big\|\frac{x}{x^2+|\mu|}\Big\|_q^q \leq \frac{1}{|\mu|^{(q-1)/2}}\int_0^\infty \frac{\xi^q}{[\xi^2+1]^q}\,d\xi \leq \frac{C}{|\mu|^{(q-1)/2}}. \tag{2.9.22}$$

Together with (2.9.22) and (2.9.20), (2.9.21) yields the existence of $C>0$

$$\Big\|\frac{x-x_\nu}{U+i\lambda}\Big\|_q^q \leq \frac{C}{|\mu|^{(q-1)/2}}. \tag{2.9.23}$$

The general case. Combining (2.9.16) and (2.9.17), (2.9.20) yields, for $q\geq 1$, the existence of $C>0$, such that for $|U(0)-\nu|<\kappa_0|\mu|$

$$\Big\|\frac{1}{U+i\lambda}\Big\|_q^q \leq \frac{C}{|\mu|^{q-1/2}}. \tag{2.9.24}$$

By (2.9.18) and (2.9.23) we may conclude, for all $q\geq 1$, that there exists $C>0$ such that, for $|U(0)-\nu|<\kappa_0|\mu|$

$$\Big\|\frac{x-x_\nu}{U+i\lambda}\Big\|_q^q \leq \frac{C}{|\mu|^{(q-1)/2}}. \tag{2.9.25a}$$

Note that

$$\begin{aligned}\Big\|\frac{x-x_\nu}{U+i\lambda}\Big\|_\infty &\leq C\Big\|\frac{x-x_\nu}{|x^2-x_\nu^2|+|\mu|}\Big\|_\infty \\ &\leq C\Big(\Big\|\mathbf{1}_{|x-x_\nu|<|\mu|^{1/2}}\frac{x-x_\nu}{|\mu|}\Big\|_\infty + \Big\|\mathbf{1}_{|x-x_\nu|\geq|\mu|^{1/2}}\frac{x-x_\nu}{|x^2-x_\nu^2|}\Big\|_\infty\Big) \leq \frac{\widehat{C}}{|\mu|^{1/2}},\end{aligned}$$

hence

$$\Big\|\frac{x-x_\nu}{U+i\lambda}\Big\|_\infty \leq \frac{\widehat{C}}{|\mu|^{1/2}}. \tag{2.9.25b}$$

Step 4. We prove (2.9.1). The proof is similar to Step 4 of the proof of Proposition 2.8.1. We estimate the right-hand side of (2.9.8) separately for $p\in[2,+\infty)$ and for $p=\infty$. Suppose first that $v\in L^p(0,1)$ for some $p\in[2,+\infty)$.

For the first term on the right-hand side, we deduce from (2.9.24),

$$|\mu|^{1/4}\Big\|\frac{v}{U+i\lambda}\Big\|_1 \leq C\frac{\|v\|_p}{|\mu|^{\frac{1}{2p}+\frac{1}{4}}}. \tag{2.9.26}$$

To estimate the second term we use (2.9.25) to obtain for all $p\geq 2$

$$\Big\|\frac{(x-x_\nu)v}{U+i\lambda}\Big\|_2 \leq \frac{C}{|\mu|^{\frac{1}{2p}+\frac{1}{4}}}\|v\|_p. \tag{2.9.27}$$

Suppose now that $v \in L^\infty(0,1)$. Then, by (2.9.24) we may conclude for the first term on the right-hand side of (2.9.8) that

$$|\mu|^{1/4}\Big\|\frac{v}{U+i\lambda}\Big\|_1 \leq \frac{C}{|\mu|^{1/4}}\|v\|_\infty. \tag{2.9.28}$$

Next, we estimate the second term on the right-hand side of (2.9.8). Using (2.9.25) we obtain that

$$\Big\|\frac{(x-x_\nu)v}{U+i\lambda}\Big\|_2 \leq \frac{C}{|\mu|^{1/4}}\|v\|_\infty.$$

Together with (2.9.28) the above yields (2.9.1) for $p=+\infty$. ∎

Remark 2.9.2. Note that, under the assumptions of Proposition 2.9.1, by (2.9.14), (2.9.15), (2.9.25a) and (2.9.24) it holds that

$$|\phi(x_\nu)|^2 \leq C\left(|\phi(x_\nu)|\,\|v\|_\infty + |\mu|^{1/4}\|\phi'\|_2\|v\|_\infty\right).$$

Using (2.9.1) for $p=+\infty$ then yields the existence of $C>0$ such that

$$|\phi(x_\nu)| \leq C\|v\|_\infty. \tag{2.9.29}$$

2.10 The case $\Im\lambda > U(0)$

In the case where $\Im\lambda = \nu > U(0)$, we get a better estimate of $\mathcal{A}_{\lambda,\alpha}^{-1}$, measured by a negative power of $|\mu| + (\nu - U(0))$. More precisely, we have the following proposition.

Proposition 2.10.1. *Let $p \in [2,+\infty]$. There exist $\mu_0 > 0$ and $C>0$ such that for all $U \in C^3([0,1])$ satisfying (2.1.3), $\nu > U(0)$, $|\mu| < \mu_0$, $\alpha \geq 0$, and $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times L^\infty(0,1)$ satisfying (2.4.22) it holds that*

$$\|\phi\|_{1,2} \leq \frac{C}{[|\mu| + |\nu - U(0)|]^{\frac{1}{2p}+\frac{1}{4}}}\|v\|_p, \tag{2.10.1}$$

Proof. We begin by restating (2.8.19)

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 + \Big\langle\frac{U''(U-\nu)\phi}{(U-\nu)^2+\mu^2},\phi\Big\rangle = \Re\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle. \tag{2.10.2}$$

Since $\nu > U(0)$, it holds by (2.1.3) that the third term on the left-hand side is positive, and hence we can conclude that

$$\|\phi'\|_2^2 \leq \Re\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle. \tag{2.10.3}$$

We split the proof into two separate parts in accordance with the magnitude of $(U(0)-\nu)^2+\mu^2$.

Step 1. The case $(U(0)-\nu)^2+\mu^2$ *small.* We consider here the case where

$$(U(0)-\nu)^2+\mu^2<\varepsilon$$

for some sufficiently small $\varepsilon>0$.

To properly bound the right-hand side of (2.10.3) in that case we need an estimate for $|\phi(0)|$. To this end we use (2.10.2) once again to obtain

$$\Big\langle\frac{U''(U-\nu)\phi}{(U-\nu)^2+\mu^2},\phi\Big\rangle\leq\Re\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle,$$

from which we can conclude that

$$\begin{aligned}&\Big\langle\frac{U''(U-\nu)}{(U-\nu)^2+\mu^2},1\Big\rangle|\phi(0)|^2\\&\qquad\leq 2\Big\langle\frac{U''(U-\nu)(\phi-\phi(0))}{(U-\nu)^2+\mu^2},\phi-\phi(0)\Big\rangle+2\Re\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle.\end{aligned}\tag{2.10.4}$$

We continue by bounding from below the left-hand side of (2.10.4). To this end we observe that since

$$\min_{x\in[0,1]}|U''(x)|\geq\frac{1}{C_0}>0\quad\text{and}\quad U(0)-U(x)\geq\frac{1}{2C_0}x^2,$$

we can conclude that

$$\begin{aligned}\Big\langle\frac{U''(U-\nu)}{(U-\nu)^2+\mu^2},1\Big\rangle&\geq\frac{1}{C}\int_0^1\frac{x^2+\nu-U(0)}{[x^2+\nu-U(0)]^2+\mu^2}\,dx\\&\geq\frac{1}{C}\int_0^1\frac{x^2}{[x^2+\nu-U(0)]^2+\mu^2}\,dx\\&\geq\frac{1}{2C}\int_0^1\frac{x^2}{x^4+(\nu-U(0)^2+\mu^2)}\,dx\\&\geq\frac{1}{2C}\Big[\int_0^\infty\frac{x^2}{x^4+[\nu-U(0)]^2+\mu^2}\,dx-1\Big].\end{aligned}$$

Setting

$$x=([\nu-U(0)]^2+\mu^2)^{1/4}s$$

yields

$$\begin{aligned}\Big\langle\frac{U''(U-\nu)}{(U-\nu)^2+\mu^2},1\Big\rangle&\geq\frac{1}{2C}\Big(\frac{1}{([\nu-U(0)]^2+\mu^2)^{1/4}}\int_0^\infty\frac{s^2}{s^4+1}\,ds-1\Big)\\&\geq\frac{1}{\widehat{C}\{[\nu-U(0)]^2+\mu^2\}^{1/4}}-\widehat{C}.\end{aligned}$$

For sufficiently small ε we obtain that

$$\Big\langle \frac{U''(U-\nu)}{(U-\nu)^2+\mu^2}, 1\Big\rangle \geq \frac{1}{2\widehat{C}\{[\nu-U(0)]^2+\mu^2\}^{1/4}}. \tag{2.10.5}$$

To estimate the first term on the right-hand side of (2.10.4) we first write

$$\Big\langle \frac{U''(U-\nu)(\phi-\phi(0))}{(U-\nu)^2+\mu^2}, \phi-\phi(0)\Big\rangle \leq \Big\|\frac{U''(U-\nu)x^2}{(U-\nu)^2+\mu^2}\Big\|_\infty \Big\|\frac{\phi-\phi(0)}{x}\Big\|_2^2.$$

As

$$\Big\|\frac{U''(U-\nu)x^2}{(U-\nu)^2+\mu^2}\Big\|_\infty \leq C\Big\|\frac{x^4+[\nu-U(0)]x^2}{[x^2+\nu-U(0)]^2+\mu^2}\Big\|_\infty \leq \widehat{C},$$

we may use Hardy's inequality to obtain that

$$\Big\langle \frac{U''(U-\nu)(\phi-\phi(0))}{(U-\nu)^2+\mu^2}, \phi-\phi(0)\Big\rangle \leq C\,\|\phi'\|_2^2. \tag{2.10.6}$$

Equation (2.10.6) together with (2.10.3) and (2.10.5) yields, when substituted into (2.10.4),

$$|\phi(0)|^2 \leq C(|\mu|^{1/2}+|\nu-U(0)|^{1/2})\Re\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle. \tag{2.10.7}$$

To complete the proof we now estimate the right-hand side of (2.10.3). To this end we write

$$\Re\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle \leq |\phi(0)|\Big\|\frac{v}{U+i\lambda}\Big\|_1 + \Big\|\frac{\phi-\phi(0)}{x}\Big\|_2\Big\|\frac{xv}{U+i\lambda}\Big\|_2.$$

Using Hardy's inequality together with (2.10.7) yields

$$\Re\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle \leq C\Big[(|\mu|^{1/2}+|\nu-U(0)|^{1/2})\Big\|\frac{v}{U+i\lambda}\Big\|_1^2 + \|\phi'\|_2\Big\|\frac{xv}{U+i\lambda}\Big\|_2\Big].$$

Using (2.10.3) once again yields

$$\|\phi'\|_2 \leq C\Big[(|\mu|^{1/4}+|\nu-U(0)|^{1/4})\Big\|\frac{v}{U+i\lambda}\Big\|_1 + \Big\|\frac{xv}{U+i\lambda}\Big\|_2\Big]. \tag{2.10.8}$$

The proof of (2.10.1) is now verified by following the same path as in the proof of Step 4 of Propositions 2.8.1 and 2.9.1. Thus, since

$$(U-\nu) \geq C(x^2+\nu-U(0)),$$

we obtain as in (2.9.20) that for all $q \geq 1$

$$\begin{aligned}\Big\|\frac{1}{U+i\lambda}\Big\|_q^q &\leq \int_0^1 \frac{C}{[x^2+\nu-U(0)+|\mu|]^q}\,dx \\ &\leq \frac{C}{[|\mu+\nu-U(0)]|^{q-1/2}}\int_0^\infty \frac{d\xi}{[\xi^2+1]^q} \\ &\leq \frac{\widehat{C}}{[|\mu|+\nu-U(0)]^{q-1/2}}.\end{aligned} \tag{2.10.9}$$

Hence, for all $p \in [2, +\infty]$,

$$\left\| \frac{v}{U + i\lambda} \right\|_1 \leq C[|\mu| + \nu - U(0)]^{-\frac{1}{2p}-\frac{1}{2}} \|v\|_p. \tag{2.10.10}$$

As in (2.9.21) and (2.9.22) we then write for all $q > 1$

$$\begin{aligned} \left\| \frac{x}{U + i\lambda} \right\|_q^q &\leq C \left\| \frac{x}{x^2 + \nu - U(0) + |\mu|} \right\|_q^q \\ &\leq \frac{\widehat{C}}{[|\mu| + \nu - U(0)]^{(q-1)/2}} \int_0^\infty \frac{\xi^q}{[\xi^2 + 1]^q} \, d\xi \\ &\leq \frac{\widetilde{C}}{[|\mu| + \nu - U(0)]^{(q-1)/2}}. \end{aligned}$$

Note also that for $q = +\infty$ we have

$$\left\| \frac{x}{U + i\lambda} \right\|_\infty \leq \frac{C}{[|\mu| + \nu - U(0)]^{1/2}}.$$

Consequently, for all $p \in [2, +\infty]$

$$\left\| \frac{xv}{U + i\lambda} \right\|_2 \leq \frac{C}{[|\mu| + \nu - U(0)]^{\frac{1}{2p}+\frac{1}{4}}} \|v\|_p. \tag{2.10.11}$$

Substituting the above, together with (2.10.10), into (2.10.8) yields (2.10.1) for sufficiently small $\varepsilon > 0$.

Step 2. The case where, for some $\varepsilon > 0$,

$$(U(0) - \nu)^2 + \mu^2 \geq \varepsilon.$$

By Poincaré's inequality and (2.10.10), we obtain

$$\begin{aligned} \left| \Re\left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| &\leq \|\phi\|_\infty \left\| \frac{v}{U + i\lambda} \right\|_1 \\ &\leq \frac{C}{[(U(0) - \nu)^2 + \mu^2]^{1/2}} \|\phi'\|_2 \|v\|_p \\ &\leq \frac{C}{\varepsilon^{\frac{2-p}{4}} [(U(0) - \nu)^2 + \mu^2]^{\frac{1}{2p}+\frac{1}{4}}} \|\phi'\|_2 \|v\|_p, \end{aligned}$$

which together with (2.10.3) readily gives (2.10.1) in this case. ■

Remark 2.10.2. As in Remark 2.9.2 we may use (2.10.7), (2.10), and (2.10.8), under the assumptions of Proposition 2.10.1, to obtain

$$|\phi(0)|^2 \leq C(|\mu| + |\nu - U(0)|) \left\| \frac{v}{U + i\lambda} \right\|_1^2 + (|\mu| + |\nu - U(0)|)^{1/2} \left\| \frac{xv}{U + i\lambda} \right\|_2^2.$$

From (2.10.10) we then obtain the existence of $C > 0$ such that

$$|\phi(0)| \leq C\|v\|_\infty. \tag{2.10.12}$$

2.11 The case $|\Re\lambda| \geq \mu_1 > 0$

The results in the preceding subsections were all obtained under the assumption that $|\mu| \leq \mu_0$ for some sufficiently small μ_0. Hence, it remains to treat the case when $|\mu| \geq \mu_1$ where $\mu_1 > 0$ is arbitrary. We complement Proposition 2.6.1 by addressing the large $|\mu|$ case.

Proposition 2.11.1. *For any $\mu_1 > 0$ and $p > 1$ there exists $C > 0$ such that for all $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ satisfying $\mathcal{A}_{\lambda,\alpha}\phi = v$, and for all $|\mu| \geq \mu_1$, $\nu \in \mathbb{R}$, and $\alpha \geq 0$,*

$$\|\phi\|_{H^1(0,1)} \leq C\|(1-x)^{1/2}v\|_1. \tag{2.11.1}$$

Proof. Since (2.6.17) holds true for any $\mu \neq 0$ we can conclude that

$$\|\phi'\|_2 \leq \Big(1 + \frac{C}{|\mu|^{1/2}}\Big)\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2}. \tag{2.11.2}$$

Hence, there exists C such that for $|\mu| \geq \mu_1$

$$\|\phi'\|_2^2 \leq C\,\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|. \tag{2.11.3}$$

Consequently, as $|\phi(x)| \leq \|\phi'\|_2(1-x)^{1/2}$ and $|\mu| > \mu_1$,

$$\|\phi'\|_2^2 \leq C\|\phi'\|_2\|(1-x)^{1/2}v\|_1\Big\|\frac{1}{U+i\lambda}\Big\|_\infty \leq \frac{C}{\mu_1}\|\phi'\|_2\|(1-x)^{1/2}v\|_1,$$

from which (2.11.1) follows with the aid of Poincaré's inequality. ∎

Chapter 3

Neumann–Dirichlet Schrödinger operators

The first part of this chapter is devoted to resolvent estimates for the operator

$$\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} = -\frac{d^2}{dx^2} + i\beta U, \tag{3.0.1a}$$

which is defined on

$$D(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}) = \{u \in H^2(0,1) \mid u(1) = u'(0) = 0\}. \tag{3.0.1b}$$

We note that to estimate the inverse norm of the Orr–Sommerfeld operator (1.1.7b) we need a bound of both $\mathcal{A}_{\lambda,\alpha}^{-1}$ and $(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \lambda)^{-1}$, and in addition, to obtain resolvent estimates for the special Schrödinger operators of the next chapter.

For convenience of notation we omit in the sequel the reference to the Dirichlet condition at $x = 1$ and use $\mathcal{L}_\beta^{\mathfrak{N}}$ instead of $\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}$.

Let $\lambda = \mu + i\nu$. Recall the definition of x_ν from (2.4.5). From [2], for instance, we know that the main contribution to the resolvent norm comes from a small region near $x = x_\nu$. We begin this section by estimating the resolvent norm of $\mathcal{L}_\beta^{\mathfrak{N}}$ in the case where $U(0) - \nu \gg \beta^{-1/2}$. In this case one may approximate $U - \nu$ by a linear potential of the form $U'(x_\nu)(x - x_\nu)$. In Section 3.2, we consider the case $|U(0) - \nu| \lesssim \beta^{-1/2}$ where $U - \nu$ will be approximated by the quadratic potential $U''(0)x^2/2 + U(0) - \nu$. Finally, Section 3.3 is devoted to some L^1 estimates that are necessary in Chapter 4.

3.1 Resolvent estimates for $U(0) - \Im\lambda \gg \beta^{-1/2}$

With x_ν defined in (2.4.5), we introduce

$$\mathfrak{J}_\nu = |U'(x_\nu)|. \tag{3.1.1}$$

We further define $\hat{\kappa}_1 \in \mathbb{C}$ to be the leftmost eigenvalue of

$$\mathcal{L}_+ = -\frac{d^2}{dx^2} + ix \tag{3.1.2}$$

in $H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+)$. The first proposition is similar to [3, Proposition 5.2]. Unlike [3] which defines the problem on $(-1, +1)$ with Dirichlet conditions at $x = \pm 1$, we consider the operator on $(0, 1)$ with a Neumann condition at $x = 0$, which corresponds to a restriction to the space of even functions on $(-1, 1)$, and a Dirichlet condition at $x = 1$. Furthermore, the velocity field is not strictly monotone as in [3]. Nevertheless, for $x_\nu \gg \beta^{-1/4}$ (or equivalently for $U(0) - \nu \gg \beta^{-1/2}$) we can still make good use of the estimates in [3].

Proposition 3.1.1. *Let $U \in C^2([0,1])$ satisfy (2.1.3) and $p \in (1,2]$. Then there exist positive Υ, a, C, C_p, and β_0 such that, for all $\beta \geq \beta_0$, $U(0) - \nu > a\beta^{-1/2}$, and $f \in L^\infty(0,1)$,*

$$\sup_{\mu \leq \Upsilon \mathfrak{J}_\nu{}^{2/3}\beta^{-1/3}} \|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1} f\|_2 \leq C \min([\mathfrak{J}_\nu\beta]^{-2/3}\|f\|_2, [\mathfrak{J}_\nu\beta]^{-5/6}\|f\|_\infty). \tag{3.1.3a}$$

Furthermore, for $f \in L^2(0,1)$,

$$\sup_{\mu \leq \Upsilon \mathfrak{J}_\nu{}^{2/3}\beta^{-1/3}} \left\| \frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1} f \right\|_p \leq \frac{C_p}{[\mathfrak{J}_\nu\beta]^{\frac{2+p}{6p}}}\|f\|_2. \tag{3.1.3b}$$

Proof. Step 1. For $\Upsilon > 0$, we prove that there exist positive β_0, a_0, and C such that for all $\beta \geq \beta_0$, $U(0) - \nu \geq a\beta^{-\frac{1}{2}}$, $a \geq a_0$ and $\mu \leq \Upsilon \mathfrak{J}_\nu{}^{2/3}\beta^{-1/3}$ we have

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1} f\|_2 + [\mathfrak{J}_\nu\beta]^{-1/3}\|\frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1} f\|_2 \leq C\,[\mathfrak{J}_\nu\beta]^{-2/3}\|f\|_2. \tag{3.1.4}$$

As $U(0) - \nu > a\beta^{-1/2}$, it holds that

$$x_\nu \geq \frac{1}{C}\, a^{1/2}\beta^{-\frac{1}{4}}. \tag{3.1.5}$$

For future reference we note, in addition, that

$$\frac{1}{C}x_\nu \leq \mathfrak{J}_\nu \leq C x_\nu. \tag{3.1.6}$$

We split the proof of (3.1.4) into two parts according to the sign of $\nu - U(\frac{1}{2})$.

Step 1.1. The case when $U(1/2) < \nu < U(0) - a\beta^{-1/2}$. We note that in this case $x_\nu \in (0, \frac{1}{2})$. Let $\hat{\chi}$ be given as in (2.4.16) by

$$\hat{\chi}(x) = \begin{cases} 0 & |x| < \frac{1}{4}, \\ 1 & |x| > \frac{1}{2}. \end{cases} \tag{3.1.7}$$

Set further for $x \in [0,1]$

$$\chi_\nu^\pm(x) = \hat{\chi}(x/x_\nu - 1)\mathbf{1}_{\mathbb{R}_+}(\pm(x - x_\nu)). \tag{3.1.8}$$

and $\hat{\chi}$ can be chosen such that

$$\tilde{\chi}_\nu := \sqrt{1 - (\chi_\nu^+)^2 - (\chi_\nu^-)^2} \in C^\infty([0,1]). \tag{3.1.9}$$

Note that χ_ν^- is supported on $(0, 3x_\nu/4)$ whereas χ_ν^+ is supported on $(5x_\nu/4, 1)$. The complementary cutoff function $\tilde{\chi}_\nu$ is supported on $(x_\nu/2, 3x_\nu/2)$. The cutoff functions defined in (3.1.8) allow us to obtain estimates for v separately on the intervals $(0, x_\nu/2)$, $(3x_\nu/2, 1)$ (via integration by parts), and $(x_\nu/2, 3x_\nu/2)$.

More precisely, let $(v, f) \in D(\mathcal{L}_\beta^{\mathfrak{N}}) \times L^2(0,1)$ satisfy $(\mathcal{L}_\beta^{\mathfrak{N}} - \lambda)v = f$. Set

$$\widehat{U}_\nu(x) = \begin{cases} U(x) & x \in (x_\nu/2, 3x_\nu/2), \\ U(x_\nu/2) + U'(x_\nu/2)(x - x_\nu) & x \leq \frac{x_\nu}{2}, \\ U(3x_\nu/2) + U'(3x_\nu/2)(x - x_\nu) & x \geq \frac{3x_\nu}{2}. \end{cases} \tag{3.1.10}$$

Let then

$$\hat{\mathcal{L}}_{\beta,\nu,\mathbb{R}} = -\frac{d^2}{dx^2} + i\beta\widehat{U}_\nu, \tag{3.1.11a}$$

be defined on the domain

$$D(\hat{\mathcal{L}}_{\beta,\nu,\mathbb{R}}) = \{u \in H^2(\mathbb{R}) \mid xu \in L^2(\mathbb{R})\}. \tag{3.1.11b}$$

We now apply the unitary dilation operator (corresponding to the change of variable $y = x/x_\nu$)

$$\mathcal{T}_\nu u(x) = x_\nu^{-1/2} u(x/x_\nu), \tag{3.1.12}$$

to obtain

$$\mathcal{T}_\nu^{-1} \hat{\mathcal{L}}_{\beta,\nu,\mathbb{R}} \mathcal{T}_\nu = x_\nu^{-2}\Big(-\frac{d^2}{dy^2} + i\tilde{\beta}_\nu \widetilde{U}_\nu(y)\Big), \tag{3.1.13}$$

where

$$\widetilde{U}_\nu(y) = \frac{\widehat{U}_\nu(x_\nu y)}{x_\nu^2} \quad \text{and} \quad \tilde{\beta}_\nu = \beta x_\nu^4. \tag{3.1.14}$$

As there exist positive m, M such that for ν satisfying the assumptions of our proposition

$$0 < m \leq |\widetilde{U}_\nu'(y)| \leq M \quad \forall y \in \mathbb{R}, \tag{3.1.15}$$

and in view of the uniform bound

$$\|\widetilde{U}_\nu''\|_{L^\infty(\mathbb{R})} \leq C, \tag{3.1.16}$$

we may apply [3, Proposition 5.1] to the family of operators

$$\tilde{\mathcal{L}}_{\tilde{\beta},\mathbb{R}} := -\frac{d^2}{dy^2} + i\tilde{\beta}\,\widetilde{U}_\nu(y) \tag{3.1.17}$$

to obtain for $\widehat{\beta} \geq \widehat{\beta}_0$

$$\sup_{\Re\tilde{\lambda} \leq \Upsilon\tilde{\beta}^{-1/3}} \|(\tilde{\mathcal{L}}_{\tilde{\beta},\mathbb{R}} - \tilde{\beta}\tilde{\lambda})^{-1}\| + \tilde{\beta}^{-1/3}\Big\|\frac{d}{dy}(\tilde{\mathcal{L}}_{\tilde{\beta},\mathbb{R}} - \tilde{\beta}\tilde{\lambda})^{-1}\Big\| \leq \frac{C}{\tilde{\beta}^{2/3}},$$

where

$$\widetilde{\beta}=\widetilde{\beta}_\nu=\beta x_\nu^4 \quad \text{and} \quad \tilde{\lambda}=x_\nu^{-2}\lambda. \tag{3.1.18}$$

We observe that, by (3.1.5), for any given $\widehat{\beta}_0>0$ there exists $a_0>0$ such that $\widetilde{\beta}>\widehat{\beta}_0$ is satisfied for $a\geq a_0$. Taking the inverse dilation transformation, we obtain

$$\sup_{\Re\lambda\leq\Upsilon x_\nu^{2/3}\beta^{-1/3}}\|(\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}}-\beta\lambda)^{-1}\|+(\beta x_\nu)^{-1/3}\Big\|\frac{d}{dx}(\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}}-\beta\lambda)^{-1}\Big\|\leq\frac{C}{(x_\nu\beta)^{2/3}}.$$

Given that [3, Proposition 5.1] allows for an arbitrary Υ, we can replace x_ν by $\mathfrak{J}_\nu$ to obtain

$$\sup_{\Re\lambda\leq\Upsilon \mathfrak{J}_\nu^{2/3}\beta^{-1/3}}\|(\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}}-\beta\lambda)^{-1}\|+(\beta \mathfrak{J}_\nu)^{-1/3}\Big\|\frac{d}{dx}(\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}}-\beta\lambda)^{-1}\Big\|\leq\frac{C}{(\mathfrak{J}_\nu\beta)^{2/3}}. \tag{3.1.19}$$

We now write

$$(\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}}-\beta\lambda)(\tilde{\chi}_\nu v)=\tilde{\chi}_\nu f-2\tilde{\chi}_\nu' v'-\tilde{\chi}_\nu'' v. \tag{3.1.20}$$

We can then conclude from (3.1.19) and (3.1.20) that

$$\|\tilde{\chi}_\nu v\|_2+[\beta\mathfrak{J}_\nu]^{-1/3}\|(\tilde{\chi}_\nu v)'\|_2\leq\frac{C}{[\beta\mathfrak{J}_\nu]^{2/3}}(\|\tilde{\chi}_\nu f\|_2+2\|\tilde{\chi}_\nu' v'\|_2+\|\tilde{\chi}_\nu'' v\|_2). \tag{3.1.21}$$

Note that (3.1.21) implies, by (3.1.6), that

$$\|\tilde{\chi}_\nu v\|_2+[\beta x_\nu]^{-1/3}\|(\tilde{\chi}_\nu v)'\|_2\leq\frac{C}{[\beta x_\nu]^{2/3}}(\|f\|_2+x_\nu^{-1}\|v'\|_2+x_\nu^{-2}\|v\|_2). \tag{3.1.22}$$

To estimate v on $(0,3x_\nu/4)$ and $(5x_\nu/4,1)$ we write

$$(\mathcal{L}_\beta^{\mathfrak{N}}-\beta\lambda)(\chi_\nu^{\pm}v)=\chi_\nu^{\pm}f-2(\chi_\nu^{\pm})'v'-(\chi_\nu^{\pm})''v. \tag{3.1.23}$$

The real part of the inner product with $\chi_\nu^{\pm}v$, after integration by parts, is given by

$$\|(\chi_\nu^{\pm}v)'\|_2^2=\|(\chi_\nu^{\pm})'v\|_2^2+\mu\beta\|\chi_\nu^{\pm}v\|_2^2+\Re\langle\chi_\nu^{\pm}v,\chi_\nu^{\pm}f\rangle, \tag{3.1.24}$$

whereas the imaginary part assumes the form

$$\mp\beta\||U-\nu|^{1/2}\chi_\nu^{\pm}v\|_2^2+2\Im\langle(\chi_\nu^{\pm})'v,(\chi_\nu^{\pm}v)'\rangle=+\Im\langle\chi_\nu^{\pm}v,\chi_\nu^{\pm}f\rangle. \tag{3.1.25}$$

As, by (2.8.8), $|U-\nu|^{1/2}\chi_\nu^{\pm}\geq\frac{1}{C}x_\nu\chi_\nu^{\pm}$, (3.1.25) yields first

$$\frac{1}{C}\beta x_\nu^2\|\chi_\nu^{\pm}v\|_2^2\leq\|\chi_\nu^{\pm}v\|_2\,\|\chi_\nu^{\pm}f\|_2+\frac{C}{x_\nu}\|v\|_2\,\|(\chi_\nu^{\pm}v)'\|_2.$$

Combining the above with (3.1.24) we obtain

$$\frac{1}{C}\beta x_\nu^2 \|\chi_\nu^\pm v\|_2^2 \leq \|\chi_\nu^\pm v\|_2 \, \|\chi_\nu^\pm f\|_2 + \frac{C}{x_\nu}\|v\|_2 \left(\frac{1}{x_\nu}\|v\|_2 + \mu_{\beta,+}^{\frac{1}{2}}\|\chi_\nu^\pm v\|_2 + \sqrt{\|\chi_\nu^\pm v\|_2 \, \|\chi_\nu^\pm f\|_2}\right),$$

where

$$\mu_{\beta,+} = \max(\mu\beta, 0). \tag{3.1.26}$$

For $\mu \leq \Upsilon \mathfrak{J}_\nu{}^{2/3}\beta^{-1/3}$, we may conclude, using (3.1.6), that

$$\frac{1}{C}\beta x_\nu^2 \|\chi_\nu^\pm v\|_2^2 \leq \|\chi_\nu^\pm v\|_2 \, \|\chi_\nu^\pm f\|_2 + \frac{C}{x_\nu}\|v\|_2\Big(\frac{1}{x_\nu}\|v\|_2 + (x_\nu\beta)^{\frac{1}{3}}\|\chi_\nu^\pm v\|_2 + \sqrt{\|\chi_\nu^\pm v\|_2 \, \|\chi_\nu^\pm f\|_2}\Big),$$

which implies

$$\|\chi_\nu^\pm v\|_2^2 \leq \frac{C}{(\beta x_\nu^2)^2}\|\chi_\nu^\pm f\|_2^2 + C\frac{1}{\beta x_\nu^2}\left(\frac{1}{x_\nu^2} + \frac{1}{\beta^{\frac{1}{3}} x_\nu^{\frac{10}{3}}}\right)\|v\|_2^2.$$

By (3.1.5), there exists $C > 0$ such that

$$\frac{1}{\beta^{\frac{1}{3}} x_\nu^{\frac{10}{3}}} \leq C a^{-2/3} x_\nu^{-2}.$$

Hence, we may conclude that there exists $C > 0$ such that, if $a \geq 1$ and $U(0) - \nu > a\beta^{-1/2}$,

$$\|\chi_\nu^\pm v\|_2 \leq \frac{C}{\beta x_\nu^2}[\|\chi_\nu^\pm f\|_2 + \beta^{1/2}\|v\|_2]. \tag{3.1.27}$$

Combining (3.1.27) and (3.1.22) leads to, with the aid of (3.1.5),

$$\begin{aligned}\|v\|_2 &\leq C\big(([\beta x_\nu^2]^{-1} + [\beta x_\nu]^{-2/3})\|f\|_2 \\ &\qquad + ([\beta x_\nu^4]^{-1/3} + [\beta x_\nu^4]^{-2/3})\|v\|_2 + [\beta^2 x_\nu^5]^{-1/3}\|v'\|_2\big) \\ &\leq \check{C}\big([\beta x_\nu]^{-2/3}\|f\|_2 + [\beta^2 x_\nu^5]^{-1/3}\|v'\|_2\big) + \hat{C}a^{-\frac{2}{3}}\|v\|_2.\end{aligned}$$

Thus, there exists $a_0 \geq 1$ and $C > 0$ such that for $a \geq a_0$ we obtain

$$\|v\|_2 \leq C\big([\beta x_\nu]^{-2/3}\|f\|_2 + [\beta^2 x_\nu^5]^{-1/3}\|v'\|_2\big). \tag{3.1.28}$$

We now use (3.1.22) together with (3.1.5) to establish that

$$\|(\tilde{\chi}_\nu v)'\|_2 \leq C([\beta\, x_\nu]^{-1/3}\|f\|_2 + [\beta x_\nu^7]^{-1/3}\|v\|_2) + \hat{C}a^{-\frac{2}{3}}\,\|v'\|_2. \tag{3.1.29}$$

By (3.1.24), as $\mu\beta \le C(x_\nu\beta)^{\frac{2}{3}}$, it holds that

$$\|(\chi_\nu^\pm v)'\|_2 \le C\Big(\frac{1}{x_\nu}\|v\|_2 + x_\nu^{\frac{1}{3}}\beta^{\frac{1}{3}}\|\chi_\nu^\pm v\|_2 + |\Re\langle\chi_\nu^\pm v, \chi_\nu^\pm f\rangle|^{\frac{1}{2}}\Big),$$

which leads to

$$\|(\chi_\nu^\pm v)'\|_2 \le \widehat{C}\Big(\frac{1}{x_\nu}\|v\|_2 + (\beta x_\nu)^{\frac{1}{3}}\|\chi_\nu^\pm v\|_2 + (\beta x_\nu)^{-\frac{1}{3}}\|f\|_2\Big). \tag{3.1.30}$$

Then we use (3.1.27), to get first that

$$\|(\chi_\nu^\pm v)'\|_2 \le C\Big(\frac{1}{x_\nu} + \frac{\beta^{\frac{1}{2}}(\beta x_\nu)^{\frac{1}{3}}}{\beta x_\nu^2}\Big)\|v\|_2 + C\Big((\beta x_\nu)^{-\frac{1}{3}} + \frac{(\beta x_\nu)^{\frac{1}{3}}}{\beta x_\nu^2}\Big)\|f\|_2,$$

and then conclude from (3.1.5) that

$$\|(\chi_\nu^\pm v)'\|_2 \le C(x_\nu^{-1}\|v\|_2 + (\beta x_\nu)^{-\frac{1}{3}}\|f\|_2). \tag{3.1.31}$$

Combining (3.1.29) and (3.1.31) yields

$$\|v'\|_2 \le C\big((x_\nu^{-1} + [\beta x_\nu^7]^{-1/3})\|v\|_2 + [\beta x_\nu]^{-1/3}\|f\|_2\big) + \widehat{C}a^{-\frac{2}{3}}\|v'\|_2.$$

Using again (3.1.5) we obtain the existence of a_0 that for $a \ge a_0$

$$\|v'\|_2 \le C(x_\nu^{-1}\|v\|_2 + [\beta x_\nu]^{-1/3}\|f\|_2).$$

Substituting (3.1.28) into the above yields the existence of $a_0 > 0$ that for all $a \ge a_0$

$$\|v'\|_2 \le C[\beta x_\nu]^{-1/3}\|f\|_2. \tag{3.1.32}$$

By (3.1.28) and (3.1.32) we then obtain

$$\|v\|_2 \le C\big([\beta x_\nu]^{-2/3} + [\beta^2 x_\nu^5]^{-1/3}[\beta x_\nu]^{-1/3}\big)\|f\|_2,$$

which implies, using (3.1.5),

$$\|v\|_2 \le C\,[\beta x_\nu]^{-2/3}\|f\|_2. \tag{3.1.33}$$

Having in mind (3.1.6) we finally obtain from (3.1.32) and (3.1.33)

$$\|v\|_2 + [\mathfrak{J}_\nu\beta]^{-1/3}\|v'\|_2 \le C[\mathfrak{J}_\nu\beta]^{-2/3}\|f\|_2, \tag{3.1.34}$$

which is precisely (3.1.4).

Step 1.2. The case $\nu \le U(1/2)$. We recall that $x_\nu = 1$ for $\nu < 0$ and observe that $x_\nu \ge \frac{1}{2}$ in this step. Hence, we need to define only a pair of cutoff functions. We thus set

$$\chi_2(x) = \widehat{\chi}(2x), \tag{3.1.35}$$

and $\tilde{\chi}_2 = \sqrt{1-\chi_2^2}$, which is supported on $[0, 1/4]$.

Then, we may write as in (3.1.27)

$$\|\tilde{\chi}_2 v\|_2 \leq C\beta^{-1}[\|f\|_2 + \beta^{1/2}\|v\|_2]. \tag{3.1.36}$$

Similarly, we obtain as in (3.1.31)

$$\|(\tilde{\chi}_2 v)'\|_2 \leq \frac{C}{\beta^{1/2}}[\|f\|_2 + \beta\|v\|_2].$$

Suppose that

$$\Upsilon < [U'(1/4)/|U'(1)|]\Re\hat{\kappa}_1. \tag{3.1.37}$$

For later reference we note that (3.1.37) implies that $\Upsilon < [U'(1/4)/\mathfrak{J}_\nu]^{2/3}\Re\hat{\kappa}_1$. As in (3.1.21) we can also conclude, from [3, Proposition 5.2], that

$$\|\chi_2 v\|_2 + \beta^{-1/3}\|(\chi_2 v)'\|_2 \leq \frac{C}{\beta^{2/3}}(\|f\|_2 + \|v'\|_2 + \|v\|_2).$$

Combining the above we may proceed as in the Step 1.1 to conclude (3.1.4) and the L^2-bound on the right-hand side in (3.1.3a).

To complete the proof of (3.1.3a) we need to establish an $\mathcal{L}(L^2, L^\infty)$ bound for $(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}$.

Step 2. For

$$\Upsilon < [|U'(1/16)|/|U'(1)|]^{2/3}\Re\hat{\kappa}_1, \tag{3.1.38}$$

we prove that there exist positive β_0, a_0, and C such that for all $\beta \geq \beta_0$, $U(0) - \nu \geq a\beta^{-\frac{1}{2}}$, $a \geq a_0$ and $\mu \leq \Upsilon\mathfrak{J}_\nu{}^{2/3}\beta^{-1/3}$ we have

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}f\|_2 + [\mathfrak{J}_\nu\beta]^{-1/3}\|\frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}f\|_2 \leq C[\mathfrak{J}_\nu\beta]^{-5/6}\|f\|_\infty. \tag{3.1.39}$$

Step 2.1. We consider the case $0 < x_\nu \leq 1/2$. Considering a pair $(v, f) \in D(\mathcal{L}_\beta^{\mathfrak{N}}) \times L^\infty(0,1)$ satisfying $(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)v = f$, we then write as in (3.1.20)

$$(\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}} - \beta\lambda)(\tilde{\chi}_\nu v) = \tilde{\chi}_\nu f - 2\tilde{\chi}_\nu' v' - \tilde{\chi}_\nu'' v. \tag{3.1.40}$$

Let $w_1 \in D(\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}})$ satisfy

$$(\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}} - \beta\lambda)w_1 = \tilde{\chi}_\nu f. \tag{3.1.41}$$

Let $\widetilde{\mathcal{L}}_{\tilde{\beta},\mathbb{R}}$ be defined by (3.1.17). We now apply [3, Lemma 5.5] to the operator $\widetilde{\mathcal{L}}_{\tilde{\beta},\mathbb{R}} - i\tilde{\beta}\widetilde{U}_\nu(1)$. Note that, due to (3.1.15) and (3.1.16), there exists $r > 1$ such that the potential $y \mapsto \widetilde{U}_\nu(y) - \widetilde{U}_\nu(1)$ belongs to $\mathcal{S}_r^2$ (see [3, equation (2.32)] for the definition of this class). Note further that [3, Lemma 5.5] holds under the assumption $\Upsilon > 0$. Hence, for any $\tilde{a} > 0$ there exists of $\widetilde{C}$ such that

$$\sup_{\Re\tilde{\lambda}\leq\Upsilon\tilde{\beta}^{-1/3}} \|(\widetilde{\mathcal{L}}_{\tilde{\beta},\mathbb{R}} - \tilde{\beta}\tilde{\lambda})^{-1}\tilde{g}\|_{L^2(-\tilde{a},\tilde{a})} \leq \frac{\widetilde{C}}{\tilde{\beta}^{5/6}}\|\tilde{g}\|_\infty. \tag{3.1.42}$$

We apply (3.1.42) with $\tilde{a} = 2$, $\tilde{g}(y) = x_\nu^{1/2}(\chi_\nu f)(x_\nu y)$, $\tilde{\lambda} = x_\nu^{-2}\lambda$ and $\tilde{\beta} = \beta x_\nu^4$ to establish after a change of variable that

$$\|w_1\|_{L^2(0,2x_\nu)} \leq \frac{C}{[\beta\mathfrak{J}_\nu]^{5/6}}\|f\|_\infty. \tag{3.1.43}$$

Let further $w_2 \in D(\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}})$ satisfy

$$(\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}} - \beta\lambda)w_2 = -2\tilde{\chi}_\nu' v' - \tilde{\chi}_\nu'' v.$$

From (3.1.13) we get that

$$\|w_2\|_2 \leq \frac{C}{[\beta\mathfrak{J}_\nu]^{2/3}}(\|\tilde{\chi}_\nu' v'\|_2 + \|\tilde{\chi}_\nu'' v\|_2). \tag{3.1.44}$$

Combining (3.1.44) with (3.1.43) yields as $\tilde{\chi}_\nu v = w_1 + w_2$ and Supp $\tilde{\chi}_\nu \subset (0, 2x_\nu)$

$$\begin{aligned}\|\tilde{\chi}_\nu v\|_2 &= \|\tilde{\chi}_\nu v\|_{L^2(0,2x_\nu)} \\ &\leq C\Big([\beta x_\nu]^{-5/6}\|f\|_\infty + [\beta^2 x_\nu^5]^{-1/3}[\|\mathbf{1}_{(\frac{x_\nu}{2},\frac{3x_\nu}{4})}v'\|_2 \\ &\qquad + \|\mathbf{1}_{(\frac{5x_\nu}{4},\frac{3x_\nu}{2})}v'\|_2 + x_\nu^{-1}\|v\|_2]\Big).\end{aligned} \tag{3.1.45}$$

Given the support of χ_ν^- it holds that

$$\|\chi_\nu^- f\|_2 \leq C x_\nu^{1/2}\|f\|_\infty,$$

and hence we can conclude from (3.1.27) that

$$\|\chi_\nu^- v\|_2 \leq \frac{C}{\beta x_\nu^2}\left[x_\nu^{1/2}\|f\|_\infty + \beta^{1/2}\|v\|_2\right]. \tag{3.1.46}$$

To obtain a similar bound for $\chi_\nu^+ v$ we obtain with the aid (2.8.9)

$$\|\chi_\nu^+|U-\nu|^{-1/2}\|_2^2 \leq C\int_{\frac{3x_\nu}{2}}^1 \frac{dx}{x^2 - x_\nu^2} \leq \frac{C}{x_\nu}.$$

Consequently, we can conclude that

$$\|(\chi_\nu^+)^2 v\|_1 \leq \|\chi_\nu^+|U-\nu|^{-1/2}\|_2\|\chi_\nu^+|U-\nu|^{1/2}v\|_2 \leq \frac{C}{x_\nu^{1/2}}\|\chi_\nu^+|U-\nu|^{1/2}v\|_2. \tag{3.1.47}$$

We now use (3.1.25) to obtain that

$$\|\chi_\nu^+ |U-\nu|^{1/2} v\|_2^2 \leq C\beta^{-1}(x_\nu^{-1}\|v\|_2\|(\chi_\nu^+ v)'\|_2 + \|(\chi_\nu^+)^2 v\|_1 \|f\|_\infty),$$

which implies that for any $\eta > 0$, we have

$$\|\chi_\nu^+ |U-\nu|^{1/2} v\|_2^2 \leq C\beta^{-1}\Big(\eta x_\nu^{-2}\|v\|_2^2 + \frac{1}{\eta}\|(\chi_\nu^+ v)'\|_2^2 + \|(\chi_\nu^+)^2 v\|_1 \|f\|_\infty\Big). \tag{3.1.48}$$

By (3.1.24) and (3.1.5) we have that

$$\|(\chi_\nu^+ v)'\|_2^2 \leq C[\beta x_\nu]^{2/3}\|v\|_2^2 + \|(\chi_\nu^+)^2 v\|_1 \|f\|_\infty. \tag{3.1.49}$$

Note that by (3.1.5) we can conclude that $x_\nu^{-2} \leq [\beta x_\nu]^{2/3}$. Substituting (3.1.49) into (3.1.48) yields for any $\eta > 0$

$$\begin{aligned}&\|\chi_\nu^+ |U-\nu|^{1/2} v\|_2^2 \\ &\quad\leq C\beta^{-1}\Big((\eta x_\nu^{-2} + \frac{1}{\eta}(\beta x_\nu)^{2/3})\|v\|_2^2 + \Big(\frac{1}{\eta}+1\Big)\|(\chi_\nu^+)^2 v\|_1 \|f\|_\infty\Big).\end{aligned}$$

Setting $\eta_\nu = x_\nu(\beta x_\nu)^{\frac{1}{3}}$ we observe that $\eta_\nu \geq \frac{1}{C}$ by (3.1.5), and hence

$$\|\chi_\nu^+ |U-\nu|^{1/2} v\|_2^2 \leq C\beta^{-1}(\beta^{1/3}x_\nu^{-2/3}\|v\|_2^2 + \|(\chi_\nu^+)^2 v\|_1 \|f\|_\infty). \tag{3.1.50}$$

We then obtain, for any $\rho > 0$,

$$\|\chi_\nu^+ |U-\nu|^{1/2} v\|_2^2 \leq C\Big([\beta x_\nu]^{-2/3}\|v\|_2^2 + \rho\beta^{-2}\|(\chi_\nu^+)^2 v\|_1 + \frac{1}{\rho}\|f\|_\infty^2\Big)$$

which, with the aid of (3.1.47), leads to

$$\begin{aligned}&\|\chi_\nu^+ |U-\nu|^{1/2} v\|_2^2 \\ &\quad\leq \widehat{C}\Big([\beta x_\nu]^{-2/3}\|v\|_2^2 + \rho\beta^{-2}x_\nu^{-1}\|\chi_\nu^+ |U-\nu|^{1/2} v\|_2^2 + \frac{1}{\rho}\|f\|_\infty^2\Big).\end{aligned}$$

Setting $\rho = [2\widehat{C}]^{-1}\beta^2 x_\nu$ finally leads to

$$\|\chi_\nu^+ |U-\nu|^{1/2} v\|_2^2 \leq C\left([\beta x_\nu]^{-2/3}\|v\|_2^2 + [\beta^2 x_\nu]^{-1}\|f\|_\infty^2\right) \tag{3.1.51}$$

Since for some positive C it holds by (2.8.9) that $\chi_\nu^+|U-\nu|^{1/2} \geq C^{-1}x_\nu\chi_\nu^+$ we can conclude that

$$\|\chi_\nu^+ v\|_2 \leq C(\beta^{-1/3}x_\nu^{-4/3}\|v\|_2 + \beta^{-1}x_\nu^{-3/2}\|f\|_\infty). \tag{3.1.52}$$

Combining (3.1.46) with (3.1.45) and (3.1.52) then yields with the aid of (3.1.5)

$$\begin{aligned}\|v\|_2 &\leq C\,\big(([\beta x_\nu^{3/2}]^{-1}+[\beta x_\nu]^{-5/6})\|f\|_\infty+[\beta x_\nu^4]^{-1/3}\|v\|_2\\&\qquad+[\beta^2x_\nu^5]^{-1/3}[\|\mathbf{1}_{(\frac{x_\nu}{2},\frac{3x_\nu}{4})}v'\|_2+\|\mathbf{1}_{(\frac{5x_\nu}{4},\frac{3x_\nu}{2})}v'\|_2]\big)\\&\leq \hat{C}\,\big([\beta x_\nu]^{-5/6}\|f\|_\infty+a^{-1}\|v\|_2+[\beta^2x_\nu^5]^{-1/3}[\|\mathbf{1}_{(\frac{x_\nu}{2},\frac{3x_\nu}{4})}v'\|_2\\&\qquad+\|\mathbf{1}_{(\frac{5x_\nu}{4},\frac{3x_\nu}{2})}v'\|_2]\big).\end{aligned}$$

Hence, there exist $a_0>0$ and $C>0$ such that for all $a\geq a_0$

$$\|v\|_2\leq C\big([\beta x_\nu]^{-5/6}\|f\|_\infty+[\beta^2x_\nu^5]^{-1/3}[\|\mathbf{1}_{(\frac{x_\nu}{2},\frac{3x_\nu}{4})}v'\|_2+\|\mathbf{1}_{(\frac{5x_\nu}{4},\frac{3x_\nu}{2})}v'\|_2]\big). \tag{3.1.53}$$

Set

$$\check{\chi}_\nu^\pm(x)=\hat{\chi}(2(x/x_\nu-1))\mathbf{1}_{\mathbb{R}_+}(\pm(x-x_\nu)),$$

where $\hat{\chi}$ is defined by (2.4.16). We note that by its definition $\check{\chi}_\nu^-=1$ on $[0,\frac{3x_\nu}{4})$ and $\operatorname{supp}\check{\chi}_\nu^-\subset(-\infty,\frac{7x_\nu}{8})$. Similarly, $\check{\chi}_\nu^+=1$ on $[\frac{5x_\nu}{4},1]$ and $\operatorname{supp}\check{\chi}_\nu^+\subset(\frac{9x_\nu}{8},+\infty)$.

Proceeding as in the proof of (3.1.24) integration by parts yields, since v satisfies a Neumann condition at $x=0$ and Dirichlet condition at $x=1$, and since we have $(\check{\chi}_\nu^-)'(0)=0$,

$$\|(\check{\chi}_\nu^\pm v)'\|_2^2=\|(\check{\chi}_\nu^\pm)'v\|_2^2+\mu\beta\|\check{\chi}_\nu^\pm v\|_2^2+\Re\langle\check{\chi}_\nu^\pm v,\check{\chi}_\nu^\pm f\rangle.$$

The above identity implies, as $\mu<\Upsilon\mathfrak{J}_\nu\beta^{-\frac{1}{3}}$,

$$\|(\check{\chi}_\nu^\pm v)'\|_2^2\leq C([x_\nu\beta]^{2/3}+x_\nu^{-2})\|v\|_2^2+\||U-\nu|^{1/2}\check{\chi}_\nu^\pm v\|_2\||U-\nu|^{-1/2}\check{\chi}_\nu^\pm\|_2\|f\|_\infty.$$

By (2.8.9) there exists $0<\nu_1<U(0)$ such that for all $\nu_1<\nu<U(0)-a\beta^{-1/2}$ it holds that

$$\||U-\nu|^{-1/2}\check{\chi}_\nu^-\|_2^2\leq C\int_0^{7x_\nu/8}\frac{dx}{x_\nu^2-x^2}\leq\frac{C}{x_\nu}. \tag{3.1.54}$$

Similarly,

$$\||U-\nu|^{-1/2}\check{\chi}_\nu^+\|_2^2\leq C\int_{9x_\nu/8}^1\frac{dx}{x_\nu^2-x^2}\leq\frac{C}{x_\nu}. \tag{3.1.55}$$

For $0<\nu<\nu_1$ (3.1.54) and (3.1.55) still hold given the support of $\check{\chi}_\nu^\pm$. Consequently, we may conclude that

$$\|(\check{\chi}_\nu^\pm v)'\|_2^2\leq C([x_\nu\beta]^{2/3}+x_\nu^{-2})\|v\|_2^2+x_\nu^{-1/2}\||U-\nu|^{1/2}\check{\chi}_\nu^\pm v\|_2\|f\|_\infty. \tag{3.1.56}$$

As in (3.1.25) it holds that

$$\mp\beta\||U-\nu|^{1/2}\check{\chi}_\nu^\pm v\|_2^2+2\Im\langle(\check{\chi}_\nu^\pm)'v,(\check{\chi}_\nu^\pm v)'\rangle=\Im\langle\check{\chi}_\nu^\pm v,\check{\chi}_\nu^\pm f\rangle,$$

which implies

$$\begin{aligned}\beta\||U-\nu|^{1/2}\check{\chi}_\nu^\pm v\|_2^2 \\ \leq \frac{C}{x_\nu}\|v\|_2\|(\check{\chi}_\nu^\pm v)'\|_2 + \||U-\nu|^{1/2}\check{\chi}_\nu^\pm v\|_2\||U-\nu|^{-1/2}\check{\chi}_\nu^\pm\|_2\|f\|_\infty.\end{aligned}$$

Consequently, by (3.1.54),

$$\||U-\nu|^{1/2}\check{\chi}_\nu^\pm v\|_2^2 \leq C([\beta x_\nu]^{-1}\|v\|_2\|(\check{\chi}_\nu^\pm v)'\|_2 + [\beta^2 x_\nu]^{-1}\|f\|_\infty^2). \quad (3.1.57)$$

Substituting (3.1.57) into (3.1.56) then yields, with the aid of (3.1.5),

$$\begin{aligned}&\|(\check{\chi}_\nu^\pm v)'\|_2^2 \\ &\leq C\big([\beta x_\nu]^{2/3}\|v\|_2^2 + [\beta x_\nu]^{-1}\|f\|_\infty^2 + [\beta^{1/2}x_\nu]^{-1}\|v\|_2^{1/2}\|(\check{\chi}_\nu^\pm v)'\|_2^{1/2}\|f\|_\infty\big).\end{aligned}$$

By the above inequality we may conclude, first, that

$$\|(\check{\chi}_\nu^\pm v)'\|_2^2 \leq C\big([\beta x_\nu]^{2/3}\|v\|_2^2 + 2[\beta x_\nu]^{-1}\|f\|_\infty^2 + x_\nu^{-1}(\|(\check{\chi}_\nu^\pm v)'\|_2\|v\|_2)\big),$$

and then, for any $\eta > 0$,

$$\|(\check{\chi}_\nu^\pm v)'\|_2^2 \leq C\Big([\beta x_\nu]^{2/3}\|v\|_2^2 + 2[\beta x_\nu]^{-1}\|f\|_\infty^2 + \eta\|(\check{\chi}_\nu^\pm v)'\|_2^2 + \frac{1}{\eta}x_\nu^{-2}\|v\|_2^2\Big).$$

Using again (3.1.5), for η small enough, we finally obtain

$$\|(\check{\chi}_\nu^\pm v)'\|_2^2 \leq C\big([\beta x_\nu]^{2/3}\|v\|_2^2 + 2[\beta x_\nu]^{-1}\|f\|_\infty^2\big).$$

From the above it can be easily verified that

$$\begin{aligned}&\|\mathbf{1}_{(\frac{x_\nu}{2},\frac{3x_\nu}{4})}v'\|_2 + \|\mathbf{1}_{(\frac{5x_\nu}{4},\frac{3x_\nu}{2})}v'\|_2 \\ &\quad\leq \|(\check{\chi}_\nu^+ v)'\|_2 + \|(\check{\chi}_\nu^- v)'\|_2 \leq C([\beta x_\nu]^{1/3}\|v\|_2 + [\beta x_\nu]^{-1/2}\|f\|_\infty). \quad (3.1.58)\end{aligned}$$

Substituting (3.1.58) into (3.1.53) yields, using (3.1.5)

$$\|v\|_2 \leq C\big([\beta x_\nu]^{-5/6}\|f\|_\infty + [\beta x_\nu^4]^{-1/3}\|v\|_2\big).$$

Hence, there exists $a_0 > 0$ such that we obtain for $a \geq a_0$

$$\|v\|_2 \leq C\,[\beta x_\nu]^{-5/6}\|f\|_\infty. \quad (3.1.59)$$

Step 2.2. The case $x_\nu \geq 1/8$. Let

$$\Upsilon < [|U'(1/16)|/|U'(1)|]^{2/3}\Re\hat{\kappa}_1. \quad (3.1.60)$$

We set

$$\hat{\eta}_\nu = \chi(-(x - x_\nu)/x_\nu), \quad (3.1.61)$$

which is supported on $(x_\nu/4, 1]$ and satisfies $\widehat{\eta}_\nu \equiv 1$ on $(x_\nu/2, 1]$. Then, as

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)(\widehat{\eta}_\nu v) = \widehat{\eta}_\nu f - 2\widehat{\eta}_\nu' v' - \widehat{\eta}_\nu'' v, \tag{3.1.62}$$

we may use [3, Propositions 5.2 and 5.4] (both hold for $U \in \mathcal{S}_r^2$ though stated for $U \in \mathcal{S}_r^4$), to obtain that

$$\|\widehat{\eta}_\nu v\|_2 \leq \frac{C}{\beta^{2/3}}(\beta^{-1/6}\|f\|_\infty + \|\widehat{\eta}_\nu' v'\|_2 + \|\widehat{\eta}_\nu'' v\|_2). \tag{3.1.63}$$

We note that (3.1.60) implies $\Upsilon < [|U'(x_\nu/4)|/\mathfrak{J}_\nu]^{2/3}\Re\hat{\kappa}_1$ for all $1/8 \leq x_\nu \leq 1$. Let $\widetilde{\eta}_\nu = \sqrt{1 - \widehat{\eta}_\nu^2} \in C^\infty(\mathbb{R}, [0, 1])$. Note that $\widetilde{\eta}_\nu$ is supported on $[0, x_\nu/2]$. Consequently, we may obtain, as in (3.1.46) but for $x_\nu \geq \frac{1}{8}$,

$$\|\widetilde{\eta}_\nu v\|_2 \leq C\beta^{-1}[\|f\|_2 + \beta^{1/2}\|v\|_2].$$

Combining the above with (3.1.63) yields

$$\|v\|_2 \leq \frac{C}{\beta^{2/3}}\big(\beta^{-1/6}\|f\|_\infty + \|\mathbf{1}_{(x_\nu/4, x_\nu/2)} v'\|_2\big).$$

We now use a variant of equation (3.1.58) (which is valid also for $x_\nu > 1/8$) to bound $\|\mathbf{1}_{(x_\nu/4, x_\nu/2)} v'\|_2$ to obtain, with the aid of (3.1.5),

$$\|v\|_2 \leq \frac{C}{\beta^{5/6}}\,\|f\|_\infty.$$

Together with (3.1.4) the above inequality establishes (3.1.3a).

Step 3. We prove (3.1.3b), when

$$\Upsilon < \inf_{x_\nu \in [0,1]}(|U'(x_\nu/2)|/|U'(x_\nu)|)^{2/3}\Re\hat{\kappa}_1. \tag{3.1.64}$$

Note that for $p = 2$ (3.1.3b) readily follows from (3.1.4). In the following we then assume $p \in (1, 2)$. Suppose first that $x_\nu < 1/2$. As above, we consider a pair (v, f) in $D(\mathcal{L}_\beta^{\mathfrak{N}}) \times L^2(0, 1)$ satisfying $(\mathcal{L}_\beta^{\mathfrak{N}} - \lambda\beta)v = f$. Let

$$\widehat{\mathcal{L}}_\beta^{\mathfrak{D}} : H^2(x_\nu/2, 3x_\nu/2) \cap H_0^1(x_\nu/2, 3x_\nu/2) \to L^2(x_\nu/2, 3x_\nu/2)$$

be associated with the same differential operator as $\mathcal{L}_\beta^{\mathfrak{N}}$. Let

$$\widetilde{\mathcal{L}}_{\widetilde{\beta}}^{\mathfrak{D}} : H^2(1/2, 3/2) \cap H_0^1(1/2, 3/2) \to L^2(1/2, 3/2)$$

be associated with the same differential operator as $\widetilde{\mathcal{L}}_{\widetilde{\beta},\mathbb{R}}$ in (3.1.17). We recall from [3, Proposition 5.2] that for any $g \in L^2(1/2, 3/2)$ it holds

$$\sup_{\Re\widetilde{\lambda} \leq \Upsilon\widetilde{\beta}^{-1/3}} \widetilde{\beta}^{-1/3}\Big\|\frac{d}{dy}(\widetilde{\mathcal{L}}_{\widetilde{\beta}}^{\mathfrak{D}} - \widetilde{\beta}\widetilde{\lambda})^{-1}g\Big\|_p \leq \frac{C}{\widetilde{\beta}^{\frac{2+p}{6p}}}\|g\|_2. \tag{3.1.65}$$

As in (3.1.20) it holds that

$$(\widehat{\mathcal{L}}_\beta^{\mathfrak{D}} - \beta\lambda)(\tilde{\chi}_\nu v) = \tilde{\chi}_\nu f - 2\tilde{\chi}_\nu' v' - \tilde{\chi}_\nu'' v,$$

and hence by applying the inverse of the dilation (3.1.12) to (3.1.65) we can conclude that

$$\|(\tilde{\chi}_\nu v)'\|_p \leq C[\beta x_\nu]^{-\frac{2+p}{6p}}(\|f\|_2 + \|\tilde{\chi}_\nu' v'\|_2 + \|\tilde{\chi}_\nu'' v\|_2). \tag{3.1.66}$$

By (3.1.4) we then obtain

$$\|(\tilde{\chi}_\nu v)'\|_p \leq C[\beta x_\nu]^{-\frac{2+p}{6p}}(1 + \beta^{-1/3}x_\nu^{-4/3} + \beta^{-2/3}x_\nu^{-8/3})\|f\|_2.$$

From (3.1.5) we easily conclude that

$$\|(\tilde{\chi}_\nu v)'\|_p \leq C[\beta x_\nu]^{-\frac{2+p}{6p}}\,\|f\|_2. \tag{3.1.67}$$

We now seek an estimate for $\chi_\nu^{\pm} v'$. To this end, we use integration by parts to obtain

$$\begin{aligned}
&\Re\langle(\chi_\nu^{\pm})^2(U-\nu)v, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)v\rangle \\
&\quad = \mp\|\chi_\nu^{\pm}|U-\nu|^{1/2}v'\|_2^2 + \Re\langle\chi_\nu^{\pm}(U'\chi_\nu^{\pm} + 2(U-\nu)(\chi_\nu^{\pm})')v, v'\rangle \\
&\qquad - \mu\beta\|\chi_\nu^{\pm}|U-\nu|^{1/2}v\|_2^2.
\end{aligned} \tag{3.1.68}$$

Since

$$|U'(x)| \leq C|U(x) - U(0)|^{1/2} \leq C(|U(x)-\nu|^{1/2} + |U(0)-\nu|^{1/2}), \tag{3.1.69}$$

we can conclude that

$$\|(\chi_\nu^{\pm})^2 U' v\|_2 \leq C(x_\nu\|v\|_2 + \|\chi_\nu^{\pm}|U-\nu|^{1/2}v\|_2).$$

Furthermore, given the support of $(\chi_\nu^{\pm})'$ we obtain by (2.8.6)

$$\|\chi_\nu^{\pm}(U-\nu)(\chi_\nu^{\pm})'\|_\infty \leq C\|(x^2 - x_\nu^2)(\chi_\nu^{\pm})'\|_\infty \leq Cx_\nu.$$

Combining the above with (3.1.68) yields that

$$\begin{aligned}
&\|\chi_\nu^{\pm}|U-\nu|^{1/2}v'\|_2^2 \\
&\quad \leq \|(U-\nu)v\|_2\|f\|_2 + C\big([\beta x_\nu]^{2/3}\|\chi_\nu^{\pm}|U-\nu|^{1/2}v\|_2^2 + [x_\nu\|v\|_2 \\
&\qquad\qquad + \|\chi_\nu^{\pm}|U-\nu|^{1/2}v\|_2]\|v'\|_2\big).
\end{aligned} \tag{3.1.70}$$

As

$$\Im\langle(U-\nu)v, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)v\rangle = \beta\|(U-\nu)v\|_2^2 + \Im\langle U'v, v'\rangle, \tag{3.1.71}$$

we obtain by (3.1.69) that

$$\beta\|(U-\nu)v\|_2^2 \leq C\big(\beta^{-1}\|f\|_2^2 + [\||U(x)-\nu|^{1/2}v\|_2 + x_\nu\|v\|_2]\|v'\|_2\big).$$

Furthermore, since

$$\||U-\nu|^{1/2}v\|_2^2 \leq \frac{1}{2}\big[x_\nu^{-2}\|(U-\nu)v\|_2^2 + x_\nu^2\|v\|_2^2\big], \tag{3.1.72}$$

we can write

$$\beta\|(U-\nu)v\|_2^2 \leq C\big(\beta^{-1}\|f\|_2^2 + [x_\nu\|v\|_2 + x_\nu^{-1}\|(U-\nu)v\|_2]\|v'\|_2\big).$$

Hence,

$$\beta\|(U-\nu)v\|_2^2 \leq C\big(\beta^{-1}\|f\|_2^2 + x_\nu\|v\|_2\|v'\|_2 + \beta^{-1}x_\nu^{-2}\|v'\|_2^2\big). \tag{3.1.73}$$

Using (3.1.4) gives the following estimates for $\|v\|_2$ and $\|v'\|_2$

$$\|v'\|_2 \leq C\,[x_\nu\beta]^{-1/3}\|f\|_2 \quad \text{and} \quad \|v\|_2 \leq C\,[x_\nu\beta]^{-2/3}\|f\|_2. \tag{3.1.74}$$

Substituting the above into (3.1.73) yields

$$\beta\|(U-\nu)v\|_2^2 \leq C(\beta^{-1} + \beta^{-1}x_\nu^{-2}[x_\nu\beta]^{-2/3})\|f\|_2^2.$$

From the above, recalling that $\beta^{\frac{1}{4}}x_\nu$ is bounded from below, we conclude that

$$\|(U-\nu)v\|_2 \leq C\beta^{-1}\|f\|_2. \tag{3.1.75}$$

Next, we write, using (3.1.74) and (3.1.75),

$$\||U-\nu|^{1/2}v\|_2^2 \leq \frac{1}{2}[\beta^{1/3}x_\nu^{-2/3}\|(U-\nu)v\|_2^2 + \beta^{-1/3}x_\nu^{2/3}\|v\|_2^2] \leq \frac{C}{\beta^{5/3}x_\nu^{2/3}}\|f\|_2^2. \tag{3.1.76}$$

Substituting (3.1.76) together with (3.1.75) into (3.1.70) yields with the aid of (3.1.74)

$$\|\chi_\nu^{\pm}|U-\nu|^{1/2}v'\|_2^2 \leq C\beta^{-1}\|f\|_2^2. \tag{3.1.77}$$

We now observe that

$$\||U-\nu|^{-1/2}\chi_\nu^-\|_q^q \leq C\int_0^{3x_\nu/4}\frac{dx}{[x_\nu^2-x^2]^{q/2}} \leq \frac{C}{x_\nu^{q-1}}. \tag{3.1.78}$$

Similarly,

$$\||U-\nu|^{-1/2}\chi_\nu^+\|_2^2 \leq C\int_{5x_\nu/4}^1\frac{dx}{[x_\nu^2-x^2]^{q/2}} \leq \frac{C}{x_\nu^{q-1}}, \tag{3.1.79}$$

which is obtained with the aid of (2.8.9) for $\nu > \nu_1$ (for $\nu \leq \nu_1$ the above bounds are trivial). Consequently, we obtain that

$$\begin{aligned}\|(\chi_\nu^{\pm})v'\|_p &\leq \|\chi_\nu^{\pm}|U-\nu|^{1/2}v'\|_2\|[\mathbf{1}_{[0,3x_\nu/4]} + \mathbf{1}_{[5x_\nu/4,1]}|U-\nu|^{-1/2}]\|_{\frac{2p}{2-p}} \\ &\leq \frac{C}{\beta^{1/2}x_\nu^{\frac{3p-2}{2p}}}\|f\|_2.\end{aligned} \tag{3.1.80}$$

Similarly, we write that

$$\begin{aligned}\|(\chi_\nu^\pm)'v\|_p &\leq \|(\chi_\nu^\pm)'|U-\nu|^{1/2}v\|_2\|[\mathbf{1}_{[0,3x_\nu/4]} + \mathbf{1}_{[5x_\nu/4,1]}|U-\nu|^{-1/2}]\|_{\frac{2p}{2-p}} \\ &\leq \frac{C}{x_\nu^{\frac{17}{6}-\frac{1}{p}}\beta^{5/6}}\|f\|_2.\end{aligned}$$

To obtain the second inequality we have used (3.1.76). Together with (3.1.80) and (3.1.5) the above yields

$$\|(\chi_\nu^\pm v)'\|_p \leq C\beta^{-1/2}x_\nu^{-\frac{3p-2}{2p}}\|f\|_2. \tag{3.1.81}$$

Combining the above with (3.1.67) yields the existence of a_0 such that (3.1.3b) holds for all $a \geq a_0$.

Consider next the case $x_\nu \geq 1/2$ (in which no dilation transformation is necessary). Let

$$\Upsilon < (|U'(1/4)|/|U'(1)|)^{2/3}\Re\hat{\kappa}_1. \tag{3.1.82}$$

We now set

$$\iota_\nu = \sqrt{\tilde{\chi}_\nu^2 + (\chi_\nu^+)^2}.$$

Then, as

$$(\hat{\mathcal{L}}_\beta^{\mathfrak{D}} - \beta\lambda)(\iota_\nu v) = \iota_\nu f - 2\iota_\nu' v' - \iota_\nu'' v.$$

We obtain using [3, Proposition 5.2] and (3.1.4) that

$$\|(\iota_\nu v)'\|_p \leq \frac{C}{\beta^{\frac{2+p}{6p}}}\|f\|_2.$$

Since (3.1.81) holds true for $\chi_\nu^- v$ in the case $x_\nu \geq 1/2$, we can combine it with the above to extend the validity (3.1.3b) to this case as well.

Given (3.1.37), (3.1.38), (3.1.60), (3.1.64), and (3.1.82) it follows that there exists $\Upsilon > 0$ for which Proposition 3.1.1 holds true. ■

Remark 3.1.2. As in [3, Proposition 5.1] we can obtain better estimates for the case where $\mu < 0$. Thus, setting $\chi_\nu^\pm \equiv 1$ in (3.1.24) yields for $\mu < 0$

$$\|v'\|_2^2 + |\mu|\beta\|v\|_2^2 = \Re\langle v, f\rangle. \tag{3.1.83}$$

From here we conclude that

$$\|v\|_2 \leq [|\mu|\beta]^{-1}\|f\|_2, \tag{3.1.84}$$

which is stronger that (3.1.3) when $|\mu| \gg \beta^{-1/3}$.

Note that for $\mu < 0$

$$\|v'\|_2 \leq [|\mu|\beta]^{-1/2}\|f\|_2. \tag{3.1.85}$$

3.2 Resolvent estimates for $|U(0) - \Im\lambda| = \mathcal{O}(\beta^{-1/2})$

In this case, we will approximate $U - \nu$ by its quadratic potential

$$x \mapsto U''(0)x^2/2 + U(0) - \nu$$

and then use a proper resolvent estimate established by R. Henry in [16].

More precisely, we prove the following proposition.

Proposition 3.2.1. *Let $U \in C^3([0,1])$ satisfy (2.1.3), $a > 0$ and $\Upsilon < \sqrt{-U''(0)}/2$. Then there exist $C > 0$ and $\beta_0 > 0$ such that, for all $\beta \geq \beta_0$,*

- *if $f \in L^\infty(0,1)$,*

$$\sup_{\substack{\mu \leq \Upsilon\beta^{-1/2} \\ |\nu - U(0)| < a\beta^{-1/2}}} \Big(\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1} f\|_2 + \beta^{-1/4} \|\frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1} f\|_2 + \beta^{1/8}\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1} f\|_1 \Big) \leq C \min(\beta^{-1/2}\|f\|_2, \beta^{-5/8}\|f\|_\infty), \tag{3.2.1a}$$

- *if $(x - x_\nu)^{-1} f \in L^2(0,1)$*

$$\sup_{\substack{\mu \leq \Upsilon\beta^{-1/2} \\ |\nu - U(0)| < a\beta^{-1/2}}} \Big(\Big\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1} f\Big\|_2 + \beta^{-1/4} \Big\|\frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1} f\Big\|_2 + \beta^{1/8}\Big\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1} f\Big\|_1 \Big) \leq C\beta^{-3/4}\Big\|\frac{f}{x - x_\nu}\Big\|_2. \tag{3.2.1b}$$

Proof. All the estimates established in this proof assume that

$$\mu \leq \Upsilon\beta^{-1/2} \quad \text{and} \quad U(0) - a\beta^{-1/2} < \nu < U(0) + a\beta^{-1/2}. \tag{3.2.2}$$

By the second condition it holds that

$$0 \leq x_\nu \leq C_a\beta^{-\frac{1}{4}}. \tag{3.2.3}$$

Consequently, for any $\nu_1 < U(0)$, there exists β_0 such that, for all $\beta \geq \beta_0$, we have

$$\nu > \nu_1 \quad \text{and} \quad x_\nu < 1/4. \tag{3.2.4}$$

Step 1: $\mathcal{L}(L^2, L^2)$ estimate. By [16, Theorem 1.3] we immediately obtain that, under (3.2.2),

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1} f\|_2 \leq C\beta^{-1/2}\|f\|_2. \tag{3.2.5}$$

To prove the second inequality in (3.2.1a), let $f \in L^2(0,1)$ and $v \in D(\mathcal{L}_\beta^{\mathfrak{N}})$ satisfy $(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)v = f$. Taking the scalar product with v, an integration by parts yields for the real part, with the aid of (3.2.2),

$$\|v'\|_2^2 = \mu\beta\|v\|_2^2 + \Re\langle v, f\rangle \leq C(\beta^{1/2}\|v\|_2^2 + \beta^{-1/2}\|f\|_2^2).$$

By (3.2.5) we can then conclude that

$$\left\|\frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1} f\right\|_2 \leq C\beta^{-1/4}\|f\|_2, \tag{3.2.6}$$

which together with (3.2.5) establishes the $\mathcal{L}(L^2, L^2)$ estimate in (3.2.1a).

Step 2: $\mathcal{L}(L^\infty, L^2)$ estimate. Next, we obtain an $\mathcal{L}(L^\infty, L^2)$ estimate for $(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}$ under (3.2.2). Let $\hat{\chi}$ be given (2.4.16). As before, we set

$$\chi_\gamma(x) = \hat{\chi}(\gamma\beta^{1/4}x),$$

for some positive

$$\gamma < [8C_a + 1]^{-1} < 1.$$

In particular, γ satisfies

$$0 < \gamma < [8\beta^{\frac{1}{4}}x_\nu + 1]^{-1}.$$

We note that χ_γ satisfies

$$\operatorname{supp}\chi_\gamma \subset \left[2x_\nu + \frac{1}{4}\beta^{-\frac{1}{4}}, 1\right) \quad \text{and} \quad |\chi_\gamma'| \leq C\gamma\beta^{\frac{1}{4}}. \tag{3.2.7}$$

Set further $\tilde{\chi}_\gamma = \sqrt{1 - \chi_\gamma^2} \in C^\infty(\mathbb{R})$ and note that $\tilde{\chi}_\gamma$ satisfies

$$\operatorname{supp}\tilde{\chi}_\gamma \subset \left[0, \frac{1}{2\gamma}\beta^{-\frac{1}{4}}\right) \quad \text{and} \quad |\tilde{\chi}_\gamma'| \leq C\gamma\beta^{\frac{1}{4}}. \tag{3.2.8}$$

Let $f \in L^2(0,1)$ and $v \in D(\mathcal{L}_\beta^{\mathfrak{N}})$ satisfy $(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)v = f$.

We begin by estimating $\chi_\gamma v$. An integration by parts yields

$$\|(\chi_\gamma v)'\|_2^2 - \|\chi_\gamma' v\|_2^2 - \mu\beta\|\chi_\gamma v\|_2^2 = \Re\langle\chi_\gamma v, \chi_\gamma f\rangle, \tag{3.2.9}$$

from which we conclude, given that $\mu\beta^{1/2}$ and γ are bounded from above,

$$\|(\chi_\gamma v)'\|_2^2 \leq \|\chi_\gamma v\|_1\|f\|_\infty + C\beta^{1/2}\|v\|_2^2. \tag{3.2.10}$$

Furthermore, we have that

$$-\beta\| \, |U-\nu|^{1/2}\chi_\gamma v\|_2^2 + 2\Im\langle \chi_\gamma' v, (\chi_\gamma v)'\rangle = \Im\langle \chi_\gamma v, \chi_\gamma f\rangle. \tag{3.2.11}$$

Hence, with the aid of (3.2.10), we can conclude that

$$\beta\||U-\nu|^{1/2}\chi_\gamma v\|_2^2 \leq C(\gamma\beta^{1/2}\|v\|_2^2 + \|\chi_\gamma v\|_1\|f\|_\infty). \tag{3.2.12}$$

We now write (note that $\chi_\gamma\chi_{2\gamma} = \chi_\gamma$)

$$\|\chi_\gamma v\|_1 \leq \|\chi_{2\gamma}|U-\nu|^{-1/2}\|_2\||U-\nu|^{1/2}\chi_\gamma v\|_2. \tag{3.2.13}$$

For $x \in [x_\nu + \frac{1}{8}\beta^{-\frac{1}{4}}, 1)$, it holds by (2.9.6), (2.9.7), and (3.2.4)

$$|U(x)-\nu| \geq \frac{1}{C}(x-x_\nu)^2 \geq \frac{1}{4C}\beta^{-\frac{1}{2}}, \tag{3.2.14}$$

which implies, for $\beta \geq \beta_0$,

$$\int_{x_\nu+\frac{1}{8}\beta^{-\frac{1}{4}}}^1 |U-\nu|^{-1}\,dx \leq C\int_{\frac{1}{4}\beta^{-\frac{1}{4}}}^1 y^{-2}\,dy \leq \hat{C}\beta^{1/4}.$$

We can then conclude, using (3.2.7), that

$$\|\chi_{2\gamma}|U-\nu|^{-1/2}\|_2 \leq C\beta^{1/8}. \tag{3.2.15}$$

Hence, by (3.2.12) and (3.2.13) we obtain that

$$\|\chi_\gamma v\|_1 \leq C\beta^{-3/8}(\gamma^{\frac{1}{2}}\beta^{1/4}\|v\|_2 + \|\chi_\gamma v\|_1^{1/2}\|f\|_\infty^{1/2}),$$

from which we conclude that

$$\|\chi_\gamma v\|_1 \leq C\beta^{-1/8}(\gamma^{1/2}\|v\|_2 + \beta^{-5/8}\|f\|_\infty).$$

Substituting the above into (3.2.12) then yields

$$\||U-\nu|^{1/2}\chi_\gamma v\|_2 \leq C\beta^{-1/4}(\gamma^{1/2}\|v\|_2 + \beta^{-5/8}\|f\|_\infty). \tag{3.2.16}$$

Since by (3.2.14),

$$|U-\nu|^{1/2}\chi_\gamma \geq \frac{1}{C}\beta^{-1/4}\chi_\gamma, \tag{3.2.17}$$

we may write

$$\|\chi_\gamma v\|_2 \leq C(\gamma^{1/2}\|v\|_2 + \beta^{-5/8}\|f\|_\infty). \tag{3.2.18}$$

We now attempt to estimate $\tilde{\chi}_\gamma v$. As

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)(\tilde{\chi}_\gamma v) = \tilde{\chi}_\gamma f - 2\tilde{\chi}_\gamma' v' - \tilde{\chi}_\gamma'' v, \tag{3.2.19}$$

we may conclude from [16, Theorem 1.3] (which can be used since $U \in C^3([0,1])$ and $\Upsilon < [-U''(0)]^{1/2}/2$), that

$$\begin{aligned}\|\tilde{\chi}_\gamma v\|_2 &\leq C\beta^{-1/2}(\|\tilde{\chi}_\gamma f\|_2 + \|\tilde{\chi}'_\gamma v'\|_2 + \|\tilde{\chi}''_\gamma v\|_2) \\ &\leq \hat{C}(\beta^{-5/8}\|f\|_\infty + \gamma\beta^{-1/4}\|v'\| + \gamma^2\|v\|_2).\end{aligned}$$

To obtain the second inequality we used the fact that supp $\tilde{\chi}_\gamma \subseteq (0, \check{C}\beta^{-\frac{1}{4}})$. Combining the above with (3.2.18) yields the existence of $\gamma_0 > 0$ such that for all $\gamma \in (0, \gamma_0)$,

$$\|v\|_2 \leq C(\beta^{-5/8}\|f\|_\infty + \gamma\beta^{-1/4}\|v'\|_2). \tag{3.2.20}$$

As in (3.2.10) (replacing χ_γ by 1), we obtain that

$$\|v'\|_2^2 \leq \|v\|_1\|f\|_\infty + C\beta^{1/2}\|v\|_2^2. \tag{3.2.21}$$

By (2.9.24), applied with $q = 1$ and $\mu = \beta^{-\frac{1}{2}}$, and (3.2.2) it holds that

$$\|(U - \nu + i\beta^{-1/2})^{-1/2}\|_2^2 = \|(U - \nu + i\beta^{-1/2})^{-1}\|_1 \leq C\beta^{1/4}.$$

From the above we conclude that

$$\begin{aligned}\|v\|_1 &\leq \||U - \nu + i\beta^{-1/2}|^{-1/2}\|_2\||U - \nu + i\beta^{-1/2}|^{1/2}v\|_2 \\ &\leq C\beta^{1/8}(\||U - \nu|^{1/2}\chi_\gamma v\|_2 + \||U - \nu|^{1/2}\tilde{\chi}_\gamma v\|_2 + \beta^{-1/4}\|v\|_2).\end{aligned}$$

By (2.8.9) (which is valid by (3.2.4))

$$|U - \nu|^{1/2}\tilde{\chi}_\gamma \leq C \sup_{x \in (0, \check{C}\beta^{-\frac{1}{4}})} |x^2 - x_\nu^2|^{1/2} \leq C\beta^{-1/4},$$

we obtain, with the aid of (3.2.16) that

$$\|v\|_1 \leq C(\beta^{-1/8}\|v\|_2 + \beta^{-3/4}\|f\|_\infty). \tag{3.2.22}$$

Substituting (3.2.22) into (3.2.21) yields

$$\|v'\|_2 \leq C(\beta^{1/4}\|v\|_2 + \beta^{-3/8}\|f\|_\infty),$$

which when substituted into (3.2.20) yields for sufficiently small γ_0 and $\gamma \in (0, \gamma_0)$

$$\|v\|_2 \leq C\beta^{-5/8}\|f\|_\infty, \tag{3.2.23}$$

and then

$$\|v'\|_2 \leq C\beta^{-3/8}\|f\|_\infty. \tag{3.2.24}$$

Substituting (3.2.23) into (3.2.22) yields

$$\|v\|_1 \leq C\beta^{-3/4}\|f\|_\infty. \tag{3.2.25}$$

By (3.2.11) it holds that

$$\beta \| |U-\nu|^{1/2}\chi_\gamma v\|_2^2 \le C(\gamma\beta^{1/2}\|v\|_2^2 + \|\chi_\gamma v\|_2\|f\|_2),$$

from which we conclude by combining it with (3.2.5)

$$\| |U-\nu|^{1/2}\chi_\gamma v\|_2 \le C\beta^{-3/4}\|f\|_2.$$

Consequently,

$$\|\chi_\gamma^2 v\|_1 \le \| |U-\nu|^{-1/2}\chi_\gamma\|_2 \| |U-\nu|^{1/2}\chi_\gamma v\|_2 \le C\beta^{-5/8}\|f\|_2.$$

Use of (3.2.15) has been made to obtain the second inequality.

Since by (3.2.5)

$$\|\tilde{\chi}_\gamma^2 v\|_1 \le C\beta^{-1/8}\|v\|_2 \le C\beta^{-5/8}\|f\|_2, \tag{3.2.26}$$

we may conclude that

$$\|v\|_1 \le C\beta^{-5/8}\|f\|_2, \tag{3.2.27}$$

which together with (3.2.25) completes the proof of (3.2.1a).

Step 3: Proof of (3.2.1b). To prove (3.2.1b) we set

$$f = (x-x_\nu)g,$$

where $g \in L^2(0,1)$.

Then, as in (3.2.10), we use (3.2.9) to obtain

$$\|(\chi_\gamma v)'\|_2^2 \le \|(x-x_\nu)\chi_\gamma v\|_2\|g\|_2 + C\beta^{1/2}\|v\|_2^2.$$

By (3.2.7) and (3.2.14) there exists $C > 0$ such that, for all $x \in [0,1]$,

$$0 \le (x-x_\nu)\chi_\gamma(x) \le C(\nu - U(x))^{1/2}\chi_\gamma(x). \tag{3.2.28}$$

Hence,

$$\|(\chi_\gamma v)'\|_2^2 \le C\big(\| |U-\nu|^{1/2}\chi_\gamma v\|_2\|g\|_2 + \beta^{1/2}\|v\|_2^2\big).$$

Next, we use (3.2.11) to obtain, as in (3.2.12), with the aid of the above and (3.2.28)

$$\beta\| |U-\nu|^{1/2}\chi_\gamma v\|_2^2 \le C\big(\gamma\beta^{1/2}\|v\|_2^2 + \|\chi_\gamma |U-\nu|^{1/2} v\|_2\|g\|_2\big),$$

from which we conclude that

$$\| |U-\nu|^{1/2}\chi_\gamma v\|_2 \le C\beta^{-1/4}(\gamma^{1/2}\|v\|_2 + \beta^{-3/4}\|g\|_2). \tag{3.2.29}$$

Combining the above with (3.2.17) yields

$$\|\chi_\gamma v\|_2 \le C(\gamma^{1/2}\|v\|_2 + \beta^{-3/4}\|g\|_2). \tag{3.2.30}$$

Furthermore, with the aid of (3.2.15) we can conclude that

$$\begin{aligned}\|\chi_\gamma^2 v\|_1 &\leq \| \, |U-\nu|^{-1/2}\chi_\gamma\|_2 \, \||U-\nu|^{1/2}\chi_\gamma v\|_2 \\ &\leq C(\gamma^{1/2}\beta^{-1/8}\|v\|_2 + \beta^{-7/8}\|g\|_2). \end{aligned} \tag{3.2.31}$$

We now use (3.2.19) to obtain, as in (3.2.20),

$$\|\tilde{\chi}_\gamma v\|_2 \leq C(\beta^{-3/4}\|g\|_2 + \gamma\beta^{-1/4}\|v'\| + \gamma^2\|v\|_2). \tag{3.2.32}$$

Combining (3.2.32) with (3.2.30) yields for sufficiently small γ

$$\|v\|_2 \leq C\,(\beta^{-3/4}\|g\|_2 + \gamma\beta^{-1/4}\|v'\|). \tag{3.2.33}$$

Then we write

$$\|v'\|_2^2 = \mu\beta\|v\|_2^2 + \Re\langle v, f\rangle = \mu\beta\|v\|_2^2 + \Re\langle (x-x_\nu)v, g\rangle. \tag{3.2.34}$$

To estimate the second term on the right-hand side of (3.2.34) we first note that by (3.2.28), (3.2.3) and (3.2.8),

$$\begin{aligned}\|(x-x_\nu)v\|_2 &\leq \|(x-x_\nu)\chi_\gamma v\|_2 + \|(x-x_\nu)\tilde{\chi}_\gamma v\|_2 \\ &\leq C(\||U-\nu|^{1/2}\chi_\gamma v\|_2 + \beta^{-1/4}\|v\|_2).\end{aligned}$$

With the aid of (3.2.29) we then obtain

$$\|(x-x_\nu)v\|_2 \leq C(\beta^{-1/4}\|v\|_2 + \beta^{-3/4}\|g\|_2). \tag{3.2.35}$$

Hence, by (3.2.34) and since $\mu < C\beta^{-1/2}$ we may conclude that

$$\|v'\|_2 \leq C(\beta^{1/4}\|v\|_2 + \beta^{-1/2}\|g\|_2). \tag{3.2.36}$$

Substituting (3.2.36) into (3.2.33) yields for small enough γ,

$$\|v\|_2 \leq C\beta^{-3/4}\|g\|_2. \tag{3.2.37}$$

By (3.2.37), the first inequality of (3.2.26), and (3.2.31) we obtain

$$\|v\|_1 \leq C\beta^{-7/8}\|g\|_2. \tag{3.2.38}$$

Together with (3.2.37) and (3.2.36), (3.2.38) verifies (3.2.1b). ■

3.3 L^1 estimates for $U(0) - \Im\lambda \gg \beta^{-1/2}$

It is not difficult to show that the resolvent of the operator $-d^2/dx^2 + ix$ is not bounded in $\mathcal{L}(L^\infty(\mathbb{R}), L^1(\mathbb{R}))$, a fact that can be easily established from the identity

$$\Big(-\frac{d^2}{dx^2} + ix\Big)\frac{1}{\sqrt{x^2+1}} = \frac{2x^2-1}{[x^2+1]^{5/2}} + i\,\frac{x}{x^2+1}.$$

For the resolvent of the operator $\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}$ on $(0,1)$, this unboundedness manifests itself through a logarithmic dependence on β as we can clearly see in the following proposition.

Proposition 3.3.1. *Let $U \in C^2([0,1])$ satisfy (2.1.3). There exist $\Upsilon > 0$, $a > 0$, $C > 0$, and $\beta_0 > 1$ such that, for $\beta \geq \beta_0$, $\Im\lambda < U(0) - a\beta^{-1/2}$, $\Re\lambda \leq \mathfrak{J}_\nu{}^{2/3}\Upsilon\beta^{-1/3}$, and $f \in L^\infty(0,1)$ we have*

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1} f\|_1 \leq C \min\big([\mathfrak{J}_\nu\beta]^{-5/6}\|f\|_2, [\mathfrak{J}_\nu\beta]^{-1}\log\beta\,\|f\|_\infty\big). \tag{3.3.1}$$

Proof. We assume that $\Upsilon > 0$ is sufficiently small so that Proposition 3.1.1 holds true. We begin by recalling that by (3.1.5), for any $N > 0$, there exists $a_0 > 0$ such that for all $a \geq a_0$, we have under the conditions of the proposition,

$$\beta x_\nu \geq \beta x_\nu^4 \geq N,$$

where x_ν is defined by (2.4.5). Let $(v, f) \in D(\mathcal{L}_\beta^{\mathfrak{N}}) \times L^\infty(0,1)$ satisfy $(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)v = f$. By (2.8.49) applied with $\mu = \beta^{-1/3}x_\nu^{2/3}$ and $q = 2$, it holds that

$$\|(U - \nu + i[\beta x_\nu^{-2}]^{-1/3})^{-1}\|_2^2 \leq C x_\nu^{-5/3}\beta^{1/3}.$$

We may then conclude that

$$\begin{aligned}\|v\|_1 &\leq \|(U - \nu + i[\beta x_\nu^{-2}]^{-1/3})^{-1}\|_2\|(U - \nu + i[\beta x_\nu^{-2}]^{-1/3})v\|_2\\ &\leq C\beta^{1/6}x_\nu^{-5/6}[\|(U-\nu)v\|_2 + [\beta x_\nu^{-2}]^{-1/3}\|v\|_2].\end{aligned} \tag{3.3.2}$$

By (3.1.75) and (3.1.3a)

$$\|v\|_1 \leq C[\beta x_\nu]^{-5/6}\,\|f\|_2. \tag{3.3.3}$$

By (2.8.62) we can conclude that

$$\|(U - \nu + i[\beta x_\nu^{-2}]^{-1/3})^{-1/2}\|_2^2 = \|(U - \nu + i[\beta x_\nu^{-2}]^{-1/3})^{-1}\|_1 \leq \frac{C}{x_\nu}\log(\beta x_\nu^4).$$

Hence, we can complete the proof of (3.3.1) by writing

$$\begin{aligned}\|v\|_1 &\leq \|(U - \nu + i[\beta x_\nu^{-2}]^{-1/3})^{-1/2}\|_2\|(U - \nu + i[\beta x_\nu^{-2}]^{-1/3})^{1/2}v\|_2\\ &\leq \frac{C}{x_\nu^{1/2}}[\log(\beta x_\nu^4)]^{1/2}[\||U-\nu|^{1/2}v\|_2 + [\beta x_\nu^{-2}]^{-1/6}\|v\|_2],\end{aligned}$$

which implies

$$\|v\|_1 \leq \frac{C}{x_\nu^{1/2}}[\log(\beta)]^{1/2}[\||U-\nu|^{1/2}v\|_2 + [\beta x_\nu^{-2}]^{-1/6}\|v\|_2]. \tag{3.3.4}$$

We note that by (3.1.3a) (which holds for $a \geq a_0$ with a_0 large enough) and (3.1.5) we have

$$[\beta x_\nu^{-2}]^{-1/6}\|v\|_2 \leq C\beta^{-1}x_\nu^{-\frac{1}{2}}\|f\|_\infty.$$

Hence, we obtain from (3.3.4)

$$\|v\|_1 \leq \frac{C}{x_\nu^{1/2}}[\log(\beta)]^{1/2}\big[\||U-\nu|^{1/2}v\|_2 + \beta^{-1}x_\nu^{-1/2}\|f\|_\infty\big]. \tag{3.3.5}$$

To complete the proof we need an estimate for $\||U-\nu|^{1/2}v\|_2$. In a similar manner to (3.1.8) we let

$$\chi_s^\pm(x) = \hat{\chi}(s(x-x_\nu))\mathbf{1}_{\mathbb{R}_+}(\pm(x-x_\nu)) \text{ with } s = [\beta x_\nu]^{1/3},$$

where $\hat{\chi}$ is defined by (3.1.7). An integration by parts yields, as in (3.1.24),

$$\|(\chi_s^\pm v)'\|_2^2 - \|(\chi_s^\pm)'v\|_2^2 - \mu\beta\|\chi_s^\pm v\|_2^2 = \Re\langle\chi_s^\pm v, \chi_s^\pm f\rangle,$$

from which we conclude, given that $\mu\beta \leq Cs^2$

$$\|(\chi_s^\pm v)'\|_2^2 \leq \|\chi_s^\pm v\|_1\|f\|_\infty + \hat{C}s^2\|v\|_2^2.$$

Furthermore, (see (3.1.25)), we have that

$$\mp\beta\|\,|U-\nu|^{1/2}\chi_s^\pm v\|_2^2 + 2\Im\langle(\chi_s^\pm)'v, (\chi_s^\pm v)'\rangle = \Im\langle\chi_s^\pm v, \chi_s^\pm f\rangle, \tag{3.3.6}$$

and hence, with the aid of above, we obtain that

$$\||U-\nu|^{1/2}\hat{\chi}_s v\|_2^2 \leq C\beta^{-1}([\beta x_\nu]^{2/3}\|v\|_2^2 + \|\chi_s^\pm v\|_1\|f\|_\infty).$$

By (3.1.3a), we then have

$$\||U-\nu|^{1/2}\chi_s^\pm v\|_2 \leq C(x_\nu^{-1/2}\beta^{-1}\|f\|_\infty + \beta^{-1/2}\|v\|_1^{1/2}\|f\|_\infty^{1/2}). \tag{3.3.7}$$

Let $\tilde{\chi}_s = \sqrt{1-(\chi_s^+)^2-(\chi_s^-)^2}$. Since $s = [\beta x_\nu]^{1/3}$ it holds that

$$\operatorname{supp}\tilde{\chi}_s \subset \Big[x_\nu - \frac{1}{2}(\beta x_\nu)^{-1/3}, x_\nu + \frac{1}{2}(\beta x_\nu)^{-1/3}\Big).$$

As

$$\||U-\nu|^{1/2}\tilde{\chi}_s v\|_2 \leq C\,[\beta x_\nu]^{-1/6}x_\nu^{1/2}\|\tilde{\chi}_s v\|_2 \leq C\beta^{-1/6}x_\nu^{1/3}\|v\|_2,$$

we may use (3.1.3a) once again to obtain

$$\||U-\nu|^{1/2}\tilde{\chi}_s v\|_2 \leq Cx_\nu^{-\frac{1}{2}}\beta^{-1}\|f\|_\infty.$$

Combining the above with (3.3.7) yields

$$\| |U-\nu|^{1/2} v\|_2 \leq C\big(x_\nu^{-1/2}\beta^{-1}\|f\|_\infty + \beta^{-1/2}\|v\|_1^{1/2}\|f\|_\infty^{1/2}\big). \tag{3.3.8}$$

Substituting (3.3.8) into (3.3.5) yields, for any $\eta > 0$,

$$\begin{aligned}\|v\|_1 &\leq \frac{C}{x_\nu^{1/2}}[\log(\beta)]^{1/2}\big(\beta^{-1}x_\nu^{-1/2}\|f\|_\infty + \beta^{-1/2}\|v\|_1^{1/2}\|f\|_\infty^{1/2}\big) \\ &\leq \frac{C}{x_\nu^{1/2}}[\log(\beta)]^{1/2}\Big(\beta^{-1}x_\nu^{-1/2}\|f\|_\infty + \beta^{-1/2}\Big(\frac{1}{\eta}\|f\|_\infty + \eta\|v\|_1\Big)\Big).\end{aligned}$$

Choosing $\eta > 0$ such that

$$C\,\eta\,\beta^{-1/2}x_\nu^{-1/2}[\log(\beta)]^{1/2} = \frac{1}{2},$$

we obtain

$$\|v\|_1 \leq \hat{C}\,\frac{1}{\beta x_\nu}\log(\beta)\|f\|_\infty. \tag{3.3.9}$$

Combining the above with (3.3.3), completes the proof of the proposition. ∎

If $U-\nu \neq 0$ in $[0,1]$ it can be easily verified (see (3.1.75)) that

$$\|(-d^2/dx^2 + i\beta[U-\nu])^{-1}\| \lesssim \beta^{-1}.$$

In contrast, when $U(x) = \nu$ for some $x \in (0,1)$ the best estimate we can obtain (see (3.1.3)) is

$$\|(-d^2/dx^2 + i\beta[U-\nu])^{-1}\| \lesssim \beta^{-2/3}.$$

The zero of $U-\nu$ at $x = x_\nu$, thus, has a significant effect on the resolvent norm. Nevertheless, if $f(x_\nu) = 0$ and f is small in the neighbourhood of x_ν one may expect that $\|(-d^2/dx^2 + i\beta[U-\nu])^{-1}f\|_2$ would be smaller in that case.

This heuristic argument is manifested, more precisely, in the following proposition.

Proposition 3.3.2. *Let $U \in C^2([0,1])$ satisfy (2.1.3). There exist $\Upsilon > 0$, $C > 0$, $a > 0$, and $\beta_0 > 0$ such that, for $\beta \geq \beta_0$, $\nu \leq U(0) - a\beta^{-1/2}$, $\mu < \Upsilon\mathfrak{J}_\nu{}^{2/3}\beta^{-1/3}$, and $f \in L^2(0,1)$ such that $(x-x_\nu)^{-1}f \in L^2(0,1)$, we have*

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}f\|_2 \leq C(\mathfrak{J}_\nu\beta)^{-1}\left\|\frac{f}{x-x_\nu}\right\|_2 \tag{3.3.10}$$

and

$$\left\|\frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}f\right\|_2 \leq C\beta^{-1/2}\left\|\frac{f}{x-x_\nu}\right\|_2. \tag{3.3.11}$$

Proof. Let $(v, f, \lambda) \in D(\mathcal{L}_\beta^{\mathfrak{N}}) \times L^2(0,1) \times \mathbb{C}$ satisfy $(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)v = f$.

Step 1. We prove (3.3.10).

Set

$$f = (U - \nu)g.$$

Given that $\mu < \Upsilon \Im_\nu{}^{2/3}\beta^{-1/3}$, it follows from (3.1.3) that there exists $u \in D(\mathcal{L}_\beta^{\mathfrak{N}})$ satisfying

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)u = g. \tag{3.3.12}$$

Let

$$w = (U - \nu)u.$$

Then, it holds that

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)w = f - 2U'u' - U''u.$$

Consequently,

$$v = w + (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(2U'u' + U''u). \tag{3.3.13}$$

We now recall (3.1.69)

$$|U'| \le C|U - U(0)|^{1/2} \le C(|U - \nu|^{1/2} + x_\nu).$$

By the above and (3.1.3a) it holds that

$$\begin{aligned}\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(2U'u' + U''u)\|_2 \\ \le C\,[\Im_\nu\beta]^{-2/3}\big(x_\nu\|u'\|_2 + \||U-\nu|^{1/2}u'\|_2 + \|u\|_2\big).\end{aligned} \tag{3.3.14}$$

Let $\chi_\nu^\pm$ be defined by (3.1.8). Clearly, in view of (3.1.9) and the fact that $U'(0) = 0$,

$$\||U-\nu|^{1/2}u'\|_2 \le \|\chi_\nu^+|U-\nu|^{1/2}u'\|_2 + \|\chi_\nu^-|U-\nu|^{1/2}u'\|_2 + Cx_\nu\|u'\|_2. \tag{3.3.15}$$

By (3.1.77), applied to the pair (u, g), (3.3.15), and (3.1.3b) we then have

$$x_\nu\|u'\|_2 + \||U-\nu|^{1/2}u'\|_2 \le C(\beta^{-1/2} + x_\nu^{2/3}\beta^{-1/3})\|g\|_2. \tag{3.3.16}$$

Using (3.1.3a) together with (3.3.14) and (3.3.12) then yields, as $x_\nu \ge \beta^{-1/4}$,

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(2U'u' + U''u)\|_2 \le C\beta^{-1}\|g\|_2. \tag{3.3.17}$$

By (3.1.75), applied to the pair (u, g), it holds that

$$\|w\|_2 \le C\beta^{-1}\,\|g\|_2. \tag{3.3.18}$$

Substituting the above together with (3.3.17) into (3.3.13) yields

$$\|v\|_2 \le C\beta^{-1}\|g\|_2.$$

Since

$$|U(x)-\nu| \geq \frac{1}{C}x_\nu|x-x_\nu|,$$

it holds that

$$\|g\|_2 \leq C x_\nu^{-1}\|(x-x_\nu)^{-1}f\|_2,$$

and hence, we can now conclude (3.3.10) from the above and (3.3.18).

Step 2. We prove (3.3.11).

Taking the real part of the scalar product with $\langle v, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)v\rangle$ we write

$$\|v'\|_2^2 = \mu\beta\|v\|_2^2 + \Re\langle (x-x_\nu)v, (x-x_\nu)^{-1}f\rangle.$$

From here we deduce that

$$\|v'\|_2^2 \leq \mu_+\beta\|v\|_2^2 + |\langle (x-x_\nu)v, (x-x_\nu)^{-1}f\rangle|, \tag{3.3.19}$$

where

$$\mu_+ = \max(\mu, 0). \tag{3.3.20}$$

Since $|x-x_\nu| \leq C\,|U-\nu|^{1/2}$ we obtain by (3.3.10), as $x_\nu \geq \beta^{-1/4}$ and $\mu_+ \leq C\,\mathfrak{J}_\nu^{2/3}\beta^{-1/3}$,

$$\|v'\|_2^2 \leq C\big(\beta^{-1}\|(x-x_\nu)^{-1}f\|_2 + \||U-\nu|^{1/2}v\|_2\|(x-x_\nu)^{-1}f\|_2\big). \tag{3.3.21}$$

By (3.1.25) it holds that

$$\beta\||U-\nu|^{1/2}\chi_\nu^{\pm}v\|_2^2 \leq C x_\nu^{-1}\|v\|_2\|v'\|_2 + \|(x-x_\nu)v\|_2\|(x-x_\nu)^{-1}f\|_2.$$

Hence, in view of (3.1.9),

$$\beta\||U-\nu|^{1/2}v\|_2^2 \leq C(x_\nu^{-1}\|v\|_2\|v'\|_2 + \beta x_\nu^2\|v\|_2^2) + \|(x-x_\nu)v\|_2\|(x-x_\nu)^{-1}f\|_2.$$

Since by (3.3.10) it holds that

$$\beta x_\nu^2\|v\|_2^2 \leq C\beta^{-1}\|(x-x_\nu)^{-1}f\|_2^2,$$

we may obtain that

$$\begin{aligned}\beta\||U-\nu|^{1/2}v\|_2^2 \leq C\big(&x_\nu^{-2}\beta^{-1}\|(x-x_\nu)^{-1}f\|_2\|v'\|_2 \\ &+ \|(U-\nu)^{1/2}v\|_2\|(x-x_\nu)^{-1}f\|_2 + \beta^{-1}\|(x-x_\nu)^{-1}f\|_2^2\big).\end{aligned}$$

From the above we conclude, as $x_\nu \geq \beta^{-1/4}$,

$$\||U-\nu|^{1/2}v\|_2^2 \leq C(\beta^{-1}\|v'\|_2^2 + \beta^{-2}\|(x-x_\nu)^{-1}f\|_2^2). \tag{3.3.22}$$

Substituting the above into (3.3.21) yields

$$\|v'\|_2 \leq C\beta^{-1/2}\|(x-x_\nu)^{-1}f\|_2,$$

verifying (3.3.11). ∎

We now seek an estimate for $(\mathcal{L}_\beta - \beta\lambda)^{-1}$ in $\mathcal{L}(H^1, L^1)$. To this end we write $f = f - f(x_\nu) + f(x_\nu)$, and estimate first $(\mathcal{L}_\beta - \beta\lambda)^{-1}(f - f(x_\nu))$ using (3.3.10). Then, we estimate $(\mathcal{L}_\beta - \beta\lambda)^{-1} f(x_\nu)$ by observing first that the leading order term is $-if(x_\nu)[\beta(U + i\lambda)]^{-1}$ for $|\mu| > x_\nu^{2/3}\beta^{-1/3}$.

Proposition 3.3.3. *Let* $U \in C^2([0,1])$ *satisfy* (2.1.3). *Then there exist* $\Upsilon > 0$, $C > 0$, $a > 0$, *and* $\beta_0 > 0$ *such that, for* $\beta \geq \beta_0$, $U(0) - \nu > a\beta^{-1/2}$, *and* $\mu \leq \mathfrak{J}_\nu{}^{2/3}\Upsilon\beta^{-1/3}$ *and* $f \in H^1(0,1)$ *we have*

$$\left\| (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1} f + i\frac{f(x_\nu)}{\beta[U - \nu - i\max(-\mu, x_\nu^{2/3}\beta^{-1/3})]} \right\|_1 \leq C[\mathfrak{J}_\nu\beta]^{-1}\|f\|_{1,2}. \tag{3.3.23}$$

Proof. Let $u = (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1} f$.

Step 1. We prove (3.3.23) in the case $-\mu \leq x_\nu^{2/3}\beta^{-1/3}$.

We apply the decomposition

$$v = u + i\frac{f(x_\nu)}{\beta(U - \nu - ix_\nu^{2/3}\beta^{-1/3})}. \tag{3.3.24}$$

Then,

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)v = f + i\beta^{-1} f(x_\nu)((\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)(U - \nu - ix_\nu^{2/3}\beta^{-1/3})^{-1}).$$

We next observe that

$$\begin{aligned}
&(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)(U - \nu - ix_\nu^{2/3}\beta^{-1/3})^{-1} \\
&= \beta\frac{(-\lambda + iU)}{(U - \nu - ix_\nu^{2/3}\beta^{-1/3})} - \frac{2|U'|^2}{(U - \nu - ix_\nu^{2/3}\beta^{-1/3})^3} + \frac{U''}{(U - \nu - ix_\nu^{2/3}\beta^{-1/3})^2},
\end{aligned}$$

and that

$$\begin{aligned}
&-i(\lambda - iU)\left(U - \nu - ix_\nu^{2/3}\beta^{-1/3}\right)^{-1} + 1 \\
&\qquad = (U - \nu - ix_\nu^{2/3}\beta^{-1/3})^{-1}(-i[(\mu + i\nu) - U] + U - \nu - ix_\nu^{2/3}\beta^{-1/3}).
\end{aligned}$$

Consequently, it holds that

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)v = f - f(x_\nu) + f(x_\nu)h, \tag{3.3.25}$$

where

$$\begin{aligned}
h = i&\frac{U''}{\beta(U - \nu - ix_\nu^{2/3}\beta^{-1/3})^2} - 2i\frac{|U'|^2}{\beta(U - \nu - ix_\nu^{2/3}\beta^{-1/3})^3} \\
&- i\frac{\mu + x_\nu^{2/3}\beta^{-1/3}}{U - \nu - ix_\nu^{2/3}\beta^{-1/3}}.
\end{aligned} \tag{3.3.26}$$

Since for $U \in C^2([0,1])$ satisfying (2.1.3), we have

$$|U(x) - \nu| \geq \frac{1}{C} x_\nu |x - x_\nu|, \tag{3.3.27}$$

we may conclude that for $k > 1$,

$$\int_0^1 \frac{dx}{|U - \nu - i x_\nu^{2/3} \beta^{-1/3}|^k} \leq C \int_0^1 \frac{dx}{x_\nu^k |x - x_\nu|^k + x_\nu^{2k/3} \beta^{-k/3}} \leq \hat{C} x_\nu^{-k} [\beta x_\nu]^{\frac{k-1}{3}}. \tag{3.3.28}$$

For later reference we mention that for $k = 1$

$$\begin{aligned} &\int_0^1 \frac{dx}{|U - \nu + i x_\nu^{2/3} \beta^{-1/3}|} \\ &\quad \leq C \Big[\frac{1}{x_\nu} \int_0^{2x_\nu} \frac{dx}{|x - x_\nu| + x_\nu^{-1/3} \beta^{-1/3}} + \int_{2x_\nu}^1 \frac{dx}{|x - x_\nu|^2} \Big] \\ &\quad \leq \frac{\hat{C}}{x_\nu} \big[\log(x_\nu^{4/3} \beta^{1/3}) + 1 \big]. \end{aligned} \tag{3.3.29}$$

Using (3.3.28) and the fact that $x_\nu \geq \frac{1}{C} \beta^{1/4}$, together with (3.1.69), it can be verified that there exist positive C and β_0 such that for $\beta \geq \beta_0$ and $|\mu| \leq x_\nu^{2/3} \beta^{-1/3}$,

$$\|h\|_2 \leq C\, [\beta x_\nu]^{-1/6}. \tag{3.3.30}$$

Consequently, by (3.3.1)

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1} h\|_1 \leq C(\beta x_\nu)^{-1}. \tag{3.3.31}$$

By (3.3.10) and Hardy's inequality (2.2.8) it holds that

$$\begin{aligned} &\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(f - f(x_\nu))\|_1 \leq \|(\mathcal{L}_\beta^{\mathfrak{D}} - \beta\lambda)^{-1}(f - f(x_\nu))\|_2 \\ &\quad \leq C(\beta x_\nu)^{-1} \left\| \frac{f - f(x_\nu)}{x - x_\nu} \right\|_2 \leq \hat{C}\, (\beta x_\nu)^{-1}\, \|f'\|_2. \end{aligned}$$

Substituting the above, together with (3.3.31) into (3.3.25) yields

$$\|v\|_1 \leq C(\beta x_\nu)^{-1}(\|f'\|_2 + |f(x_\nu)|) \leq \hat{C}\, (\beta x_\nu)^{-1}\, \|f\|_{1,2}.$$

Step 2. We prove (3.3.23) in the case $\mu \leq -x_\nu^{2/3} \beta^{-1/3}$.

In this case we consider instead the decomposition

$$v = u + i \frac{f(x_\nu)}{\beta(U + i\lambda)}.$$

Then we obtain

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda) v = f + i\beta^{-1} f(x_\nu)((\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)(U + i\lambda)^{-1}).$$

As

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)(U + i\lambda)^{-1} = i\beta - \frac{2|U'|^2}{(U + i\lambda)^3} + \frac{U''}{(U + i\lambda)^2},$$

we obtain that

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)v = f - f(x_\nu) + f(x_\nu)\tilde{h}, \tag{3.3.32}$$

where

$$\tilde{h} = i\frac{U''}{\beta(U + i\lambda)^2} - 2i\frac{|U'|^2}{(U + i\lambda)^3} \tag{3.3.33}$$

and proceed in a similar manner using the lower bound for $|\mu|$. ■

An immediate consequence of Proposition 3.3.3 now follows by using (3.3.29).

Corollary 3.3.4. *Under the conditions of Proposition 3.3.3, it holds (with sufficiently large a) that*

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1} f\|_1 \leq C\,[\Im_\nu\beta]^{-1}\big(\|f\|_{1,2} + |f(x_\nu)|\log(x_\nu^{4/3}\beta^{1/3})\big). \tag{3.3.34}$$

We conclude this section by another auxiliary estimate which will be useful in Sections 5.10 and 5.11.

Proposition 3.3.5. *Let $U \in C^3([0,1])$ satisfying (2.1.3), $a > 0$ and $\Upsilon < \sqrt{-U''(0)}/2$. Then there exist $C > 0$, $\beta_0 > 0$ such that, for all $\beta \geq \beta_0$,*

$$\begin{aligned}\sup_{\substack{\mu \leq \Upsilon\beta^{-1/2} \\ \nu < U(0) + a\beta^{-1/2}}} &\Big(\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U - \nu)f\|_2 \\ &\quad + \beta^{-1/2}\|\frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U - \nu)f\|_2\Big) \\ &\leq C\beta^{-1}\|f\|_2.\end{aligned} \tag{3.3.35}$$

Note that if we apply (3.3.10) (for $\nu < U(0) - a_0\beta^{-1/2}$ with some sufficiently large a_0) or (3.2.1b) (in the case $|\nu - U(0)| \leq a_0\beta^{-1/2}$) we obtain that

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U - \nu)f\|_2 \leq C\beta^{-3/4}[1 + x_\nu\beta^{1/4}]^{-1}\|f\|_2,$$

which is weaker than (3.3.35).

Proof. Let $v \in D(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}})$ such that

$$(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)v = (U - \nu)f. \tag{3.3.36}$$

Let $w \in D(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}})$ satisfy

$$(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)w = f.$$

It can be easily verified that

$$(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)([U-\nu]w) = (U-\nu)f - 2U'w' - U''w.$$

Hence,

$$v = (U-\nu)w + (\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(2U'w' + U''w). \tag{3.3.37}$$

Let $a_0 > 0$. Consider first the case where $U(0) - a_0\beta^{-1/2} < \nu < U(0) + a_0\beta^{-1/2}$. By (3.1.71) (with v replaced by w) and since by (2.1.3) we have $|U'(x)| \leq x$, it holds that

$$\beta\|(U-\nu)w\|_2^2 \leq C(\beta^{-1}\|f\|_2^2 + [\|(x-x_\nu)w\|_2 + x_\nu\|w\|_2]\|w'\|_2).$$

We can now use (3.2.35), given that $x_\nu \leq C\beta^{-1/4}$, to obtain

$$\beta\|(U-\nu)w\|_2^2 \leq C(\beta^{-1}\|f\|_2^2 + [\beta^{-3/4}\|f\|_2 + \beta^{-1/4}\|w\|_2]\|w'\|_2).$$

We may now apply (3.2.1a) to the pair (w, f) to conclude that

$$\|(U-\nu)w\|_2 \leq C\beta^{-1}\|f\|_2. \tag{3.3.38}$$

In (3.1.75) we have established that there exists $a > 0$ such that (3.3.38) holds also whenever $\nu < U(0) - a\beta^{-1/2}$ under the conditions of Proposition 3.1.1.

Next, we use once again either (3.2.1a) (in the case when $|\nu - U(0)| < a\beta^{-1/2}$) or (3.1.3a) (in the case when $\nu < U(0) - a\beta^{-1/2}$), with f replaced by $U''w$, to obtain that

$$\begin{aligned}\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U''w)\|_2 &\leq C\beta^{-1/2}[1 + x_\nu\beta^{1/4}]^{-2/3}\|w\|_2 \\ &\leq \tilde{C}\beta^{-1}[1 + x_\nu\beta^{1/4}]^{-4/3}\|f\|_2 \\ &\leq \hat{C}\beta^{-1}\|f\|_2.\end{aligned} \tag{3.3.39}$$

Finally, we write

$$\begin{aligned}\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U'w')\|_2 \leq\ & \|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}([U' - U'(x_\nu)]w')\|_2 \\ &+ \|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U'(x_\nu)w')\|_2.\end{aligned}$$

For the second term on the right-hand side we have by (3.2.1a) and (3.1.3)

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U'(x_\nu)w')\|_2 \leq C\beta^{-1/2}x_\nu[1 + x_\nu\beta^{1/4}]^{-2/3}\|w'\|_2 \leq \hat{C}\beta^{-1}\|f\|_2.$$

For the first term we use instead either (3.2.1b) or (3.3.10) with $f = [U' - U'(x_\nu)]w'$ and then (3.2.1a) and (3.1.3) for the second one to obtain

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}([U' - U'(x_\nu)]w')\|_2 \leq C\beta^{-3/4}\left\|\frac{U' - U'(x_\nu)}{x - x_\nu}w'\right\|_2 \leq \hat{C}\beta^{-1}\|f\|_2.$$

Hence,

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U'w')\|_2 \leq C\beta^{-1}\|f\|_2. \tag{3.3.40}$$

By (3.3.37), (3.3.38), (3.3.39), (3.3.39), and (3.3.40), we then conclude

$$\|v\|_2 \leq C\beta^{-1}\|f\|_2. \tag{3.3.41}$$

The estimate of v' in (3.3.35) follows immediately from the identity

$$\|v'\|_2^2 = \mu\beta\|v\|_2^2 + \Re\langle v, (U - \nu)f\rangle,$$

together with (3.3.41) and the fact that $\mu \leq \Upsilon\beta^{-1/2}$. ∎

Chapter 4

No-slip Schrödinger operators

4.1 Preliminaries

Given the fact that $-\phi'' + \alpha^2\phi$ does not necessarily belong to $D(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}})$, we establish, in this chapter, resolvent estimates for the same differential operator but with one boundary condition replaced by an integral condition which will be satisfied by $-\phi'' + \alpha^2\phi$ (cf. see also [3, Section 6] and the discussion around equation (1.2.8) in the introduction.)

This chapter follows the same path as in [3, Section 6] but this time for symmetric flows in $(-1, +1)$ (so that $U'(0) = 0$), to which end we consider the interval $(0, 1)$ and a Neumann condition at $x = 0$.

Let

$$\mathcal{L}_\beta^\zeta = -\frac{d^2}{dx^2} + i\beta U,$$

be defined on

$$D(\mathcal{L}_\beta^\zeta) = \{u \in H^2(0,1) \mid \langle \zeta, u\rangle = 0, u'(0) = 0\}, \tag{4.1.1}$$

where $\zeta \in H^2(0, 1)$.

We will later (see (4.7.1)) confine the discussion to the case where $\zeta_\alpha(x) = \cosh(\alpha x)/\cosh\alpha$.

More precisely, we introduce

$$\mathfrak{U}_0 := \{\zeta \in H^2(0,1) \mid \zeta'(0) = 0, \zeta(1) = 1\} \tag{4.1.2}$$

and for $\beta \geq 0$, $\gamma > 0$, $\theta > 0$ and $\lambda \in \mathbb{C}$, the subset

$$\mathfrak{U}_1(\beta,\lambda,\gamma,\theta) = \{\zeta \in \mathfrak{U}_0, \|\zeta\|_\infty \leq \beta^\gamma, \|\zeta'\|_{L^2(1-\beta^{-\gamma},1)} \leq \theta\beta^{1/6}\lambda_\beta^{1/4}\}, \tag{4.1.3}$$

where

$$\lambda_\beta = 1 + |\lambda|\beta^{1/3}. \tag{4.1.4}$$

Let Ai denote Airy's function (See [1, Section 10.4]) and A_0 the generalized Airy function

$$\mathbb{C} \ni z \mapsto A_0(z) = e^{i\pi/6}\int_z^{+\infty} \mathrm{Ai}(e^{i\pi/6}t)\,dt. \tag{4.1.5}$$

(See [1, 10.4] or in [3, Appendix A.2]). We then set

$$\mathcal{S} = \{z \mid A_0(iz) = 0\}.$$

It is established in [3, Appendix A.2] that $\mathcal{S}$ (which is denoted there by $\mathcal{S}_\lambda$) is non-empty. In [22] (cf. also [3, Appendix A]) it is shown that

$$\vartheta_1^r := \inf_{z\in\mathcal{S}} \Re z > 0. \tag{4.1.6}$$

4.2 Resolvent estimates for $|U(0) - \nu| \gg \beta^{-1/2}$

We can now state the following proposition.

Proposition 4.2.1. *Let $U \in C^2([0,1])$ satisfy (2.1.3). Let further $\gamma < 1/4$. Then, there exist $\Upsilon > 0$, $\beta_0 > 0$, $a > 0$, and $\theta_0 > 0$ such that, for all $\beta \geq \beta_0$, $\theta \in (0, \theta_0]$, and $\lambda \in \mathbb{C}$ satisfying*

$$\max(|U(0) - \nu|^{-1/3}, 1)\beta^{1/3}\Re\lambda \leq \Upsilon \tag{4.2.1}$$

and

$$|U(0) - \nu| > a\beta^{-1/2}, \tag{4.2.2}$$

there exists a constant $C > 0$ such that, for any $f \in H^1(0,1)$, $\zeta \in \mathfrak{U}_1(\beta,\lambda,\gamma,\theta)$, and $v \in D(\mathcal{L}_\zeta^\beta)$ satisfying

$$(\mathcal{L}_\zeta^\beta - \beta\lambda)v = f, \tag{4.2.3}$$

it holds that

$$\begin{aligned}|v(1)| &\leq C\,\beta^{1/3}\,\lambda_\beta^{1/2}\,\|\zeta\|_\infty \\ &\times \min\big([x_\nu\beta]^{-5/6}\|f\|_2, [|U(0) - \nu|^{1/2}\beta]^{-1}[\|f\|_{1,2} + |f(x_\nu)|\log(1 + x_\nu\beta^{1/4})]\big),\end{aligned} \tag{4.2.4}$$

where x_ν is defined by (2.4.5), and that

$$|v(1)| \leq C\,\beta^{1/3}\,\lambda_\beta^{1/2}\,\|\zeta(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}f\|_1. \tag{4.2.5}$$

Furthermore, for any $p > 1$ and $\nu < U(0) - a\beta^{-1/2}$ it holds that

$$|v(1)| \leq C\,\lambda_\beta^{1/2}\beta^{1/3}[x_\nu\beta]^{-1}\big[\|\zeta\|_\infty\|f\|_{1,2} + \|\zeta\|_{1,p}(1 + |\log[\nu + i\,\mathfrak{m}]|)\,|f(x_\nu)|\big], \tag{4.2.6a}$$

where

$$\mathfrak{m} := -\max(-\mu, x_\nu^{2/3}\beta^{-1/3}). \tag{4.2.6b}$$

Finally, if in addition $(x - x_\nu)^{-1}f \in L^2(0,1)$ and $\nu < U(0)$, then we have

$$|v(1)| \leq C\beta^{-2/3}\,\lambda_\beta^{1/2}\,\|\zeta\|_\infty\,|U(0) - \nu|^{-1/4}\Big\|\frac{f}{x - x_\nu}\Big\|_2. \tag{4.2.7}$$

Note that since $\mathfrak{J}_\nu = |U'(x_\nu)| \sim x_\nu \sim [U(0) - \nu]^{1/2}$ for $\nu < U(0)$, (4.2.1) is similar to the condition set on $\Re\lambda$ in (3.1.19). Here, we use a different notation since we need to address the case $\nu > U(0)$ as well.

Proof. As in [3, equation (6.20)] we begin by a decomposition of v into a boundary term associated with $x = 1$ and a solution of the same equation satisfying a Dirichlet condition at $x = 1$. We estimate the boundary term by using a linear approximation of U near $x = 1$ (recall that by (2.1.3) $U(1) = 0$ and $U'(1) = -1$). Let for $\beta > 0$ and $\lambda \in \mathbb{C}$,

$$\tilde{\psi}_{\lambda,\beta}(x) = e^{-i\pi/6} \frac{\mathrm{Ai}\big(\beta^{1/3} e^{-i\pi/6}[(1-x) - i\lambda]\big)}{\overline{A_0(i\beta^{1/3}\bar{\lambda})}}, \tag{4.2.8}$$

which is (note that $\tilde{\psi} = \psi_+$, with $J_+ = 1$, in [3, equation (6.8b)]) the decaying solution as $x \to -\infty$ of

$$\begin{cases} \Big(-\dfrac{d^2}{dx^2} + i\beta[(x-1) + i\lambda]\Big)\tilde{\psi} = 0 & \text{in } (-\infty, 1), \\ \displaystyle\int_{-\infty}^{1} \tilde{\psi}(x)\, dx = \beta^{-1/3}. \end{cases} \tag{4.2.9}$$

Note further that by (4.1.6) $\tilde{\psi}_{\lambda,\beta}$ is well defined whenever $\Re\lambda < \vartheta_1^r$. For later reference we recall from [3, equation (8.87)] that for any $\hat{\delta}_1$, there exists $C > 0$ and β_0 such that, for $\Re\lambda \le (\vartheta_1^r - \hat{\delta}_1)\beta^{-1/3}$ and $\beta \ge \beta_0$,

$$\frac{1}{C}\lambda_\beta^{1/2} \le |\tilde{\psi}_{\lambda,\beta}(1)| \le C\,\lambda_\beta^{1/2}. \tag{4.2.10}$$

To guarantee that the Neumann condition at $x = 0$ is satisfied we further set

$$\psi_{\lambda,\beta}(x) = \tilde{\psi}_{\lambda,\beta}(x)\chi(1-x), \tag{4.2.11}$$

where χ is given by (2.6.20), which we recall here for the convenience of the reader

$$\chi(t) = \begin{cases} 1 & t < 1/2, \\ 0 & t > 3/4. \end{cases}$$

Consequently, $\psi_{\lambda,\beta}$ is supported on $[1/4, +1]$ and $\psi_{\lambda,\beta} = \tilde{\psi}_{\lambda,\beta}$ on $[1/2, 1]$. We omit the subscript (λ, β) when no ambiguity is expected. Consider a pair $(\tilde{v}, h) \in L^2(0,1) \times D(\mathcal{L}_\beta^{\mathfrak{N}})$ such that

$$h = \Big(-\frac{d^2}{dx^2} + i\beta(U + i\lambda)\Big)\psi \tag{4.2.12}$$

and

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)\,\tilde{v} = h. \tag{4.2.13}$$

We note that the assumptions of the proposition, and in particular (4.2.1) and (4.2.2) allow us to apply Propositions 3.1.1, 3.3.1, 3.3.2, and 3.3.3 throughout the proof.

Step 1. We prove that

$$\beta^{-1/6}\|\tilde{v}\|_2 + \|\tilde{v}\|_1 \leq C \max(\beta^{-2/3}\lambda_\beta^{-3/4}, \beta^{-5/3}). \tag{4.2.14}$$

By (4.2.9) it holds that

$$h = i\beta\,[U - (1-x)]\,\psi + \chi''(1-x)\tilde{\psi} - 2\chi'(1-x)\tilde{\psi}' \quad \text{in } (0,1). \tag{4.2.15}$$

We note that by [3, equation (6.17)] there exists $\Upsilon > 0$ (in the statement of the proposition) such that whenever $\beta^{1/3}\Re\lambda \leq \Upsilon$

$$\|(1-x)^k\,\psi_{\lambda,\beta}\|_2 \leq \|(1-x)^k\,\tilde{\psi}_{\lambda,\beta}\|_2 \leq C\,\lambda_\beta^{\frac{1-2k}{4}}\,\beta^{-(1+2k)/6} \quad \text{for } k \in [0,4]. \tag{4.2.16}$$

Furthermore, since by [3, Proposition A.1] (or more precisely by [3, equations (A.4), (A.6), (A.19), (A.20)] and the display below [3, equation (A.29)]) it holds that

$$\Psi(x,\lambda) := \frac{\mathrm{Ai}(e^{i\pi/6}[x + i\lambda])}{\mathrm{Ai}(e^{i2\pi/3}\lambda)} = \Big[1 + \frac{i}{4}((-\lambda)^{-1/2}x^2 - \lambda^{-1}x)\Big]e^{-(-\lambda)^{1/2}x} + w_1,$$

where

- for $\mu_0 > 0$
$$\lambda \in \mathcal{V}(\mu_0) := \{\Re\lambda \leq \mu_0\} \cap \{|\lambda| > 3\mu_0\},$$
- the square root of $-\lambda$ is chosen such that
$$\Re(-\lambda)^{1/2} > 0,$$
- and the remainder $w_1 \in H^1(\mathbb{R}_+)$ satisfies
$$\|x^4 w_1\|_{L^2(\mathbb{R}_+)} + \|x^4 w_1'\|_{L^2(\mathbb{R}_+)} \leq C\,|\lambda|^{-9/4}.$$

Consequently, for all $\lambda \in \mathcal{V}(\mu_0)$ it holds by [3, equation (A.20)] that

$$\|x^4\Psi\|_2 + |\lambda|^{-1/2}\|x^4\Psi'\|_2 \leq C\,|\lambda|^{-9/4}.$$

Let $0 < \mu_0 < \hat{\kappa}_1$ (given by (3.1.2)). Then, there exists $C(\mu_0) > 0$ such that

$$\sup_{|\lambda|\leq 3\mu_0} \|x^4\,\mathrm{Ai}(e^{i\pi/6}[x + i\lambda])\|_{1,2} \leq C,$$

and, since all the zeroes of $\mathrm{Ai}(e^{i2\pi/3}\lambda)$ are located in the half-plane $\Re\lambda \geq \hat{\kappa}_1$, we have, since $\mu_0 < \hat{\kappa}_1$, that

$$\sup_{\substack{|\lambda|\leq 3\mu_0 \\ \Re\lambda < \mu_0}} \left|\frac{1}{\mathrm{Ai}(e^{i2\pi/3}\lambda)}\right| \leq C.$$

Consequently, by the above inequalities, relying on the sole condition that $\Re\lambda \le \mu_0$, there exists $C > 0$ such that

$$\|x^4\Psi\|_2 + [1+|\lambda|^2]^{-1/4}\|x^4\Psi'\|_2 \le C\,[1+|\lambda|^2]^{-9/8}.$$

Note that

$$\frac{\tilde\psi_{\lambda,\beta}(x)}{\tilde\psi_{\lambda,\beta}(1)} = \Psi(\beta^{1/3}(1-x),\beta^{1/3}\lambda).$$

Using dilation, translation, and (4.2.10) the above yields

$$\|(1-x)^k\,\tilde\psi'_{\lambda,\beta}\|_{L^2(-\infty,1)} \le C\,\lambda_\beta^{\frac{3-2k}{4}}\,\beta^{(1-2k)/6}. \tag{4.2.17}$$

Hence, rewriting (4.2.15) in the form

$$\begin{aligned} h = i\beta[U-(1-x)]\psi + ((1-x)^{-2}\chi''(1-x))(1-x)^2\tilde\psi \\ -2((1-x)^{-3}\chi'(1-x))(1-x)^3\tilde\psi', \end{aligned}$$

we obtain

$$|h(x)| \le C[\beta(1-x)^2|\psi(x)| + (1-x)^3|\tilde\psi'(x)|], \tag{4.2.18}$$

and hence by (4.2.16) with $k=2$ and (4.2.17) with $k=3$,

$$\|h\|_2 \le C\,\beta^{1/6}\lambda_\beta^{-3/4}. \tag{4.2.19}$$

Recall from (2.4.5) the definition of x_ν:

$$U(x_\nu) = \nu \text{ for } 0<\nu<U(0),\; x_\nu = 1 \text{ if } \nu \le 0 \text{ and } x_\nu = 0 \text{ if } \nu > U(0).$$

We split the proof of (4.2.14) into two steps, depending on the value of x_ν.

Step 1.1. $x_\nu > 1/4$. In this case we have by (4.2.13), (3.1.3a), (3.1.75), and (3.3.1) (note that $\mathfrak{J}_\nu \ge \mathfrak{J}_{U^{-1}(1/4)} > 0$ in this case) that

$$\beta^{-1/3}\|\tilde v\|_2 + \|(U-\nu)\tilde v\|_2 + \beta^{-1/6}\|\tilde v\|_1 \le C\,\beta^{-1}\,\|h\|_2.$$

By (4.2.19) we then obtain

$$\beta^{-1/3}\|\tilde v\|_2 + \|(U-\nu)\tilde v\|_2 + \beta^{-1/6}\|\tilde v\|_1 \le C\beta^{-5/6}\lambda_\beta^{-3/4}, \tag{4.2.20}$$

readily yielding (4.2.14).

Step 1.2. $0 < x_\nu \le 1/4$. We recall that $x_\nu \ge C a^{1/2}\beta^{-1/4}$. We write

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)(\chi\tilde v) = \chi h - 2\chi'\tilde v' - \tilde\chi''\tilde v,$$

to obtain by (3.1.3a) and (3.3.1)

$$\|\chi\tilde v\|_2 + [\beta x_\nu]^{1/6}\|\chi\tilde v\|_1 \le \frac{C}{[\beta x_\nu]^{2/3}}(\|\chi h\|_2 + \|\tilde v'\|_2 + \|\tilde v\|_2). \tag{4.2.21}$$

Let $\tilde{\chi} = \sqrt{1-\chi^2}$ where χ is chosen such that $\tilde{\chi} \in C^\infty(\mathbb{R})$. We note that the support of $\tilde{\chi}$ belongs to $[1/2, +\infty)$. Then, an integration by parts yields, as in (3.1.24)–(3.1.25)

$$\|(\tilde{\chi}\tilde{v})'\|_2^2 = \|\tilde{\chi}'\tilde{v}\|_2^2 + \mu\beta\|\tilde{\chi}\tilde{v}\|_2^2 + \Re\langle\tilde{\chi}\tilde{v}, \tilde{\chi}h\rangle, \tag{4.2.22}$$

and, observing that the sign of $(U-\nu)$ is constant on the support of $\tilde{\chi}$,

$$-\beta\||U-\nu|^{1/2}\tilde{\chi}\tilde{v}\|_2^2 + 2\Im\langle\tilde{\chi}'\tilde{v}, (\tilde{\chi}\tilde{v})'\rangle = \Im\langle\tilde{\chi}\tilde{v}, \tilde{\chi}h\rangle. \tag{4.2.23}$$

Combining (4.2.22), (4.2.23), given the support of $\tilde{\chi}$ and the fact that $x_\nu \leq 1/4$, yields

$$\begin{aligned}\beta\|\tilde{\chi}\tilde{v}\|_2^2 &\leq C(\|\tilde{v}\|_2\,\|(\tilde{\chi}\tilde{v})'\|_2 + \|\tilde{\chi}\tilde{v}\|_2\|h\|_2) \leq C(\|\tilde{v}\|_2^2 + \|(\tilde{\chi}\tilde{v})'\|_2^2 + \|\tilde{\chi}\tilde{v}\|_2\|h\|_2)\\ &\leq 2C(\|\tilde{v}\|_2^2 + \mu\beta\|\tilde{\chi}\tilde{v}\|_2^2 + \|\tilde{\chi}\tilde{v}\|_2\|h\|_2).\end{aligned}$$

Observing that $\mu \leq \Upsilon\mathfrak{J}_\nu\beta^{-\frac{1}{3}} \leq C\beta^{-\frac{1}{3}}$, we may conclude the existence of $\beta_0 > 0$ and $\widehat{C} > 0$ such that for all $\beta > \beta_0$

$$\|\tilde{\chi}\tilde{v}\|_2 \leq C\beta^{-1}\,(\|h\|_2 + \beta^{1/2}\|\tilde{v}\|_2). \tag{4.2.24}$$

Combining (4.2.24) with the control of $\|\chi\tilde{v}\|_2$ given in (4.2.21) yields for sufficiently large β_0

$$\|\tilde{v}\|_2 \leq C(\beta^{-1}\|h\|_2 + [\beta x_\nu]^{-2/3}(\|\chi h\|_2 + \|\tilde{v}'\|_2)). \tag{4.2.25}$$

By (3.2.34) (with $v = \tilde{v}$ and $f = h$) and the fact that $\mu \leq C\beta^{-1/3}x_\nu^{2/3}$ we can conclude that

$$\|\tilde{v}'\|_2 \leq C([\beta x_\nu]^{1/3}\|\tilde{v}\|_2 + \|\tilde{v}\|_2^{1/2}\|h\|_2^{1/2}). \tag{4.2.26}$$

Since by (4.2.2) βx_ν is large for sufficiently large β_0, substituting (4.2.26) into (4.2.25) yields

$$\|\tilde{v}\|_2 \leq C[(\beta^{-1} + [\beta x_\nu]^{-4/3})\|h\|_2 + [\beta x_\nu]^{-2/3}\|\chi h\|_2].$$

Hence, using the fact that $x_\nu \geq \beta^{-1/4}$,

$$\|\tilde{v}\|_2 \leq C(\beta^{-1}\|h\|_2 + \beta^{-1/2}\|\chi h\|_2). \tag{4.2.27}$$

By (4.2.15), (4.2.16), and (4.2.17), we obtain, since χ is supported on $[0, 3/4]$,

$$\|\chi h\|_2 \leq C\left[\beta\|(1-x)^4\psi\|_2 + \|(1-x)^4\psi'\|_2\right] \leq \widehat{C}\beta^{-1/2}\lambda_\beta^{-\frac{5}{4}}(\lambda_\beta^{-\frac{1}{2}} + \beta^{-2/3}). \tag{4.2.28}$$

Substituting (4.2.28) together with (4.2.19) into (4.2.27) gives

$$\|\tilde{v}\|_1 \leq \|\tilde{v}\|_2 \leq C\beta^{-5/6}\lambda_\beta^{-3/4}, \tag{4.2.29}$$

completing the proof of (4.2.14), for $0 < x_\nu \leq 1/4$ as well.

Step 1.3: $U(0)-\nu \leq -a\beta^{-1/2}$. Since

$$\Im\langle \chi^2\tilde{v}, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)\tilde{v}\rangle = -\beta\|(\nu-U)^{1/2}\chi\tilde{v}\|_2^2 + 2\Im\langle \chi'\tilde{v}, (\chi\tilde{v})'\rangle, \tag{4.2.30}$$

we may conclude that

$$\beta\|(\nu-U)^{1/2}\chi\tilde{v}\|_2^2 \leq \|\chi\tilde{v}\|_2\|\chi h\|_2 + C\|\tilde{v}\|_2\|(\chi\tilde{v})'\|_2. \tag{4.2.31}$$

As

$$\Re\langle \chi^2\tilde{v}, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)\tilde{v}\rangle = \|(\chi\tilde{v})'\|_2^2 - \mu\beta\|\chi\tilde{v}\|_2^2 - \|\chi'\tilde{v}\|_2^2,$$

we can conclude that

$$\|(\chi\tilde{v})'\|_2 \leq C\big(\mu_{\beta,+}^{1/2}\|\chi\tilde{v}\|_2 + \|\tilde{v}\|_2 + \|\chi\tilde{v}\|_2^{1/2}\|\chi h\|_2^{1/2}\big),$$

where $\mu_{\beta,+}$ is defined in (3.1.26). Substituting the above into (4.2.31) yields

$$\beta\|(\nu-U)^{1/2}\chi\tilde{v}\|_2^2 \leq \|\chi\tilde{v}\|_2\|\chi h\|_2 + C\big(\mu_{\beta,+}^{1/2}\|\chi\tilde{v}\|_2\|\tilde{v}\|_2 + \|\tilde{v}\|_2^2 + \|\chi\tilde{v}\|_2\|\chi h\|_2\big).$$

Since $\nu - U \geq \nu - U(0)$ in $[0,1]$ we may now conclude that

$$\|\chi\tilde{v}\|_2 \leq \frac{C}{\beta(\nu-U(0))}(\|\chi h\|_2 + (\mu_{\beta,+}^{1/2} + [\beta(\nu-U(0))]^{1/2})\|v\|_2). \tag{4.2.32}$$

Given that (4.2.24) remains valid for $\nu > U(0)$ it holds that

$$\|\tilde{v}\|_2 \leq \|\chi\tilde{v}\|_2 + \|\tilde{\chi}\tilde{v}\|_2 \leq \|\chi\tilde{v}\|_2 + C\beta^{-1}(\|h\|_2 + \beta^{1/2}\|\tilde{v}\|_2).$$

Consequently, there exists β_0 such that for $\beta \geq \beta_0$ we may write

$$\|\tilde{v}\|_2 \leq \|\chi\tilde{v}\|_2 + C\beta^{-1}\|h\|_2. \tag{4.2.33}$$

Substituting the above into (4.2.32) yields, for some sufficiently large β_0 and $\beta \geq \beta_0$

$$\|\chi\tilde{v}\|_2 \leq C(\beta^{-1}(\nu-U(0))^{-1}(\|\chi h\|_2 + [\nu-U(0)]^{1/2})\beta^{-1/2}\|h\|_2). \tag{4.2.34}$$

By (4.2.28) and (4.2.19) we then obtain (note that $|\lambda| > U(0)$ as $\nu - U(0) \geq a\beta^{-1/2}$)

$$\|\chi\tilde{v}\|_2 \leq C\,\beta^{-4/3}. \tag{4.2.35}$$

Substituting the above into (4.2.33) yields

$$\|\tilde{v}\|_1 \leq \|\tilde{v}\|_2 \leq C\beta^{-5/6}\lambda_\beta^{-3/4}.$$

Step 2. We prove (4.2.4) and (4.2.6).

Step 2.1. We prove (4.2.4) in the case $\nu < U(0)$.

Consider the pairs $(v, f) \in D(\mathcal{L}_\beta^\zeta) \times L^2(0,1)$ satisfying (4.2.3) and $(\psi, \tilde{v})$ satisfying (4.2.8)–(4.2.13). As in [3, equation (6.20)] there exists $(A, u) \in \mathbb{C} \times D(\mathcal{L}_\beta^{\mathfrak{N}})$ such that

$$v = A(\psi - \tilde{v}) + u, \tag{4.2.36}$$

where

$$u = (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1} f. \tag{4.2.37}$$

Taking the inner product with ζ yields in view of (4.1.1)

$$A\langle \zeta, (\psi - \tilde{v})\rangle = -\langle \zeta, u\rangle. \tag{4.2.38}$$

By (4.1.3) and (4.2.20) it holds that for any $0 < \gamma < 1/4$ there exist positive C and β_0 such that

$$|\langle \zeta, \tilde{v}\rangle| \leq C\beta^\gamma \, \|\tilde{v}\|_1 \leq \hat{C}\beta^{-(2/3-\gamma)} \tag{4.2.39}$$

for all $\beta > \beta_0$.

Then, we write

$$\langle \zeta, \psi\rangle = \langle 1, \psi\rangle + \langle \zeta - 1, \psi\rangle. \tag{4.2.40}$$

For the first term on the right-hand side of (4.2.40), we may rely on [3, equation (6.28)] to obtain

$$\langle 1, \tilde{\psi}\rangle = -\beta^{-1/3} + \mathcal{O}(\beta^{-4/3}). \tag{4.2.41}$$

Observing that

$$|\langle 1, \psi - \tilde{\psi}\rangle| \leq C\|(1-x)^4\tilde{\psi}\|_2 \leq C\lambda_\beta^{-7/4}\beta^{-3/2},$$

we obtain

$$\langle 1, \psi\rangle = -\beta^{-1/3} + \mathcal{O}(\beta^{-4/3}). \tag{4.2.42}$$

For the second term on the right-hand side of (4.2.40) we have

$$|\langle \zeta - 1, \psi\rangle| \leq \|\zeta'\|_{L^2(1-\beta^{-\gamma},1)}\|\|(1-x)^{1/2}\psi\|_1 + (1 + \|\zeta\|_\infty)\|\psi\|_{L^1(0,1-\beta^{-\gamma})}. \tag{4.2.43}$$

By [3, equation (6.27)] it holds that

$$\|\psi\|_{L^1(0,1-\beta^{-\gamma})} \leq \beta^{3\gamma}\|(1-x)^3\psi\|_1 \leq C\beta^{3\gamma-4/3},$$

and that

$$\|(1-x)^{1/2}\psi\|_1 \leq C\beta^{-1/2}\lambda_\beta^{-1/4}. \tag{4.2.44}$$

Consequently, we obtain from (4.2.43) that (recall that $\zeta \in \mathfrak{U}_1(\beta, \lambda, \gamma, \theta)$)

$$|\langle \zeta - 1, \psi\rangle| \leq C_0(\theta\beta^{-1/3} + \beta^{4\gamma-4/3}). \tag{4.2.45}$$

As

$$\langle \zeta, (\psi - \tilde{v}) \rangle = \langle 1, \psi \rangle + \langle (\zeta - 1), \psi \rangle - \langle \zeta, \tilde{v} \rangle,$$

we can conclude from (4.2.45), (4.2.42), and (4.2.39) that

$$|\langle \zeta, (\psi - \tilde{v}) \rangle| \geq \beta^{-1/3}(1 - C_0\theta - \widehat{C}\beta^{4\gamma-1} - \widehat{C}\beta^{-1} - \widehat{C}\beta^{\gamma-1/3}). \tag{4.2.46}$$

We choose $\theta_0 = 1/2C_0$, and since $\gamma < 1/4$, there exists β_0 such that under the assumptions of the proposition we can conclude from (4.2.38) and (4.2.46) that

$$|A| \leq C\beta^{1/3}|\langle \zeta, u \rangle| \leq C\beta^{1/3}\|\zeta\|_\infty \|u\|_1. \tag{4.2.47}$$

For $\nu < U(0)$ we may use (3.3.34), the fact that $x_\nu \geq C\beta^{-1/4}$, and the $\mathcal{L}(L^1, L^2)$ estimate in (3.3.1) to obtain that

$$|A| \leq C\|\zeta\|_\infty \min\big(x_\nu^{-5/6}\beta^{-1/2}\|f\|_2, x_\nu^{-1}\beta^{-2/3}[\|f\|_{1,2} + |f(x_\nu)|\log(x_\nu\beta^{1/4})]\big). \tag{4.2.48}$$

We can now conclude (4.2.4) from (4.2.48), (4.2.10), (4.2.11), and the fact (see (4.2.36)) that $v(1) = A\psi(1)$.

Step 2.2. We prove (4.2.6)

To prove (4.2.6) we now write

$$\Big\langle \zeta, \frac{1}{U - \nu + i\mathfrak{m}} \Big\rangle = \zeta(x_\nu) \int_0^1 \frac{dx}{U - \nu + i\mathfrak{m}} + \int_0^1 \frac{[\zeta - \zeta(x_\nu)]\,dx}{U - \nu + i\mathfrak{m}},$$

where $\mathfrak{m}$ is defined in (4.2.6b). For the coefficient of $\zeta(x_\nu)$ in the first term on the right-hand side we write

$$\int_0^1 \frac{dx}{U - \nu + i\mathfrak{m}} = \frac{1}{2\tilde{x}_\nu} \int_0^1 \Big[\frac{1}{[U(0) - U]^{1/2} - \tilde{x}_\nu} - \frac{1}{[U(0) - U]^{1/2} + \tilde{x}_\nu}\Big]\,dx, \tag{4.2.49}$$

where

$$\tilde{x}_\nu = [U(0) - \nu + i\mathfrak{m}]^{1/2}.$$

An integration by parts yields

$$\begin{aligned}\int_0^1 &\frac{dx}{[U(0) - U]^{1/2} \pm \tilde{x}_\nu} \\ &= \frac{2[U(0) - U]^{1/2}}{U'} \log\big([U(0) - U]^{1/2} \pm \tilde{x}_\nu\big)\Big|_0^1 \\ &\quad - \int_0^1 \Big(\frac{2[U(0) - U]^{1/2}}{U'}\Big)' \log\big([U(0) - U]^{1/2} \pm \tilde{x}_\nu\big)\,dx.\end{aligned} \tag{4.2.50}$$

Given that

$$\Big\|\Big(\frac{2[U(0) - U]^{1/2}}{U'}\Big)'\Big\|_\infty \leq C,$$

it holds, for $\mu \geq -1$ that

$$\Big|\int_0^1 \Big(\frac{2[U(0)-U]^{1/2}}{U'}\Big)' \log\big([U(0)-U]^{1/2} \pm \tilde{x}_\nu\big)\,dx\Big| \leq \hat{C}. \tag{4.2.51}$$

We now observe that

$$\begin{aligned}&\frac{2[U(0)-U]^{1/2}}{U'} \log\big([U(0)-U]^{1/2} \pm \tilde{x}_\nu\big)\Big|_0^1 \\ &\quad = -\Big[\frac{2}{U''(0)}\Big] \log\big(\pm \tilde{x}_\nu\big) + 2\sqrt{U(0)} \log\big([U(0)]^{1/2} \pm \tilde{x}_\nu\big).\end{aligned} \tag{4.2.52}$$

Given that

$$\log\big(-\tilde{x}_\nu\big) - \log\big(+\tilde{x}_\nu\big) = i\pi$$

and

$$\big|[U(0)]^{1/2} - \tilde{x}_\nu\big| \geq \frac{1}{C}\,|\nu + i\mathfrak{m}|,$$

we obtain, by substituting (4.2.52) together with (4.2.51) into (4.2.50), given that $\frac{1}{C}x_\nu \leq \tilde{x}_\nu$,

$$\Big|\int_0^1 \frac{\zeta(x_\nu)dx}{U-\nu+i\mathfrak{m}}\Big| \leq C\,x_\nu^{-1}[1 + |\log|\nu+i\mathfrak{m}|^{-1}|]\|\zeta\|_\infty. \tag{4.2.53}$$

For $\mu \leq -1$ it holds that

$$\Big|\int_0^1 \frac{\zeta(x_\nu)dx}{U-\nu+i\mathfrak{m}}\Big| \leq \frac{C}{|\mu|}\,\|\zeta\|_\infty$$

in accordance with (4.2.53).

For the second term we have by (3.3.27), for any $p > 1$

$$\Big|\int_0^1 \frac{\zeta-\zeta(x_\nu)dx}{U-\nu+i\mathfrak{m}}\Big| \leq C\|\zeta'\|_p \int_0^1 \frac{|x-x_\nu|^{\frac{p-1}{p}}\,dx}{x_\nu|x-x_\nu|} \leq \frac{\hat{C}}{x_\nu}\|\zeta'\|_p.$$

Combining the above with (4.2.53) yields

$$\Big|\Big\langle\zeta, \frac{1}{U-\nu+i\mathfrak{m}}\Big\rangle\Big| \leq \frac{C}{x_\nu}[1 + |\log|\nu+i\mathfrak{m}|^{-1}|]\|\zeta\|_{1,p}. \tag{4.2.54}$$

By the first inequality of (4.2.47) it holds that

$$|A| \leq C\beta^{1/3}\Big\|\Big[\zeta\Big[u + i\beta^{-1}\frac{f(x_\nu)}{U-\nu+i\mathfrak{m}}\Big]\Big\|_1 + \beta^{-1}|f(x_\nu)|\Big|\Big\langle\zeta, \frac{1}{U-\nu+i\mathfrak{m}}\Big\rangle\Big|\Big]. \tag{4.2.55}$$

Substituting (4.2.54) into (4.2.55) yields with the aid of (3.3.23)

$$|A| \leq C\beta^{-2/3}x_\nu^{-1}\big[\|\zeta\|_\infty\|f\|_{1,2} + (1 + |\log|\nu+i\mathfrak{m}|^{-1}|)\|\zeta\|_{1,p}|f(x_\nu)|\big]. \tag{4.2.56}$$

We can now conclude (4.2.6) from (4.2.56), (4.2.10), (4.2.11), and the fact that $v(1) = A\psi(1)$.

Step 2.3. We prove (4.2.4) in the case $\nu > U(0)$.

In this case we write

$$\|u\|_1 \leq \|(\nu-U)^{-1/2}\|_2 \, \|(\nu-U)^{1/2}u\|_2,$$

and as

$$\|(\nu-U)^{-1/2}\|_2^2 \leq C \int_0^1 \frac{dx}{x^2+\nu-U(0)} \leq \hat{C}\, |\nu-U(0)|^{-1/2},$$

we may conclude that

$$\|u\|_1 \leq \check{C}\, [\nu-U(0)]^{-1/4}\, \|(\nu-U)^{1/2}u\|_2. \tag{4.2.57}$$

We now use (4.2.30) with $\chi \equiv 1$, $\tilde{v}=u$ and $h=f$ to obtain that

$$\beta\|(\nu-U)^{1/2}u\|_2^2 = -\Im\langle u, f\rangle. \tag{4.2.58}$$

Consequently,

$$\|(\nu-U)^{1/2}u\|_2 \leq \frac{1}{\beta^{1/2}}\|u\|_1^{1/2}\|f\|_\infty^{1/2},$$

and hence, by (4.2.57) we obtain that

$$\|u\|_1 \leq \frac{C}{\beta[\nu-U(0)]^{1/2}}\|f\|_\infty \leq \frac{\hat{C}}{\beta[\nu-U(0)]^{1/2}}\|f\|_{1,2}. \tag{4.2.59}$$

Next, we use the fact that

$$\nu-U(0) \leq \nu-U(x) \quad \text{for } x\in[0,1], \tag{4.2.60}$$

to obtain from (4.2.58) that

$$\|u\|_2 \leq \frac{1}{\beta[\nu-U(0)]}\|f\|_2. \tag{4.2.61}$$

As

$$\|(\nu-U)^{-1}\|_2^2 \leq C\int_0^1 \frac{dx}{[x^2+\nu-U(0)]^2} \leq \frac{\hat{C}}{|\nu-U(0)|^{3/2}},$$

we may write

$$\|u\|_1 \leq \|(U-\nu)^{-1}\|_2\|(U-\nu)u\|_2 \leq C[\nu-U(0)]^{-3/4}\|(U-\nu)u\|_2. \tag{4.2.62}$$

By (3.1.71) (applied with $v=u$) and (3.1.69) combined with (4.2.60), it holds that

$$\begin{aligned}\beta\|(U-\nu)u\|_2^2 &\leq \|(U-\nu)u\|_2\|f\|_2 + C\|(U-\nu)^{1/2}u\|_2\|u'\|_2\\ &\leq \|(U-\nu)u\|_2\|f\|_2 + C\|(U-\nu)u\|_2^{1/2}\|u\|_2^{1/2}\|u'\|_2.\end{aligned} \tag{4.2.63}$$

Furthermore, by (3.2.34) and (4.2.1)

$$\|u'\|_2 \leq \mu_{\beta,+}^{1/2}\|u\|_2 + \|u\|_2^{1/2}\|f\|_2^{1/2} \leq C(\nu - U(0))^{1/6}\beta^{1/3}\|u\|_2 + \|u\|_2^{1/2}\|f\|_2^{1/2}.$$

We now use (4.2.61) to deduce from above

$$\|u'\|_2 \leq C\left[(\nu - U(0))^{-5/6}\beta^{-2/3} + (\nu - U(0))^{-1/2}\beta^{-1/2}\right]\|f\|_2,$$

which implies using (4.2.2)

$$\|u'\|_2 \leq \hat{C}(\nu - U(0))^{-1/2}\beta^{-1/2}\|f\|_2. \tag{4.2.64}$$

Substituting (4.2.61) and (4.2.64) into (4.2.63) yields

$$\beta\|(U-\nu)u\|_2^2 \leq \|(U-\nu)u\|_2\|f\|_2 + \frac{C}{\beta(\nu - U(0))}\|(U-\nu)u\|_2^{1/2}\|f\|_2^{3/2},$$

from which we conclude using (4.2.2) that

$$\|(U-\nu)u\|_2 \leq C\beta^{-1}[1+\beta^{-1/3}[\nu - U(0)]^{2/3}]\|f\|_2 \leq \frac{\hat{C}}{\beta}\|f\|_2.$$

Substituting the above into (4.2.62) yields together with (4.2.2)

$$\|u\|_1 \leq \frac{C}{\beta[\nu - U(0)]^{3/4}}\|f\|_2 \leq \frac{\hat{C}}{(\beta[\nu - U(0)]^{1/2})^{5/6}}\|f\|_2,$$

which, together with (4.2.59) proves that

$$\|u\|_1 \leq C\min\left((\beta[\nu - U(0)]^{1/2})^{-5/6}\|f\|_2, (\beta[\nu - U(0)]^{1/2})^{-1}\|f\|_{1,2}\right).$$

As $\psi_{\lambda,\beta}(1) = \widetilde{\psi}_{\lambda,\beta}(1)$ we may infer from (4.2.10)

$$\frac{1}{C}\lambda_\beta^{1/2} \leq |\psi_{\lambda,\beta}(1)| \leq C\,\lambda_\beta^{1/2}, \tag{4.2.65}$$

and hence we can conclude by (4.2.36) and (4.2.47) (which remains valid for $\nu > U(0)$) that

$$|v(1)| = |A\psi(1)| \leq C\,\lambda_\beta^{1/2}\min((\beta[\nu - U(0)]^{1/2})^{-5/6}\|f\|_2, x_\nu^{-1}\beta^{-2/3}\|f\|_{1,2})$$

which verifies (4.2.4) for $\nu > U(0)$.

Step 3. We prove (4.2.5).

The proof of (4.2.5) which reads

$$|v(1)| \leq C\,\lambda_\beta^{1/2}\,\beta^{1/3}\,\|\zeta u\|_1,$$

follows immediately from the first inequality in (4.2.47), from (4.2.10), and again from the fact that $v(1) = A\psi(1)$.

Step 4. We prove (4.2.7). To prove it for $x_\nu < 1/4$ we set

$$f = (x - x_\nu)g,$$

and assume that $g \in L^2(0,1)$. Recall the definition of $\chi_\nu^\pm$ and $\tilde{\chi}_\nu$ from (3.1.8). Since by (3.1.78)–(3.1.79)

$$\||U - \nu|^{-1/2}\chi_\nu^\pm\|_2^2 \le C \int_{\frac{5x_\nu}{4}}^1 \frac{dx}{x^2 - x_\nu^2} + C \int_0^{\frac{3x_\nu}{4}} \frac{dx}{x_\nu^2 - x^2} \le \frac{\hat{C}}{x_\nu},$$

we can conclude that

$$\|(\chi_\nu^\pm)^2 u\|_1 \le \||U - \nu|^{-1/2}\chi_\nu^\pm\|_2 \, \||U - \nu|^{1/2}\chi_\nu^\pm u\|_2 \le C x_\nu^{-1/2} \||U - \nu|^{1/2}\chi_\nu^\pm u\|_2.$$

By (3.3.22) and (3.3.11) we then obtain

$$\|(\chi_\nu^\pm)^2 u\|_1 \le C([\beta x_\nu]^{-1/6}\|u\|_2 + \beta^{-1} x_\nu^{-1/2}\|g\|_2).$$

Hence, by (3.3.10) (which reads $\|u\|_2 \le C\,[\Im_\nu\beta]^{-1}\,\|g\|_2$), we can conclude that

$$\|(\chi_\nu^\pm)^2 u\|_1 \le C([\beta x_\nu]^{-7/6} + [\beta^{-1} x_\nu^{-1/2}])\|g\|_2.$$

Given that $x_\nu \ge \frac{1}{C}\beta^{-1/4}$ we obtain that

$$\|(\chi_\nu^\pm)^2 u\|_1 \le C\beta^{-1} x_\nu^{-1/2}\|g\|_2. \tag{4.2.66}$$

Employing again (3.3.10) we write

$$\|\tilde{\chi}_\nu u\|_2 \le \|u\|_2 \le C[\beta x_\nu]^{-1}\|g\|_2.$$

Consequently, since $\tilde{\chi}_\nu$ is supported on $[x_\nu/2, 3x_\nu/2]$

$$\|\tilde{\chi}_\nu^2 u\|_1 \le x_\nu^{1/2}\|\tilde{\chi}_\nu u\|_2 \le C\beta^{-1} x_\nu^{-1/2}\|g\|_2. \tag{4.2.67}$$

Combining (4.2.67) with (4.2.66), (4.2.5), and (4.2.47) yields (4.2.7) for the case $x_\nu < 1/4$.

In the case $x_\nu \ge 1/4$, (4.2.7) immediately follows from (3.3.10) and the fact that $\|u\|_1 \le \|u\|_2$. ∎

4.3 Resolvent estimates for $|U(0) - \Im\lambda| = \mathcal{O}(\beta^{-1/2})$

Here, we introduce, for $\beta > 0$, $\lambda \in \mathbb{C}$, and $\theta > 0$,

$$\mathfrak{U}_2(\beta, \theta, \lambda) = \{\zeta \in \mathfrak{U}_0 \mid \|\zeta'\|_2 \le \theta\beta^{1/6}\lambda_\beta^{1/4}\}, \tag{4.3.1}$$

where $\mathfrak{U}_0$ is introduced in (4.1.2). In the present context λ lies in a bounded set and hence

$$\|\zeta'\|_2 \le C\theta\beta^{1/4}.$$

Proposition 4.3.1. *Let* $U \in C^3([0,1])$ *satisfy* (2.1.3), $\Upsilon < \sqrt{-U''(0)}/2$, $\mu_1 > 0$, *and* $a > 0$. *Then, there exist* $\beta_0 > 0$ *and* $\theta_0 > 0$ *such that for all* $\beta \geq \beta_0$, $\theta \in (0, \theta_0]$, $\lambda \in \mathbb{C}$ *satisfying*

$$U(0) - a\beta^{-1/2} < \nu < U(0) + a\beta^{-1/2} \tag{4.3.2a}$$

and

$$-\mu_1 \leq \mu < \Upsilon\beta^{-1/2} \tag{4.3.2b}$$

for any ζ *in* $\mathfrak{U}_2(\beta, \theta, \lambda)$, *and* $(f, v) \in H^1(0,1) \times D(\mathcal{L}_\zeta^\beta)$ *satisfying* (4.2.3), *it holds that*

$$|v(1)| \leq C\|\zeta\|_\infty \min(\beta^{-1/8}\|f\|_2, \beta^{-1/4}\|f\|_\infty). \tag{4.3.3}$$

Furthermore, for f *satisfying* $(x - x_\nu)^{-1} f \in L^2(0,1)$ *we have*

$$|v(1)| \leq C\beta^{-3/8}\|\zeta\|_\infty \Big\| \frac{f}{x - x_\nu} \Big\|_2. \tag{4.3.4}$$

Proof. By (4.2.12), (4.2.13), (4.2.19), and (3.2.1a) in Proposition 3.2.1 it holds that

$$\|\tilde{v}\|_2 + \beta^{1/8}\|\tilde{v}\|_1 \leq C\beta^{-1/2}\|h\|_2 \leq \hat{C}\beta^{-1/3}\lambda_\beta^{-3/4}. \tag{4.3.5}$$

Since $|\lambda| > U(0)/2$ we obtain for $\beta \geq \beta_0$ with β_0 large enough

$$\|\tilde{v}\|_2 + \beta^{1/8}\|\tilde{v}\|_1 \leq C\beta^{-1/3}[\beta^{1/3}]^{-3/4} = C\beta^{-7/12}. \tag{4.3.6}$$

Given that for $\zeta \in \mathfrak{U}_2(\beta, \theta, \lambda)$ it holds that

$$\|\zeta\|_\infty \leq (1 + C\|\zeta'\|_2) \leq C(1 + \theta\beta^{1/6}\lambda_\beta^{1/4}), \tag{4.3.7}$$

and hence we can conclude, from (4.3.3) and (4.3.4), that

$$\|\zeta\|_\infty \leq C\beta^{1/4}. \tag{4.3.8}$$

We then obtain, using (4.3.6),

$$|\langle\zeta, \tilde{v}\rangle| \leq \|\zeta\|_\infty\|\tilde{v}\|_1 \leq \hat{C}\beta^{-11/24}. \tag{4.3.9}$$

Furthermore, we have, using (4.2.44) and the fact that $\|\zeta'\|_2 \leq \theta\beta^{1/4}$ that

$$|\langle\zeta - 1, \psi\rangle| \leq \|\zeta'\|_{L^2(0,1)}\|\|(1-x)^{1/2}\psi\|_1 \leq C\theta\beta^{-1/3}. \tag{4.3.10}$$

Since v is still expressible by (4.2.36), we can now conclude, as in (4.2.47), with (4.2.39) and (4.2.45) respectively replaced by (4.3.9) and (4.3.10) that there exist θ_0 and β_0 such that, for $\theta \leq \theta_0$ and $\beta \geq \beta_0$, it holds that

$$|A| \leq C\beta^{1/3}\|\zeta\|_\infty\|u\|_1, \tag{4.3.11}$$

where u is given by (4.2.37).

We now use (3.2.1a) in Proposition 3.2.1 to obtain that

$$|A| \leq C\|\zeta\|_\infty \min(\beta^{-7/24}\|f\|_2, \beta^{-5/12}\|f\|_\infty, \beta^{-13/24}\|(x-x_\nu)^{-1}f\|_2).$$

Consequently, by (4.2.36) and (4.2.10) we obtain that

$$|v(1)| \leq C\,\lambda_\beta^{1/2}\|\zeta\|_\infty \min(\beta^{-7/24}\|f\|_2, \beta^{-5/12}\|f\|_\infty, \beta^{-13/24}\|(x-x_\nu)^{-1}f\|_2).$$

Given that (4.3.2) provides a uniform bound on $|\lambda|$, we have

$$\lambda_\beta^{1/2} \leq C\beta^{1/6},$$

hence we can conclude that

$$|v(1)| \leq C\|\zeta\|_\infty \min(\beta^{-1/8}\|f\|_2, \beta^{-1/4}\|f\|_\infty, \beta^{-3/8}\|(x-x_\nu)^{-1}f\|_2). \qquad \blacksquare$$

4.4 Resolvent estimates for negative $\Re\lambda$

Although Propositions 4.2.1 and 4.3.1 provide estimates when the spectral parameter λ belongs to domains in $\mathbb{C}$ that include $\Re\lambda \leq 0$, one can obtain a better estimate if we assume $\Re\lambda \leq -\mu_0$ for some fixed $\mu_0 > 0$, or at least $\Re\lambda \leq -C\,[U(0)-\nu]$ for $\nu < U(0)$.

Proposition 4.4.1. *Let $U \in C^2([0,1])$ satisfy (2.1.3). Let further a, μ_0, and ν_0 denote positive constants. Then, there exist $C > 0$, $\beta_0 > 0$, and $\theta_0 > 0$ such that for all $\beta \geq \beta_0$, $\theta \in (0,\theta_0]$ and $\lambda = \mu + i\nu \in \mathbb{C}$ satisfying*

$$-\nu_0 < \nu < U(0) + a\beta^{-1/2} \tag{4.4.1}$$

and

$$\mu \leq -\mu_0 \tag{4.4.2}$$

for any $\zeta \in \mathfrak{U}_2(\beta,\theta,\lambda)$, given by (4.3.1), and any pair $(f,v) \in L^2(0,1)\times D(\mathcal{L}_\zeta^\beta)$ satisfying (4.2.3), it holds that

$$|v(1)| \leq C\,\beta^{-1/2}\,\|\zeta\|_\infty\|f\|_2. \tag{4.4.3}$$

Proof. As in (4.2.36) we write

$$v = A(\psi - \tilde{v}) + u,$$

where $A \in \mathbb{C}$, $\psi = \psi_{\lambda,\beta}$ is given by (4.2.11), $\tilde{v}$ by (4.2.13), and u by (4.2.37).

As $\zeta \in \mathfrak{U}_2(\beta,\theta,\lambda)$, we obtain, given that $-\nu_0 < \nu < U(0) - a\beta^{-1/2}$, and in view of (4.4.2), (4.3.7), and (4.2.14)

$$|\langle\zeta,\tilde{v}\rangle| \leq \|\zeta\|_\infty\|\tilde{v}\|_1 \leq C\beta^{-1/2}\lambda_\beta^{-1/2} \leq \hat{C}\,\beta^{-2/3}. \tag{4.4.4}$$

Note that while both Propositions 4.2.1 and 4.3.1 assume $\mu \geq -\mu_0$, both (4.2.14) and (4.3.7) are valid for $\mu < -\mu_0$ as well.

In the case

$$U(0) - a\beta^{-1/2} < \nu < U(0) + a\beta^{-1/2},$$

we proceed as in the proof of (4.4.4) but use (4.3.5) instead of (4.2.14) and (4.2.19) which continues to hold in this case. Hence, we obtain the weaker estimate

$$|\langle\zeta, \tilde{v}\rangle| \leq \widehat{C}\,\beta^{-11/24}.$$

Combining the above with (4.4.4) yields the existence of $C > 0$ such that for any λ satisfying (4.4.1) and (4.4.2) it holds that

$$|\langle\zeta, \tilde{v}\rangle| \leq C\beta^{-11/24}. \tag{4.4.5}$$

Furthermore, as in (4.3.10) we write

$$|\langle\zeta - 1, \psi\rangle| \leq \|\zeta'\|_{L^2(0,1)}\|\|(1-x)^{1/2}\psi\|_1,$$

from which we conclude, using the fact that $\zeta \in \mathfrak{U}_2(\beta, \theta, \lambda)$ and (4.2.44)

$$|\langle\zeta - 1, \psi\rangle| \leq C\theta\beta^{1/6}\lambda_\beta^{1/4} \times \beta^{-1/2}\lambda_\beta^{-1/4}.$$

Consequently, it holds that

$$|\langle\zeta - 1, \psi\rangle| \leq C\theta\beta^{-1/3}. \tag{4.4.6}$$

Hence, as in (4.3.11) we obtain that, choosing θ_0 small enough

$$|A| \leq C\beta^{1/3}\|\zeta\|_\infty\|u\|_1. \tag{4.4.7}$$

To estimate $\|u\|_1$ we observe that

$$\Re\langle u, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)u\rangle = \|u'\|_2^2 - \mu\beta\|u\|_2^2, \tag{4.4.8}$$

where $\mathcal{L}_\beta^{\mathfrak{N}}$ is defined in (3.0.1), from which we conclude that

$$\|u\|_1 \leq \|u\|_2 \leq \frac{1}{|\mu|\beta}\|f\|_2. \tag{4.4.9}$$

The proof of the proposition can now be completed by using (4.2.10) and the fact that $v(1) = A\psi(1)$. Thus, by (4.2.10) we obtain from (4.4.7) that

$$|v(1)| \leq C\beta^{1/3}\lambda_\beta^{1/2}\|\zeta\|_\infty\,\|u\|_1. \tag{4.4.10}$$

Since for $\mu \leq -\mu_0$ it holds by (4.4.1) that

$$|\lambda| \leq |\mu| + |\nu| \leq C|\mu|,$$

we may conclude that

$$|v(1)| \leq C\beta^{1/2}|\mu|^{1/2}\|\zeta\|_\infty \|u\|_1 .$$

Hence, by (4.4.9) we can conclude that

$$|v(1)| \leq C\beta^{-1/2}|\mu|^{-1/2}\|\zeta\|_\infty\|f\|_2 \leq \hat{C}\beta^{-1/2}|\mu_0|^{-1/2}\|\zeta\|_\infty\|f\|_2 .$$

The proposition is proved. ■

We next consider the case $-\mu_0 < \mu < -\frac{|U(0)-\nu|}{\kappa_1}$ for some $\kappa_1 > 0$. While (4.2.5) and (4.3.3) hold true under this assumption, it is necessary, in the next section, to obtain better estimates since Proposition 2.8.1 is inapplicable in this case.

Proposition 4.4.2. *Let $U \in C^2([0,1])$ satisfy (2.1.3). Let further a, κ_1, ν_0 and μ_0 denote positive constants. Then, there exist $C > 0$, $\beta_0 > 0$, and $\theta_0 > 0$ such that, for all $\beta \geq \beta_0$, $\theta \in (0, \theta_0]$ and $\lambda = \mu + i\nu \in \mathbb{C}$ satisfying (see (4.2.2) and (4.4.1))*

$$-\nu_0 < \nu < U(0) + a\beta^{-1/2} \tag{4.4.11}$$

and

$$-\mu_0 < \mu < -\frac{|U(0)-\nu|}{\kappa_1} \tag{4.4.12}$$

for any $\zeta \in \mathfrak{U}_2(\beta,\theta,\lambda)$, and any pair $(v, f) \in D(\mathcal{L}_\zeta^\beta) \times L^2(0,1)$ satisfying (4.2.3), it holds that

$$|v(1)| \leq C\,\|\zeta\|_\infty \min(|\mu|^{-1/2}\beta^{-1/2}\|f\|_\infty, |\mu|^{-3/4}\beta^{-1/2}\|f\|_2). \tag{4.4.13}$$

Proof. Step 1. We prove that

$$|v(1)| \leq C\,|\mu|^{-3/4}\beta^{-1/2}\|\zeta\|_\infty\|f\|_2 . \tag{4.4.14}$$

As in the proof of Proposition 4.4.1 and since for sufficiently small θ_0, (4.4.10) still holds under the assumptions of this proposition we obtain that

$$|v(1)| \leq C\beta^{1/3}\lambda_\beta^{1/2}\|\zeta\|_\infty \|u\|_1 \leq \hat{C}\beta^{1/2}\|\zeta\|_\infty\|u\|_1 , \tag{4.4.15}$$

where u is given by (4.2.37). Note that under (4.4.11) and (4.4.12), $|\lambda|$ is bounded. To obtain an estimate for $\|u\|_1$ we now write

$$\|u\|_1 \leq \|(U+i\lambda)^{-1}\|_2\,\|(U+i\lambda)u\|_2 . \tag{4.4.16}$$

By (3.1.75) and (4.4.8) we have that

$$\|(U+i\lambda)u\|_2 \leq \|(U-\nu)u\|_2 + |\mu|\,\|u\|_2 \leq \frac{\hat{C}}{\beta}\|f\|_2 . \tag{4.4.17}$$

By (2.9.24) (with $q = 2$) it holds that

$$\|(U + i\lambda)^{-1}\|_2^2 \leq C\,|\mu|^{-3/2}.$$

Consequently, we may conclude from (4.4.16) and (4.4.17) that

$$\|u\|_1 \leq C\,|\mu|^{-3/4}\beta^{-1}\,\|f\|_2.$$

Substituting into (4.4.15) then yields (4.4.14).

Step 2. We prove that

$$|v(1)| \leq C\,|\mu|^{-1/2}\beta^{-1/2}\|\zeta\|_\infty\,\|f\|_\infty. \tag{4.4.18}$$

Suppose that $f \in L^\infty(0,1)$. Then by (4.4.8) it holds for negative values of μ that

$$\|u'\|_2^2 + |\mu|\beta\|u\|_2^2 \leq \|u\|_1\,\|f\|_\infty. \tag{4.4.19}$$

Set

$$\chi_\beta^\pm(x) = \hat{\chi}([|\mu|\beta]^{1/2}(x - x_\nu))\mathbf{1}_{\mathbb{R}_+}(\pm(x - x_\nu)),$$

where $\hat{\chi}$ be given by (2.4.16).

Note that χ_β^+ is supported in $(x_\nu + [|\mu|\beta]^{-1/2}/4, +\infty)$ whereas χ_β^- is supported in $(-\infty, x_\nu - [|\mu|\beta]^{-1/2}/4)$. Let further $\tilde{\chi}_\beta = \sqrt{1 - (\chi_\beta^+)^2 - (\chi_\beta^-)^2}$. Note that by (4.4.12) it follows that we can choose β_0 such that for all $\beta > \beta_0$

$$x_\nu - [|\mu|\beta]^{-1/2}/2 > 0. \tag{4.4.20}$$

Hence, the support of $\tilde{\chi}_\beta$ is contained in

$$\Big(x_\nu - \frac{1}{2}[|\mu|\beta]^{-1/2},\, x_\nu + \frac{1}{2}[|\mu|\beta]^{-1/2}\Big) \subset [0,1].$$

As in (3.1.25) (with $\chi_\nu^\pm$ replaced by $\chi_\beta^\pm$), we now obtain

$$\beta\||U - \nu|^{1/2}\chi_\beta^\pm u\|_2^2 \leq 2\,\|(\chi_\beta^\pm)'u\|_2\|\chi_\beta^\pm u'\|_2 + \|u\|_1\|f\|_\infty. \tag{4.4.21}$$

Consequently, by (4.4.21) and the definition of $\chi_\beta^\pm$ we conclude that

$$\beta\||U - \nu|^{1/2}\chi_\beta^\pm u\|_2^2 \leq C|\mu\beta|^{-1/2}\|u\|_2\|u'\|_2 + \|u\|_1\|f\|_\infty.$$

By (4.4.19) we then have

$$\||U - \nu|^{1/2}\chi_\beta^\pm u\|_2^2 \leq C\,\beta^{-1}\,\|u\|_1\|f\|_\infty. \tag{4.4.22}$$

By (4.4.11) and (4.4.12) we have that

$$\frac{1}{C}\,\beta^{-1/2} \leq x_\nu^2 \leq C|\mu|. \tag{4.4.23}$$

Given the definition of $\tilde{\chi}_\beta$ we obtain that

$$|U(x) - \nu| \leq C\Big(\sup_{x\in\operatorname{supp}\tilde{\chi}_\beta} |U'(x)|\Big)\,|\mu\beta|^{-1/2}.$$

Hence, by (4.4.23) and (4.4.20) we can conclude that

$$\||U-\nu|^{1/2}\tilde{\chi}_\beta u\|_2^2 \leq C x_\nu [|\mu|\beta]^{-1/2}\|u\|_2^2.$$

Combining the above with (4.4.22) now yields

$$\||U-\nu|^{1/2}u\|_2 \leq C\,\beta^{-1/2}\,\|u\|_1^{1/2}\|f\|_\infty^{1/2}.$$

By (2.9.24) (with $q = 1$) it holds that

$$\|(U + i\lambda)^{-1/2}\|_2^2 = \|(U + i\lambda)^{-1}\|_1 \leq C\,|\mu|^{-1/2}.$$

Consequently,

$$\begin{aligned}\|u\|_1 &\leq \|(U + i\lambda)^{-1/2}\|_2\,\|(U + i\lambda)^{1/2}u\|_2 \\ &\leq C\,|\mu|^{-1/4}\big[\||U-\nu|^{1/2}u\|_2 + |\mu|^{1/2}\|u\|_2\big] \\ &\leq \hat{C}\,|\mu|^{-1/4}\,\beta^{-1/2}\,\|u\|_1^{1/2}\|f\|_\infty^{1/2}.\end{aligned}$$

Hence,

$$\|u\|_1 \leq C\,|\mu|^{-1/2}\beta^{-1}\,\|f\|_\infty,$$

which, when substituted into (4.4.15), establishes (4.4.18).

Together with (4.4.14) the above inequality completes the proof of the proposition. ∎

4.5 Rapidly decaying functions

When considering large values of α in the next section, it is useful to consider, as in [3], the operator $\mathcal{L}_\zeta^\beta$ where ζ decays rapidly away from the boundary at $x = 1$. Set then for $\lambda \in \mathbb{C}$ and positive β, θ, α

$$\mathfrak{U}_3(\beta,\theta,\alpha,\lambda) = \big\{\zeta \in \mathfrak{U}_2(\beta,\theta,\lambda) \mid |\zeta(x)| \leq e^{-\alpha(1-x)}\|\zeta\|_\infty, \forall x \in [0,1]\big\}, \quad (4.5.1)$$

where $\mathfrak{U}_2(\beta,\theta,\lambda)$ is introduced in (4.3.1).

Proposition 4.5.1. *Let $U \in C^3([0,1])$ satisfy (2.1.3). Let further $a > 0$, $\mu_0 > 0$, and $\Upsilon < \sqrt{-U''(0)}/2$. Then, there exist $C > 0$, $\beta_0 > 0$, and $\theta_0 > 0$ such that for all $\beta \geq \beta_0$, $\theta \in (0,\theta_0]$, all $\alpha \geq 1$ and $\lambda \in \mathbb{C}$ satisfying*

$$-\mu_0 \leq \Re\lambda \leq \Upsilon\beta^{-1/2} \quad (4.5.2)$$

and

$$\frac{1}{2}U(0) < \nu < U(0) + a\beta^{-1/2}, \tag{4.5.3}$$

for all $\zeta \in \mathfrak{U}_3(\beta,\theta,\alpha,\lambda)$*, and for all pair* $(f,v) \in L^2(0,1) \times D(\mathcal{L}_\zeta^\beta)$ *satisfying* (4.2.3)*, it holds that*

$$|v(1)| \leq C\alpha^{-1/2}\,(\beta^{-1/2} + e^{-\alpha/C})\|\zeta\|_\infty\,\|f\|_2. \tag{4.5.4}$$

Proof. Step 1. Control of $v(1)$. Since (4.3.9) and (4.3.10) remain valid under our assumptions (4.5.2) and (4.5.3), we can follow the same procedure as in Proposition 4.3.1 to obtain

$$|v(1)| \leq C\,\beta^{1/3}\lambda_\beta^{1/2}|\langle\zeta,u\rangle| \leq C\beta^{1/2}|\langle\zeta,u\rangle|, \tag{4.5.5}$$

where u is given by (4.2.37).

Step 2. We prove under (4.5.2) and (4.5.3) that

$$\|u\|_2 \leq C\,\beta^{-1/2}\,\|f\|_2. \tag{4.5.6}$$

We first consider the case where

$$U(0)/2 < \nu < U(0) - a_1\beta^{-1/2}, \tag{4.5.7}$$

where $a_1 \geq a$ will be determined in the sequel. In this case, we can use (3.1.3) which reads (for $\mathfrak{J}_\nu = |U'(x_\nu)|$)

$$\|u\|_2 \leq C(\mathfrak{J}_\nu\beta)^{-2/3}\|f\|_2$$

and holds under the condition $\mu \leq \Upsilon_0\,{\mathfrak{J}_\nu}^{2/3}\,\beta^{-1/3}$ for some sufficiently small $\Upsilon_0 > 0$. Note that for $\nu - U(0) > a_1\beta^{-1/2}$ there exists $\widehat{C} > 0$ such that

$$\mathfrak{J}_\nu \geq \frac{1}{\widehat{C}}\,a_1^{1/2}\,\beta^{-1/4}.$$

Consequently, there exists $C > 0$ such that (3.1.3) is applicable for all

$$\mu \leq Ca_1^{1/3}\beta^{-1/2}.$$

For sufficiently large $a_1 \geq a$ the above set of μ values contains (4.5.2) and hence we can conclude (4.5.6) when (4.5.7) holds true.

We now look at the case

$$U(0) - a_1\beta^{-1/2} < \nu < U(0) + a\beta^{-1/2},$$

Here, we can apply (3.2.1a) (with a replaced by a_1) to obtain (4.5.6) which, combined with (4.5.5), leads to

$$|v(1)| \leq C\,\|\zeta\|_\infty\|f\|_2.$$

Note that at this stage it is sufficient to assume that $\zeta \in \mathfrak{U}_2(\beta,\theta,\alpha,\lambda)$.

Step 3. With $\hat{x}_\nu \in (0,1)$ satisfying

$$U(\hat{x}_\nu) = \frac{\nu}{2}, \tag{4.5.8}$$

we prove that

$$|\langle \zeta, u\rangle| \le C\,\alpha^{-1/2}\,\beta^{-1/2}(\beta^{-1/2} + e^{-\alpha(1-\hat{x}_\nu)})\,\|\zeta\|_\infty\,\|f\|_2. \tag{4.5.9}$$

Consider the decomposition

$$\langle \zeta, u\rangle = \langle \mathbf{1}_{L^2(0,\hat{x}_\nu)}\zeta, u\rangle + \langle \mathbf{1}_{(\hat{x}_\nu,1)}\zeta, u\rangle.$$

We first obtain using (4.5.6) that

$$|\langle \mathbf{1}_{L^2(0,\hat{x}_\nu)}\zeta, u\rangle| \le \|\mathbf{1}_{L^2(0,\hat{x}_\nu)}\zeta\|_2\,\|u\|_2 \le C\,\alpha^{-1/2}\,\beta^{-1/2}\,e^{-\alpha(1-\hat{x}_\nu)}\,\|\zeta\|_\infty\|f\|_2. \tag{4.5.10}$$

Moreover, it holds that

$$|\langle \mathbf{1}_{(\hat{x}_\nu,1)}\zeta, u\rangle| \le C\,\|\zeta\|_2\,\|\mathbf{1}_{(\hat{x}_\nu,1)}u\|_2 \le \frac{C}{\alpha^{1/2}}\|\zeta\|_\infty\,\|\mathbf{1}_{(\hat{x}_\nu,1)}u\|_2. \tag{4.5.11}$$

Let

$$\check{\chi}_\nu(x) = \chi\Big(\frac{x-x_\nu}{\hat{x}_\nu - x_\nu}\Big)\mathbf{1}_{\mathbb{R}_+}(x-x_\nu), \tag{4.5.12}$$

where χ is given by (2.4.16). Note that $\check{\chi}_\nu$ is supported in the interval $(x_\nu + (\hat{x}_\nu - x_\nu)/4, +\infty)$ and equals 1 on $[(x_\nu + \hat{x}_\nu)/2, 1]$. Integration by part yields

$$\|(\check{\chi}_\nu u)'\|_2^2 = \|\check{\chi}_\nu' u\|_2^2 + \mu\beta\|\check{\chi}_\nu u\|_2^2 + \Re\langle \check{\chi}_\nu u, \check{\chi}_\nu f\rangle, \tag{4.5.13a}$$

and, given that $(U-\nu)$ has constant sign on the support of $\check{\chi}_\nu$,

$$-\beta\||U-\nu|^{1/2}\check{\chi}_\nu u\|_2^2 + 2\Im\langle \check{\chi}_\nu' u, (\check{\chi}_\nu u)'\rangle = \Im\langle \check{\chi}_\nu u, \check{\chi}_\nu f\rangle. \tag{4.5.13b}$$

Combining the above we obtain, given the support of $\check{\chi}_\nu$

$$\begin{aligned}\|\check{\chi}_\nu u\|_2^2 &\le C\,\||U-\nu|^{1/2}\check{\chi}_\nu u\|_2^2 \le C\beta^{-1}\big[\|u\|_2\|(\check{\chi}_\nu u)'\|_2 + \|\check{\chi}_\nu u\|_2\|\check{\chi}_\nu f\|_2\big]\\ &\le C\beta^{-1}\big[\|u\|_2(\|u\|_2 + \mu_{\beta,+}^{1/2}\|\check{\chi}_\nu u\|_2 + \|\check{\chi}_\nu u\|_2^{1/2}\|\check{\chi}_\nu f\|_2^{1/2})\\ &\qquad + \|\check{\chi}_\nu u\|_2\|\check{\chi}_\nu f\|_2\big].\end{aligned} \tag{4.5.14}$$

From here we deduce that

$$\|\check{\chi}_\nu u\|_2^2$$
$$\le C\beta^{-1}\big[\|u\|_2(\|u\|_2 + \beta^{1/4}\|\check{\chi}_\nu u\|_2 + \|\check{\chi}_\nu u\|_2^{1/2}\|\check{\chi}_\nu f\|_2^{1/2}) + \|\check{\chi}_\nu u\|_2\|\check{\chi}_\nu f\|_2\big],$$

which implies

$$\|\check{\chi}_\nu u\|_2^2 \leq C\Big(\frac{1}{\beta^2}\|\check{\chi}_\nu f\|_2^2 + \frac{1}{\beta}\|u\|_2^2\Big). \tag{4.5.15}$$

Combining (4.5.15) and (4.5.6) leads to

$$\|\mathbf{1}_{(\hat{x}_\nu,1)}u\|_2 \leq \|\check{\chi}_\nu u\|_2 \leq C\,\beta^{-1}\,\|f\|_2. \tag{4.5.16}$$

For later reference we note that by (4.5.13a) and (4.5.16) it holds that

$$\|\mathbf{1}_{(\hat{x}_\nu,1)}u'\|_2 \leq \|(\check{\chi}_\nu u)'\|_2 \leq C\,\beta^{-1/2}\,\|f\|_2. \tag{4.5.17}$$

Combining (4.5.16) with (4.5.10) and (4.5.11) yields (4.5.9), which, together with (4.5.5) yields (4.5.4). ■

4.6 Auxiliary estimates

We recall that for $(\lambda,\beta) \in \mathbb{C}\times\mathbb{R}_+$, $\psi = \psi_{\lambda,\beta}$ is given by (4.2.11). We now set, for $x \in (0,1)$

$$\widehat{\psi}_{\lambda,\beta}(x) = \frac{\psi_{\lambda,\beta}(x)}{\psi_{\lambda,\beta}(1)}. \tag{4.6.1}$$

The following auxiliary estimate will become useful in the next section.

Lemma 4.6.1. *Let $\widetilde{\nu}_1 \in (0, U(0))$, $a > 0$, and $\Upsilon < \sqrt{-U''(0)}/2$. Then there exist C and β_0 such that, for $\beta \geq \beta_0$,*

$$\nu \in (U(0) - \widetilde{\nu}_1, U(0) + a\beta^{-1/2}) \quad \textit{and} \quad \mu < \Upsilon\beta^{-1/2} \tag{4.6.2}$$

such that

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}\widehat{\psi}_{\lambda,\beta}\|_2 + \beta^{-1/2}\Big\|\frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}\widehat{\psi}_{\lambda,\beta}\Big\|_2 \leq C\,\beta^{-5/4}. \tag{4.6.3}$$

For convenience of omit the subscript (λ,β) from $\widehat{\psi}_{\lambda,\beta}$ in the sequel.

Proof. Let $v \in D(\mathcal{L}_\beta^{\mathfrak{N}})$ satisfy

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)v = \widehat{\psi}. \tag{4.6.4}$$

Let $\mu_0 > 0$. We begin by considering the case $\mu \geq -\mu_0$. Let further $\check{\chi}_\nu$ be given by (4.5.12). As in (4.5.15) we obtain

$$\|\check{\chi}_\nu v\|_2 \leq C\,[\beta^{-1/2}\|v\|_2 + \beta^{-1}\|\widehat{\psi}\|_2].$$

By (4.5.6) it holds that

$$\|v\|_2 \leq C\beta^{-1/2}\|\widehat{\psi}\|_2.$$

Furthermore, using (4.2.16) (with $k = 0$) and (4.2.10) we have for $|\lambda| \geq |\nu| \geq U(0)/2$

$$\|\widehat{\psi}\|_2 \leq C\,\lambda_\beta^{-1/4}\beta^{-1/6} \leq \widehat{C}\,\beta^{-1/4}. \tag{4.6.5}$$

Hence, we obtain that

$$\|\check{\chi}_\nu v\|_2 \leq C\beta^{-5/4}. \tag{4.6.6}$$

Furthermore, since by (4.5.13a)

$$\|(\check{\chi}_\nu v)'\|_2 \leq [\mu_{\beta,+}]^{1/2}\|\check{\chi}_\nu v\|_2 + C(\|v\|_2 + \|\check{\chi}_\nu v\|_2^{1/2}\|\widehat{\psi}\|_2^{1/2}), \tag{4.6.7}$$

where $\mu_{\beta,+}$ is given by (3.1.26), we can conclude from (4.6.6) that

$$\|(\check{\chi}_\nu v)'\|_2 \leq C(\beta^{-3/4} + \|v\|_2). \tag{4.6.8}$$

Let

$$\hat{\chi}_\nu = \sqrt{1 - \check{\chi}_\nu^2}. \tag{4.6.9}$$

Clearly,

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)(\hat{\chi}_\nu v) = -\hat{\chi}_\nu'' v - 2\hat{\chi}_\nu' v' + \hat{\chi}_\nu\widehat{\psi}. \tag{4.6.10}$$

By (4.6.6) it holds that

$$\|v\|_2 \leq C(\|\hat{\chi}_\nu v\|_2 + \beta^{-5/4}).$$

Furthermore, we have, by (4.6.8) together with (4.6.6) for the last line, that

$$\begin{aligned}\|v'\|_2 &\leq \|(\check{\chi}_\nu v)'\|_2 + \|(\hat{\chi}_\nu v)'\| \\ &\leq C(\beta^{-3/4} + \|v\|_2 + \|(\hat{\chi}_\nu v)'\|) \\ &\leq \widehat{C}(\beta^{-3/4} + \|\hat{\chi}_\nu v\|_2 + \|(\hat{\chi}_\nu v)'\|).\end{aligned}$$

We now apply either (3.2.1a) or (3.1.3) (see the proof of Proposition 4.5.1, Step 2) to (4.6.10) to obtain

$$\begin{aligned}&\|\hat{\chi}_\nu v\|_2 + \beta^{-1/4}\|(\hat{\chi}_\nu v)'\|_2 \\ &\quad\leq C\beta^{-1/2}(\|v\|_2 + \|v'\|_2 + \|\hat{\chi}_\nu\widehat{\psi}\|_2) \\ &\quad\leq \widehat{C}\beta^{-1/2}(\beta^{-3/4} + \|(\hat{\chi}_\nu v)'\|_2 + \|(\hat{\chi}_\nu v)'\|_2 + \|\hat{\chi}_\nu\widehat{\psi}\|_2).\end{aligned}$$

Hence,

$$\|\hat{\chi}_\nu v\|_2 + \beta^{-1/4}\|(\hat{\chi}_\nu v)'\|_2 \leq C\beta^{-1/2}(\beta^{-3/4} + \|\hat{\chi}_\nu\widehat{\psi}\|_2).$$

By (4.2.16) and (4.2.10) we obtain that

$$\|\hat{\chi}_\nu\widehat{\psi}\|_2 \leq C\beta^{-3/4},$$

and hence
$$\|\hat{\chi}_\nu v\|_2 + \beta^{-1/4}\|(\hat{\chi}_\nu v)'\|_2 \leq C\beta^{-5/4}.$$
Combining the above with (4.6.6) and (4.6.10) yields (4.6.3).

Consider now the case where $\mu < -\mu_0$. Here, we use (3.1.84) and (3.1.85), applied to the pair $(v, \widehat{\psi})$, and then (4.6.5) to obtain that

$$\|v\|_2 + \beta^{-1/2}\|v'\|_2 \leq C\beta^{-1}\|\widehat{\psi}\|_2 \leq \widehat{C}\,\beta^{-5/4},$$

establishing, thereby, (4.6.3) for the case $\mu < -\mu_0$. ■

Remark 4.6.2. Let for $(\lambda, \beta) \in \mathbb{C} \times \mathbb{R}_+$, $\hat{g}(x) = h(x)\chi(1-x)/\psi_{\lambda,\beta}(1)$ where h is given by (4.2.15). By the same arguments used to establish (4.6.3) we may conclude under the assumptions of Lemma 4.6.1 that there exists C and β_0 such that for $\beta \geq \beta_0$,

$$\nu \in (U(0) - \widetilde{\nu}_1, U(0) + a\beta^{-1/2}) \quad \text{and} \quad -\mu_0 \leq \mu < \Upsilon\beta^{-1/2},$$

we have

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}\hat{g}\|_2 + \beta^{-1/2}\Big\|\frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}\hat{g}\Big\|_2 \leq C\,\beta^{-5/4}. \tag{4.6.11}$$

4.7 Resolvent estimates for large α

In this section, we will adapt the results of Section 6.3 in [3] to the present setting, involving a Neumann condition at $x = 0$. For $\alpha > 0$, we consider $\mathfrak{z}_\alpha$ to be the solution of

$$\begin{cases} -\mathfrak{z}'' + \alpha^2\mathfrak{z} = 0 & \text{for } x \in (0,1), \\ \mathfrak{z}(1) = 1 \text{ and } \mathfrak{z}'(0) = 0. \end{cases} \tag{4.7.1}$$

The solution of (4.7.1) is given by

$$\mathfrak{z}_\alpha(x) = \cosh(\alpha x)/\cosh(\alpha), \tag{4.7.2}$$

and hence, for large α decays exponentially fast away from $x = 1$.

Proposition 4.7.1. *Let $\theta_1 > 0$, $U \in C^2([0,1])$ satisfy (2.1.3), and $\Upsilon < \sqrt{-U''(0)}/2$. Let further $\widehat{\mu}_m > 0$ be given by [3, equation (6.57)], Then, for any $\widehat{\Upsilon} > 0$, there exist $\beta_0 > 0$ and $C > 0$ such that, for $\beta \geq \beta_0$ and $\alpha \geq \theta_1\beta^{1/3}$,*

$$\sup_{\substack{\Re\lambda \leq \min(\Upsilon\beta^{-1/2}, \\ \beta^{-1/3}[\widehat{\mu}_m - \widehat{\Upsilon} - \alpha^2\beta^{-2/3}/2])}} \|(\mathcal{L}_\beta^{\mathfrak{z}_\alpha} - \beta\lambda)^{-1}\| \leq \frac{C}{\beta^{1/2}[1 + \beta^{1/6}|U(0) - \nu|^{1/3}]}. \tag{4.7.3}$$

Proof. The proof follows the same lines of the proof of [3, Proposition 6.11], and hence we bring only its main ingredients.

Let $\theta = \alpha\beta^{-1/3}$ and

$$F(\lambda,\theta) = \int_{\mathbb{R}_+} e^{-\theta x}\ \mathrm{Ai}(e^{i\pi/6}(x+i\lambda))\,dx.$$

Let further

$$\omega(\beta,\lambda,\theta) := \frac{F(\beta^{1/3}\lambda,0)}{F(\beta^{1/3}\lambda,\theta)}.$$

We then define

$$\psi_\theta = \omega(\beta,\lambda,\theta)\psi,$$

where $\psi = \psi_{\lambda,\beta}$ is defined in (4.2.11), and

$$h_\theta = \omega(\beta,\lambda,\theta)h,$$

where h is defined by (4.2.12). Set

$$\tilde{v}_\theta = (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}h_\theta.$$

By [3, equation (6.79)] for any $\widehat{\Upsilon} > 0$ there exists $C > 0$ such that

$$\sup_{\Re\lambda\leq\beta^{-1/3}[\widehat{\mu}_m-\widehat{\Upsilon}-\alpha^2\beta^{-2/3}/2]} |\omega(\beta,\lambda,\theta)| \leq C\,\theta \quad \forall\theta\geq\theta_1, \tag{4.7.4}$$

and hence by (4.2.14) and (4.3.6) we obtain that

$$\|\tilde{v}_\theta\|_1 \leq C\theta\beta^{-2/3}. \tag{4.7.5}$$

Note that for any $\Upsilon > 0$, there exist β_0 and a such that for $\beta \geq \beta_0$ and $\nu < U(0) - a\beta^{-1/2}$, we have

$$\frac{\sqrt{-U''(0)}}{2}\beta^{-1/2} \leq \Upsilon\min(1,|U(0)-\nu|^{1/3})\beta^{-1/3},$$

which allows for the application of (4.2.14).

Suppose now that $(v,g) \in D(\mathcal{L}_\beta^{\mathfrak{z}\alpha}) \times L^2(0,1)$ satisfy

$$(\mathcal{L}_\beta^{\mathfrak{z}\alpha} - \beta\lambda)v = g.$$

Then as in [3, equation (6.20)] or (4.2.36)–(4.2.37) we may write v in the form

$$v = A(\psi_\theta - \tilde{v}_\theta) + u, \tag{4.7.6}$$

where

$$u = (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}g.$$

Taking the inner product of (4.7.6) with $\mathfrak{z}_\alpha$ yields

$$A\langle \mathfrak{z}_\alpha, (\psi_\theta - \tilde{v}_\theta)\rangle = \langle \mathfrak{z}_\alpha, u\rangle.$$

As in [3], using the approximation $\mathfrak{z}_\alpha \approx e^{-\alpha(1-x)}$ (see the display below [3, equation (6.95)] and its proof with minor changes), we then show that

$$\langle \mathfrak{z}_\alpha, (\psi_\theta - \tilde{v}_\theta)\rangle = \beta^{-1/3}[1 + \mathcal{O}(\beta^{-1/3})].$$

Consequently,

$$|A| \leq C\beta^{1/3}|\langle \mathfrak{z}_\alpha, u\rangle|. \tag{4.7.7}$$

To complete the proof we need an estimate for $|\langle \mathfrak{z}_\alpha, u\rangle|$. The proof in the case $\nu \leq U(1/2)$ is identical with the derivation of [3, equation (6.93)], given that (3.1.3) together with (3.1.84) give [3, equation (5.4)] in this case. Hence, we consider here only the case where $\nu > U(1/2)$.

We thus write, with the aid of either (3.1.3) or (3.2.5)

$$|\langle \mathfrak{z}_\alpha, \mathbf{1}_{[0,\hat{x}_\nu]}u\rangle| \leq Ce^{-\alpha/2}\|\mathbf{1}_{[0,\hat{x}_\nu]}u\|_2 \leq Ce^{-\theta\beta^{1/3}/2}\|g\|_2, \tag{4.7.8}$$

where $\hat{x}_\nu$ is given by (4.5.8) so that $U(\hat{x}_\nu) = \nu/2$. Furthermore,

$$|\langle \mathfrak{z}_\alpha, \mathbf{1}_{[\hat{x}_\nu,1]}u\rangle| \leq \|\mathfrak{z}_\alpha\|_1\|\mathbf{1}_{[\hat{x}_\nu,1]}u\|_\infty \leq \frac{C}{\theta\beta^{1/3}}\|\mathbf{1}_{[\hat{x}_\nu,1]}u\|_\infty. \tag{4.7.9}$$

Since $u(1) = 0$ it holds that

$$\|\mathbf{1}_{[\hat{x}_\nu,1]}u\|_\infty \leq \|\mathbf{1}_{[\hat{x}_\nu,1]}u\|_2\|\mathbf{1}_{[\hat{x}_\nu,1]}u'\|_2.$$

By (4.5.16) and (4.5.17), and (4.7.9) we then have

$$|\langle \mathfrak{z}_\alpha, \mathbf{1}_{[\hat{x}_\nu,1]}u\rangle| \leq \frac{C}{\theta\beta^{13/12}}\|g\|_2.$$

Substituting the above, together with (4.7.8) into (4.7.7) yields

$$|A| \leq \frac{C}{\theta\beta^{3/4}}\|g\|_2.$$

By (4.7.6), (4.2.16), (4.2.14), and (4.3.6) we then have

$$\|v\|_2 \leq \frac{C}{\theta\beta^{3/4}}(\|\psi_\theta\|_2 + \|\tilde{v}_\theta\|_2)\|g\|_2 + \|u\|_2. \tag{4.7.10}$$

By (4.2.16) with $k = 0$ and (4.7.4) it holds that

$$\frac{1}{\theta\beta^{3/4}}\|\psi_\theta\|_2 \leq C\,\lambda_\beta^{\frac{1}{4}}\beta^{-11/12}.$$

By (4.3.6) and (4.7.4) it holds that

$$\frac{1}{\theta\beta^{3/4}}\|\tilde{v}_\theta\|_2 \leq C\,\beta^{-4/3}.$$

Finally, (3.1.3) and (3.2.1a) establish the existence of $\Upsilon > 0$ such that

$$\|u\|_2 \leq \frac{C}{\beta^{1/2}[1+\beta^{1/6}|U(0)-\nu|^{1/3}]}\|g\|_2 \tag{4.7.11}$$

for all $U(0)/2 < \nu < U(0) + a\beta^{-1/2}$.

For $\nu \geq U(0) + a\beta^{-1/2}$ we can use (4.5.13b) with $\check{\chi}_\nu \equiv 1$ and $f = g$ to obtain

$$\|u\|_2 \leq \frac{C}{\beta|U(0)-\nu|}\|g\|_2,$$

and hence (4.7.11) holds true for all $\nu > U(0)/2$.

Combining the above yields

$$\|v\|_2 \leq \frac{C}{\beta^{1/2}[1+\beta^{1/6}|U(0)-\nu|^{1/3}]}\|g\|_2,$$

verifying thereby (4.7.3). ∎

Chapter 5

The Orr–Sommerfeld operator

5.1 Introduction

In this chapter, we prove Theorems 1.1.1 and 1.1.2 by obtaining inverse estimates for the Orr–Sommerfeld operator (1.1.7b). As in [3], we use the estimates for the inviscid operator $\mathcal{A}_{\lambda,\alpha}$ from Chapter 2 together with the resolvent estimates for the Schrödinger operators $\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}$ and $\mathcal{L}_\zeta^\beta$ from Chapters 3 and 4. In contrast with [3] we need to consider here many different cases depending on the values of $\Im\lambda$ and α.

Figure 5.1 presents a rough sketch of the various domains where each estimate is valid in the (α,ν) plane ($\nu = \Im\lambda$). The blank domain denotes the domain in the (α,ν) plane where resolvent estimates have not been obtained. We refer the reader to Section 1.2 for a brief explanation of the methods of the proof.

In the following we explain why the division of the (α,ν) plane into 10 subdomains is necessary. Propositions 5.12.1 and 5.12.2 deal with the case where $\nu \notin [0, U(0)]$ making use of the invertibility of $\mathcal{A}_{i\nu,\alpha}$ in these cases. The necessity of Proposition 5.7.1 which deals with the case $\alpha \gtrsim \beta^{1/3}$, and Proposition 5.8.1 (and Proposition 5.8.2) which deals with the case $\alpha \ll \beta^{-1/6}$ is explained in Section 1.2. Proposition 5.6.1 deals with the case $|\nu| < \nu_0 < U(0)$ and $1 \ll \alpha \ll \beta^{1/3}$. In this range of α values we may effectively use the fact that $\|(-d^2/dx^2 + \alpha^2)^{-1}\|$ is small at the conclusion of the proof. Proposition 5.4.1 deals with the case $\nu \geq \beta^{-1/5+\delta}$ for any $0 < \delta < 1/5$ and $\alpha \lesssim 1$. In the proof we use the same methods as in [3], till the value of ν becomes to small due to the non-invertibility of $\mathcal{A}_{0,0}$. For $|\nu| \leq \beta^{-1/5+\delta}$ and $\beta^{-1/10+\delta/2} \ll \alpha \lesssim 1$ we use Proposition 5.5.1. This range of α values allows the application of Proposition 2.5.1 towards the end of the proof. Proposition 5.10.1 deals with the case where $|U(0) - \nu| \lesssim \beta^{-1/2}$. Here, we can approximate U by a quadratic potential near $x = 0$ and use the estimates in Sections 2.9, 3.2, and 4.3. Finally, Proposition 5.11.1 deals with the transition from a linear behavior of $U - U(x_\nu)$ (x_ν is defined in (2.4.5)) to a quadratic behavior near x_ν.

5.2 Preliminaries

We begin by recalling from (4.6.1) the definition of the boundary terms

$$\hat{\psi}_{\lambda,\beta}(x) = \frac{\mathrm{Ai}\big(\beta^{1/3}e^{-i\pi/6}[(1-x) - i\lambda]\big)}{\mathrm{Ai}\big(e^{-i2\pi/3}\beta^{1/3}\lambda\big)}\chi(1-x), \tag{5.2.1}$$

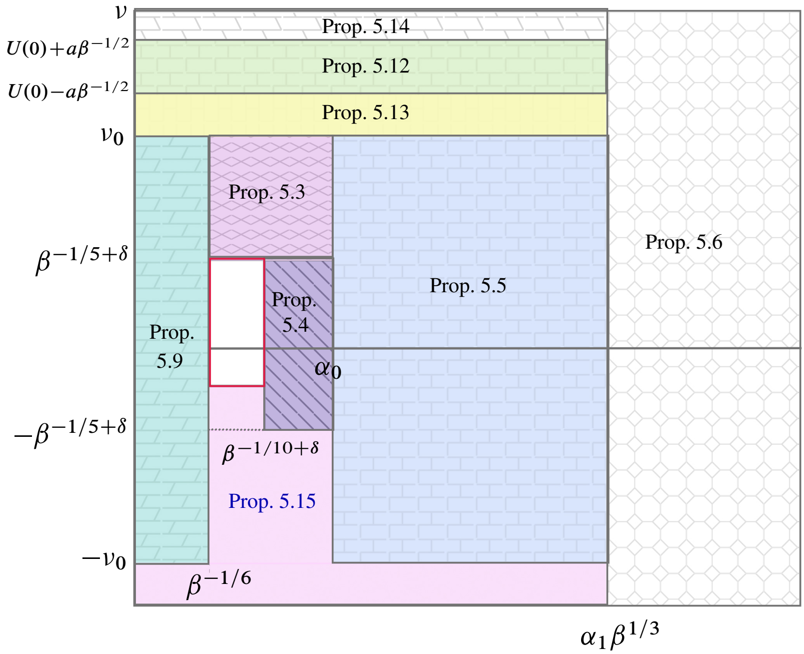

Figure 5.1. Summary of the results in Chapter 5.

where we recall that χ is given by (2.6.20). We also recall from [3, Section 8.3.2, equation (8.91)] that there exists $\Upsilon > 0$ such that, for all $\beta \geq 1$ and $\Re\lambda < \Upsilon\beta^{-1/3}$, it holds that

$$\|(1-x)^s \widehat{\psi}_{\lambda,\beta}\|_1 \leq C\, \lambda_\beta^{-(s+1)/2}\, \beta^{-(s+1)/3}, \quad s \in [0,3]. \tag{5.2.2}$$

Similarly, from [3, Proposition A.8 and equation (A.43c,d)], we can conclude the existence of $C > 0$ such that

$$\|(1-x)^s \widehat{\psi}_{\lambda,\beta}\|_\infty \leq C\, \lambda_\beta^{-s/2}\, \beta^{-s/3}, \quad s \in [0,3]. \tag{5.2.3}$$

We further recall the definition of the inviscid operator in equation (2.1.1), which is the Neumann–Dirichlet realization in $(0,1)$ of

$$\mathcal{A}_{\lambda,\alpha} \stackrel{\text{def}}{=} (U + i\lambda)\Big(-\frac{d^2}{dx^2} + \alpha^2\Big) + U''$$

for $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{R}$. We note that $\mathcal{A}_{\lambda,\alpha}$ is invertible when either $\nu \notin [0, U(0)]$ or $|\mu| > 0$, either by Proposition 2.6.1 or by Proposition 4.13 in [3] which holds true since $|U''| > 0$. We introduce in addition

$$\phi_{\lambda,\beta,\alpha} := \mathcal{A}_{\lambda,\alpha}^{-1}(U + i\lambda)\widehat{\psi}_{\lambda,\beta}. \tag{5.2.4}$$

We dedicate this section to two extensions of [3, Lemma 8.1]. These are useful in order to establish the contribution of the boundary terms in (1.2.6). The reader is referred to Section 1.2 for more details on the necessity of these estimates. The first of them is the following lemma which considers the case $\nu > \beta^{-1/5}$. The proof is significantly more complex than the proof of [3, Lemma 8.1] in view of the non-injectivity of $\mathcal{A}_{0,0}$.

Lemma 5.2.1. *Let $U \in C^3([0,1])$ satisfy (2.1.3). There exist positive constants Υ, C, $\widehat{C}$, ν_0, and β_0 such that, for all $\alpha \geq 0$, $\beta \geq \beta_0$, $\lambda \in \mathbb{C}$ for which $\mu < \Upsilon\beta^{-1/3}$, $\mu \neq 0$, and $\beta^{-1/5} < \nu < \nu_0$ it holds that*

$$\|\phi_{\lambda,\beta,\alpha}\|_{1,2} \leq C\nu^{-1}\,[|\lambda|\beta]^{-3/4} \tag{5.2.5}$$

and

$$|\phi_{\lambda,\beta,\alpha}(x_\nu)| \leq \widehat{C}\,[|\lambda|\beta]^{-3/4}. \tag{5.2.6}$$

Proof. Step 1. We prove (5.2.5) and (5.2.6) for the case $\alpha^2 < \nu^{-1}$ and $0 < |\mu| < 1$.

By (2.6.4) applied to the pair (ϕ, v) with $v = (U + i\lambda)\widehat{\psi}$ (see (5.2.4)), it holds that

$$|\phi(x_\nu)|^2 \leq C\,|\langle\phi, \widehat{\psi}\rangle|. \tag{5.2.7}$$

Let $\tilde{x}_\nu = (1 + x_\nu)/2$. To estimate the right-hand side of (5.2.7), we first obtain a bound for $\|\phi'\|_{L^\infty(\tilde{x}_\nu,1)}$. To this end, we integrate the balance $(U + i\lambda)^{-1}\mathcal{A}_{\lambda,\alpha}\phi = \widehat{\psi}$ over $(\tilde{x}_\nu, 1)$ to obtain

$$\|\phi''\|_{L^1(\tilde{x}_\nu,1)} \leq \Big(\alpha^2 + \frac{C}{\nu}\Big)\|\phi\|_{L^1(\tilde{x}_\nu,1)} + \|\widehat{\psi}\|_{L^1(\tilde{x}_\nu,1)}. \tag{5.2.8}$$

Since $\phi(1) = 0$ it holds that

$$\|\phi\|_{L^1(\tilde{x}_\nu,1)} \leq \|\phi'\|_{L^\infty(\tilde{x}_\nu,1)}\|1 - x\|_{L^1(\tilde{x}_\nu,1)} \leq C\,\|\phi'\|_{L^\infty(\tilde{x}_\nu,1)}\nu^2. \tag{5.2.9}$$

Using (5.2.8), (5.2.9), (5.2.2) for $s = 0$, and the fact that (in this step) $\alpha^2 \leq |\nu|^{-1}$, we obtain that

$$\|\phi''\|_{L^1(\tilde{x}_\nu,1)} \leq C\,(\nu\|\phi'\|_{L^\infty(\tilde{x}_\nu,1)} + [|\lambda|\beta]^{-1/2}). \tag{5.2.10}$$

Clearly, there exists $z_\nu \in [\tilde{x}_\nu, 1]$ such that

$$|\phi'(z_\nu)| \leq |1 - \tilde{x}_\nu|^{-1/2}\|\phi'\|_{L^2(\tilde{x}_\nu,1)} \leq \widehat{C}\,|\nu|^{-1/2}\|\phi'\|_{L^2(\tilde{x}_\nu,1)}.$$

Since for any $x \in (\tilde{x}_\nu, 1)$ it holds that

$$|\phi'(x) - \phi'(z_\nu)| \leq \|\phi''\|_{L^1(\tilde{x}_\nu,1)},$$

we can deduce that

$$\|\phi'\|_{L^\infty(\tilde{x}_\nu,1)} \leq C\nu^{-1/2}\|\phi'\|_{L^2(\tilde{x}_\nu,1)} + \|\phi''\|_{L^1(\tilde{x}_\nu,1)}. \tag{5.2.11}$$

We can then conclude, using (5.2.10), that we can choose $\nu_0 > 0$ and $C > 0$ such that for all $0 < \nu < \nu_0$

$$\|\phi'\|_{L^\infty(\tilde{x}_\nu,1)} \leq C(\nu^{-1/2}\|\phi'\|_{L^2(\tilde{x}_\nu,1)} + [|\lambda|\beta]^{-1/2}). \tag{5.2.12}$$

We now write

$$\begin{aligned} |\langle\phi,\widehat{\psi}\rangle| &\leq |\langle\phi,\widehat{\psi}\rangle_{L^2(\tilde{x}_\nu,1)}| + |\langle\phi,\widehat{\psi}\rangle_{L^2(0,\tilde{x}_\nu)}| \\ &\leq \|\phi'\|_{L^\infty(\tilde{x}_\nu,1)}\|(1-x)\widehat{\psi}\|_{L^1(\tilde{x}_\nu,1)} + \|\phi'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_{L^1(0,\tilde{x}_\nu)}, \end{aligned} \tag{5.2.13}$$

and then observe that

$$\begin{aligned} \|(1-x)^{1/2}\widehat{\psi}\|_{L^1(0,\tilde{x}_\nu)} &= \|(1-x)^{-5/2}(1-x)^3\widehat{\psi}\|_{L^1(0,\tilde{x}_\nu)} \\ &\leq (1-\tilde{x}_\nu)^{-5/2}\|(1-x)^3\widehat{\psi}\|_{L^1(0,\tilde{x}_\nu)} \\ &\leq C\,\nu^{-5/2}\,\|(1-x)^3\widehat{\psi}\|_1. \end{aligned} \tag{5.2.14}$$

By (2.6.2) with $v = (U + i\lambda)\widehat{\psi}$ together with (2.6.1) it holds that

$$\|\phi'\|_2 \leq \frac{C}{\nu^{3/2}}\|[(1-x)^{1/2} + \nu^{-1/2}(1-x)]\widehat{\psi}\|_1. \tag{5.2.15}$$

Hence, by (5.2.2) with $s = 1/2, 1, 3$, (5.2.14), and (5.2.15) we obtain that

$$\begin{aligned} &\|\phi'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_{L^1(0,\tilde{x}_\nu)} \\ &\quad\leq \frac{C}{\nu^4}\|[(1-x)^{1/2} + \nu^{-1/2}(1-x)]\widehat{\psi}\|_1\|(1-x)^3\widehat{\psi}\|_1 \\ &\quad\leq \frac{\widehat{C}}{\nu^4}([|\lambda|\beta]^{-11/4} + \nu^{-1/2}[|\lambda|\beta]^{-3}). \end{aligned}$$

Then, since by assumption $\nu > \beta^{-1/5}$, we obtain that

$$\|\phi'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_{L^1(0,\tilde{x}_\nu)} \leq \widetilde{C}\beta^{-1/5}\,[|\lambda|\beta]^{-3/2}. \tag{5.2.16}$$

By (5.2.2) with $s = 1$ and (5.2.12) we have that

$$\|\phi'\|_{L^\infty(\tilde{x}_\nu,1)}\|(1-x)\widehat{\psi}\|_{L^1(\tilde{x}_\nu,1)} \leq C\,(\nu^{-1/2}\|\phi'\|_{L^2(\tilde{x}_\nu,1)} + [|\lambda|\beta]^{-1/2})\,[|\lambda|\beta]^{-1}. \tag{5.2.17}$$

Substituting (5.2.17), together with (5.2.16) into (5.2.13) then yields

$$|\langle\phi,\widehat{\psi}\rangle| \leq C(|\nu|^{-1/2}\|\phi'\|_{L^2(x_\nu,1)} + [|\lambda|\beta]^{-1/2})[|\lambda|\beta]^{-1}. \tag{5.2.18}$$

We now use (2.6.19) and (2.6.1) to obtain that

$$\|\phi'\|_{L^2(x_\nu,1)} \leq C\big[\nu^{-1/2}|\langle\phi,\widehat{\psi}\rangle|^{1/2} + \|[(1-x)^{1/2} + \nu^{-1/2}(1-x)]\widehat{\psi}\|_1\big]. \tag{5.2.19}$$

Substituting (5.2.19) into (5.2.18) yields, with the aid of (5.2.2) and the fact that $\nu > \beta^{-1/5}$

$$|\langle \phi, \widehat{\psi} \rangle| \le C(\nu^{-1}|\langle \phi, \widehat{\psi} \rangle|^{1/2} + [|\lambda|\beta]^{-1/2})[|\lambda|\beta]^{-1},$$

which immediately implies

$$|\langle \phi, \widehat{\psi} \rangle| \le C(\nu^{-2}[|\lambda|\beta]^{-1} + [|\lambda|\beta]^{-1/2})[|\lambda|\beta]^{-1} \le \widetilde{C}\,[|\lambda|\beta]^{-3/2}. \tag{5.2.20}$$

From the above inequality, with the aid of (5.2.7), we conclude (5.2.6). Hence, by (5.2.19), we also get that

$$\|\phi'\|_{L^2(x_\nu,1)} \le C\,\nu^{-1/2}[|\lambda|\beta]^{-3/4}. \tag{5.2.21}$$

To obtain an effective bound for $\|\phi'\|_2$ we use (2.6.53) and (2.6.1) to obtain, with the aid of (5.2.19), (5.2.21), and (5.2.2)

$$\begin{aligned} \|\phi'\|_2 &\le C\big(\|[(1-x)^{1/2} + \nu^{-1/2}(1-x)]\widehat{\psi}\|_1 + \nu^{-1/2}\|\phi'\|_{L^2(x_\nu,1)}\big) \\ &\le \widetilde{C}\,\nu^{-1}[|\lambda|\beta]^{-3/4}, \end{aligned} \tag{5.2.22}$$

from which we conclude (5.2.5) by using Poincaré's inequality.

Step 2. We prove (5.2.5) and (5.2.6) for the case $\alpha^2 > \nu^{-1}$ and $|\mu| \le 1$.

To obtain (5.2.5) for $\alpha^2 > |\nu|^{-1}$ and $\nu < \nu_0$, we observe that for any $A_0 > 0$ we can choose ν_0 such that $\alpha^2 \ge A_0$ for $\nu < \nu_0$, and consequently use (2.6.86) in the form (with $v = (U + i\lambda)\widehat{\psi}$)

$$\begin{aligned} \|\phi\|_{1,2} &\le C\left\|[(1-x)^{1/2} + \nu^{-1/2}(1-x)]\frac{v}{U+i\lambda}\right\|_1 \\ &= C\,\|[(1-x)^{1/2} + \nu^{-1/2}(1-x)]\widehat{\psi}\|_1. \end{aligned}$$

Using (5.2.2) and the fact that $\nu > \beta^{-1/5}$, we obtain,

$$\|\phi\|_{1,2} \le \widehat{C}([|\lambda|\beta]^{-3/4} + \nu^{-1/2}[|\lambda|\beta]^{-1}) \le \widetilde{C}\,[|\lambda|\beta]^{-3/4},$$

which implies (5.2.5) for $\alpha^2 > \nu^{-1}$. We now use (5.2.7) to obtain

$$|\phi(x_\nu)|^2 \le C\,\|\phi'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_1. \tag{5.2.23}$$

We may then conclude (5.2.6) as well by using (5.2.5) and (5.2.2).

Step 3. We prove (5.2.5) and (5.2.6) for $|\mu| \ge 1$.

The proof of (5.2.5) in this case follows from (2.6.17) which yields

$$\|\phi'\|_2^2 \le C\,|\langle \phi, \widehat{\psi} \rangle|.$$

Consequently, by (5.2.2) we obtain that

$$\|\phi'\|_2^2 \le C\,\|\phi'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_1 \le \widetilde{C}\,[|\lambda|\beta]^{-3/4}\|\phi'\|_2,$$

yielding, thereby,

$$\|\phi'\|_2 \leq \widehat{C}\,[\|\lambda|\beta]^{-3/4}.$$

We can then complete the proof of (5.2.5) by using Poincaré's inequality. The proof of (5.2.6) follows from (5.2.5) and Sobolev embeddings. ∎

We next consider (as in Proposition 2.5.1) the case $|\lambda| \ll 1$ and α large enough, which will be sufficient to guarantee a satisfactory estimate of $\|\mathcal{A}_{\lambda,\alpha}^{-1}\|$ despite the fact that $\mathcal{A}_{0,0}$ is not injective.

Lemma 5.2.2. *Let $\delta > 0$, and $U \in C^3([0,1])$ satisfy (2.1.3). There exist positive constants Υ_0, λ_0, C, and β_0 such that, for all $\beta \geq \beta_0$, all $\lambda \in \mathbb{C} \setminus \{0\}$ for which $-\lambda_0 < \Re\lambda \leq \beta^{-1/3}\Upsilon_0$, $|\Im\lambda| < \lambda_0$, and $\alpha \geq \alpha_{\lambda,\delta}$, it holds that*

$$|c_\parallel(\lambda,\beta,\alpha)| \leq \frac{C}{|\alpha^2\|U\|_2^2 + i\lambda|}(\lambda_\beta^{-1}\beta^{-2/3} + \lambda_\beta^{-1/2}|\lambda|\beta^{-1/3}) \tag{5.2.24a}$$

and

$$\begin{aligned}&\Big\|\phi_{\lambda,\beta,\alpha} - c_\parallel(\lambda,\beta,\alpha)U\Big\|_{1,2}\\ &\leq C\Big[\lambda_\beta^{-3/4}\beta^{-1/2} + \frac{|\lambda|}{|\alpha^2\|U\|_2^2 + i\lambda|}(\lambda_\beta^{-1}\beta^{-2/3} + \lambda_\beta^{-1/2}|\lambda|\beta^{-1/3})\Big],\end{aligned} \tag{5.2.24b}$$

where, as in (2.3.1),

$$c_\parallel(\lambda,\beta,\alpha) = \frac{\langle U, \phi_{\lambda,\beta,\alpha}\rangle}{\|U\|_2^2},$$

and as in (2.5.1)

$$\alpha_{\lambda,\delta} = \|U\|_2^{-1}(|\Im\lambda|(1+2\delta))^{1/2}.$$

Proof. We write as in (2.3.1) with $\phi = \phi_{\lambda,\beta,\alpha}$

$$\phi = c_\parallel\, U + \phi_\perp.$$

Then by (2.5.3a) and (2.5.2b), there exists λ_1 such that for $0 < |\lambda| < \lambda_1$,

$$|c_\parallel| \leq \frac{1 + C|\lambda|^2\log|\lambda|^{-1}}{|\alpha^2\|U\|_2^2 + i\lambda|}(\|(U + i\lambda)\widehat{\psi}\|_1 + C|\lambda|\|\widehat{\psi}\|_1). \tag{5.2.25}$$

It follows from (5.2.2) (with $s = 0$) that for some positive C

$$\|\widehat{\psi}\|_1 \leq C\,\lambda_\beta^{-1/2}\,\beta^{-1/3}. \tag{5.2.26}$$

Furthermore, by (5.2.26) and (5.2.2) (with $s = 1$)

$$\|(U + i\lambda)\widehat{\psi}\|_1 \leq |\lambda|\,\|\widehat{\psi}\|_1 + C\|(1-x)\widehat{\psi}\|_1 \leq \widehat{C}(|\lambda|\lambda_\beta^{-1/2}\beta^{-1/3} + \lambda_\beta^{-1}\beta^{-2/3}). \tag{5.2.27}$$

Consequently, for β_0 large enough, there exists $\lambda_0 < \lambda_1/\sqrt{2}$ such that for any $|\lambda| \leq \sqrt{2}\lambda_0$ it holds by (5.2.25), that

$$|c_\| | \leq \frac{C}{|\alpha^2 \, \|U\|_2^2 + i\lambda|}\big(\lambda_\beta^{-1}\beta^{-2/3} + |\lambda|\lambda_\beta^{-1/2}\beta^{-1/3}\big),$$

readily verifying (5.2.24a).

We now apply (2.5.3b) to obtain

$$\|\phi_\perp\|_{1,2} \leq C\Big[\|(1-x)^{1/2}\widehat{\psi}\|_1 + \frac{|\lambda|}{|\alpha^2\|U\|_2^2 + i\lambda|}(\|U\widehat{\psi}\|_1 + C|\lambda|\|\widehat{\psi}\|_1)\Big]$$

which, combined with (5.2.2) (with $s = 1/2$ and $s = 1$), (5.2.26), yields

$$\|\phi_\perp\|_{1,2} \leq C\Big[\lambda_\beta^{-3/4}\beta^{-1/2} + \frac{|\lambda|}{|\alpha^2\|U\|_2^2 + i\lambda|}(\lambda_\beta^{-1}\beta^{-2/3} + \lambda_\beta^{-1/2}|\lambda|\beta^{-1/3})\Big], \tag{5.2.28}$$

establishing, thereby, (5.2.24b). ∎

5.3 Resolvent estimates and Fredholm property

We recall from the introduction that

$$\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta} = (\mathcal{L}_\beta - \beta\lambda)\Big(\frac{d^2}{dx^2} - \alpha^2\Big) - i\beta U'' \tag{5.3.1}$$

on $(0, 1)$ with domain (see (1.1.11))

$$D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}) = \big\{u \in H^4(0,1), u'(0) = u^{(3)}(0) = 0 \text{ and } u(1) = u'(1) = 0\big\} \tag{5.3.2}$$

and

$$\mathcal{L}_\beta = -\frac{d^2}{dx^2} + i\beta U. \tag{5.3.3}$$

Note that this domain is independent of the parameters (λ, α, β), i.e., $D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}) = D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{0,0,0})$.

It can be easily verified that $\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{0,0,0}$ is invertible. Next, we observe that

$$\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\big(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{0,0,0}\big)^{-1} = I + K_{\lambda,\alpha,\beta},$$

where $K_{\lambda,\alpha,\beta}$ is a compact operator from $L^2(0,1)$ to $L^2(0,1)$. Hence, $I + K_{\lambda,\alpha,\beta}$ is a Fredholm operator. Considering again the family $\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta} = (I + K_{\lambda,\alpha,\beta})\,\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{0,0,0}$, we can conclude that it is a Fredholm family from $D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{0,0,0})$ into $L^2(0,1)$.

Since its index depends continuously on (α, β, λ) and vanishes for $(\alpha, \beta, \lambda) = (0, 0, 0)$, it must be zero for all $(\lambda, \alpha, \beta) \in \mathbb{C} \times \mathbb{R}^2$.

The rest of Chapter 5 is dedicated to the estimation of $(\mathcal{B}^{\mathfrak{D},\mathfrak{N}}_{\lambda,\alpha,\beta})^{-1}$. In practice, we first show in each section that for some subset of parameters (λ,α,β),

$$\phi \in D(\mathcal{B}^{\mathfrak{D},\mathfrak{N}}_{\lambda,\alpha,\beta}) \Rightarrow \|\phi\|_{1,2} \leq C(\lambda,\alpha,\beta)\|f\|_2, \tag{5.3.4}$$

where $f = \mathcal{B}^{\mathfrak{D},\mathfrak{N}}_{\lambda,\alpha,\beta}\phi$. It follows that $\mathcal{B}^{\mathfrak{D},\mathfrak{N}}_{\lambda,\alpha,\beta}$ is injective and since its index vanishes, we can conclude the existence of $(\mathcal{B}^{\mathfrak{D},\mathfrak{N}}_{\lambda,\alpha,\beta})^{-1} : L^2(0,1) \to D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{0,0,0})$ together with the estimate

$$\|(\mathcal{B}^{\mathfrak{D},\mathfrak{N}}_{\lambda,\alpha,\beta})^{-1}\| + \left\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{D},\mathfrak{N}}_{\lambda,\alpha,\beta})^{-1}\right\| \leq C(\lambda,\alpha,\beta). \tag{5.3.5}$$

5.4 Resolvent estimates for $\beta^{-1/5} \ll |\Im\lambda| < U(0)$

The next proposition is somewhat similar to [3, Lemma 8.8] albeit with a significant difference: the fact that $\mathcal{A}_{0,0}$ is not invertible, which makes the estimates become significantly more complex.

Proposition 5.4.1. *Let $U \in C^4([0,1])$ satisfy (2.1.3) and the assumption $U'''(0) = 0$. Let $0 < \delta < 1/5$, $\nu_0 < U(0)$, and α_0 denote positive constants. There exist $C > 0$, and $\beta_0 > 0$ such that for all $\beta \geq \beta_0$, it holds that*

$$\sup_{\substack{0\leq\alpha\leq\alpha_0 \\ \Re\lambda\leq\beta^{-2/5-\delta} \\ \beta^{-1/5+\delta}\leq\Im\lambda<\nu_0}} \Im\lambda\Big(\|(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\| + \left\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\right\|\Big) \leq C\beta^{-1/2+\delta}. \tag{5.4.1}$$

Proof. We assume throughout the proof, without any loss of generality, that $0 < \delta \leq 1/30$.

Step 1. Preliminaries. Let $\phi \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})$ and $f = \mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\phi$. Let further $v_{\mathfrak{D}} \in H^2(0,1)$ be defined by

$$v_{\mathfrak{D}} = \mathcal{A}_{\lambda,\alpha}\phi + (U + i\lambda)\phi''(1)\hat{\psi}, \tag{5.4.2}$$

where $\hat{\psi} = \hat{\psi}_{\lambda,\beta}$ is given by (5.2.1). Note that by (1.1.11), (2.1.3) and the fact that $U'''(0) = 0$, we have

$$v_{\mathfrak{D}}(1) = v'_{\mathfrak{D}}(0) = 0, \tag{5.4.3}$$

and hence $v_{\mathfrak{D}} \in D(\mathcal{L}^{\mathfrak{N},\mathfrak{D}}_\beta)$ and we may introduce, as in [3, Lemma 8.8],

$$g_{\mathfrak{D}} := (\mathcal{L}^{\mathfrak{N},\mathfrak{D}}_\beta - \beta\lambda)v_{\mathfrak{D}}, \tag{5.4.4a}$$

which is expressible in the alternative form (using the fact that $f = \mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\phi$)

$$g_{\mathfrak{D}} = (U + i\lambda)(-f + \phi''(1)\hat{g}) - (U''\phi)'' - 2U'\tilde{v}'_{\mathfrak{D}} - U''\tilde{v}_{\mathfrak{D}}, \tag{5.4.4b}$$

wherein

$$\hat{g} := (\mathcal{L}_\beta - \beta\lambda)\widehat{\psi} \tag{5.4.5}$$

and

$$\tilde{v}_{\mathfrak{D}} := \frac{v_{\mathfrak{D}} - U''\phi}{U + i\lambda} = -\phi'' + \alpha^2\phi + \phi''(1)\widehat{\psi}. \tag{5.4.6}$$

We note that

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)\tilde{v}_{\mathfrak{D}} = i\beta U''(\phi - \phi(x_\nu)) + i\beta U''\phi(x_\nu) - f + \phi''(1)\hat{g}, \tag{5.4.7}$$

where x_ν is given by (2.4.5). Recall that $\mathcal{L}_\beta^{\mathfrak{N}}$ stands for $\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}$, which is defined in (3.0.1).

As in the proof of [3, Lemma 8.8] (see equation (8.90) there) we can integrate by parts to obtain

$$\begin{aligned}
&\Re\langle (U'')^{-1}\tilde{v}_{\mathfrak{D}}, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)\tilde{v}_{\mathfrak{D}} - i\beta U''\phi\rangle \\
&\quad = \|(U'')^{-1/2}\tilde{v}_{\mathfrak{D}}'\|_2^2 + \Re\langle \big((U'')^{-1}\big)'\tilde{v}_{\mathfrak{D}}, \tilde{v}_{\mathfrak{D}}'\rangle - \beta\mu\, \|(U'')^{-1/2}\tilde{v}_{\mathfrak{D}}\|_2^2 \\
&\qquad + \beta\, \Re\langle \phi''(1)\widehat{\psi}, i\phi\rangle.
\end{aligned} \tag{5.4.8}$$

We begin the estimation of $\tilde{v}_{\mathfrak{D}}'$ by obtaining a bound for the last term on the right-hand side of (5.4.8).

Step 2: Estimate of $\beta\Re\langle\phi''(1)\widehat{\psi}, i\phi\rangle$. We begin by writing ϕ as the sum

$$\phi = \hat{w} + \phi''(1)w$$

with

$$w(x) = \int_x^1 (\xi - x)\widehat{\psi}(\xi)\, d\xi,$$

and the remainder

$$\hat{w}(x) := \int_x^1 (\xi - x)[\phi''(\xi) - \phi''(1)\widehat{\psi}(\xi)]\, d\xi.$$

Then, we separately estimate the contribution of the terms $\beta\Re\langle\phi''(1)\widehat{\psi}, i\phi''(1)w\rangle$ and $\beta\Re\langle\phi''(1)\widehat{\psi}, i\hat{w}\rangle$. By (5.2.3) it holds that

$$|w(x)| \leq C\, |1 - x|^2.$$

Consequently,

$$|\Re\langle\phi''(1)\widehat{\psi}, i\phi''(1)w\rangle| \leq C\, |\phi''(1)|^2 \|(1-x)^2\widehat{\psi}\|_1,$$

and hence, by (5.2.2) with $s = 2$, we then obtain that

$$\beta|\Re\langle\phi''(1)\widehat{\psi}, i\phi''(1)w\rangle| \leq C\, [1 + |\lambda|\beta^{1/3}]^{-3/2}\, |\phi''(1)|^2 \tag{5.4.9}$$

(see [3, equation (8.90)]).

To estimate $\beta\Re\langle\phi''(1)\widehat{\psi}, i\hat{w}\rangle$, we first obtain by (5.4.6), for $x \in (0,1)$,

$$\begin{aligned}\left|\overline{\widehat{\psi}(x)}\hat{w}(x)\right| &= \left|\overline{\widehat{\psi}(x)}\int_x^1 (\xi - x)[-\tilde{v}_{\mathfrak{D}}(\xi) + \alpha^2\phi(\xi)]\,d\xi\right| \\ &\leq C(1-x)^{5/2}\,|\widehat{\psi}(x)|(\|\tilde{v}_{\mathfrak{D}}'\|_2 + \alpha^2\|\phi'\|_2),\end{aligned} \tag{5.4.10}$$

where to obtain the last inequality we used the fact that

$$|\phi(x)| \leq (1-x)^{1/2}\|\phi'\|_2 \quad \text{and} \quad |\tilde{v}_{\mathfrak{D}}(x)| \leq (1-x)^{1/2}\|\tilde{v}_{\mathfrak{D}}'\|_2.$$

Using (5.2.2) with $s = 5/2$, we obtain

$$\beta|\Re\langle\phi''(1)\widehat{\psi}, i\hat{w}\rangle| \leq C|\phi''(1)|\lambda_\beta^{-7/4}\beta^{-1/6}(\|\tilde{v}_{\mathfrak{D}}'\|_2 + \alpha^2\|\phi'\|_2). \tag{5.4.11}$$

Consequently, from (5.4.9) and (5.4.11), we thus get, as $\alpha \leq \alpha_0$,

$$\begin{aligned}\beta|\Re\langle\phi''(1)\widehat{\psi}, i\phi\rangle| \leq C\big(&[1 + |\lambda|\beta^{1/3}]^{-3/2}|\phi''(1)|^2 \\ &+ \lambda_\beta^{-7/4}\beta^{-1/6}|\phi''(1)|(\|\tilde{v}_{\mathfrak{D}}'\|_2 + \|\phi'\|_2)\big).\end{aligned} \tag{5.4.12}$$

(See [3, equation (8.93)].)

Step 3: Estimate $|\phi''(1)|$. Let $\mathfrak{z} = \mathfrak{z}_\alpha$ be given by (4.7.2) and recall that $\phi \in D(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})$. As $\langle\mathfrak{z}, \phi'' - \alpha^2\phi\rangle = 0$, $\phi'' - \alpha^2\phi$ belongs to the domain of $\mathcal{L}_\beta^{\mathfrak{z}} - \beta\lambda$. Hence, we may write

$$(\mathcal{L}_\beta^{\mathfrak{z}} - \beta\lambda)(\phi'' - \alpha^2\phi) = i\beta U''\phi + f. \tag{5.4.13}$$

We separately solve in $D(\mathcal{L}_\beta^{\mathfrak{z}})$ the equations $(\mathcal{L}_\beta^{\mathfrak{z}} - \beta\lambda)v = f$ and $(\mathcal{L}_\beta^{\mathfrak{z}} - \beta\lambda)v_1 = i\beta U''\phi := f_1$, so that $(\phi'' - \alpha^2\phi) = v_1 + v_2$, and then apply (4.2.6) to the pair (v_1, f_1) (note that the assumptions of Proposition 4.2.1 are satisfied) and (4.2.4) to the pair (v, f) to obtain

$$|\phi''(1)| \leq C\,\lambda_\beta^{1/2}\left(\beta^{1/3}[\|\phi\|_{1,2} + |\phi(x_\nu)|\,\big|\log|\nu + i\mathfrak{m}|^{-1}\big|] + \beta^{-1/2}\|f\|_2\right), \tag{5.4.14}$$

where $\mathfrak{m}$ is defined in 4.2.6b. Note that since $\alpha \leq \alpha_0$ it holds that $\|\mathfrak{z}\|_{1,p} \leq C(\alpha_0)$. Note further that

$$|\phi(x_\nu)|\,\big|\log|\nu + i\mathfrak{m}|^{-1}\big| \leq C\,\nu^{1/2}\big|\log|\nu + i\mathfrak{m}|^{-1}\big|\,\|\phi'\|_2 \leq \widehat{C}\,\|\phi'\|_2.$$

Then, for any $\widehat{\nu}_0 > 0$, there exists a constant $C > 0$ such that for $|\lambda| \geq \widehat{\nu}_0\beta^{-1/3}$, (in particular it holds for $\nu > \beta^{-1/5}$ for sufficiently large β_0).

$$|\phi''(1)| \leq C\,|\lambda|^{1/2}\big(\beta^{1/2}[\|\phi\|_{1,2} + \beta^{-1/3}]\|f\|_2\big). \tag{5.4.15}$$

Substituting (5.4.14) into (5.4.12) yields

$$\begin{aligned}\beta|\Re\langle\phi''(1)\widehat{\psi}, i\phi\rangle| \leq C\big[&\lambda_\beta^{-1/2}\left(\beta^{2/3}\|\phi\|_{1,2}^2 + \beta^{-1}\|f\|_2^2\right) \\ &+ \lambda_\beta^{-5/4}\big(\beta^{1/3}\|\phi\|_{1,2} + \beta^{-1/2}\|f\|_2\big)\big(\|\tilde{v}_{\mathfrak{D}}'\|_2 + \|\phi'\|_2\big)\big],\end{aligned}$$

from which, we conclude that for any $\hat{\delta} > 0$ there exists $C_{\hat{\delta}} > 0$ such that

$$\beta|\,\Re\langle\phi''(1)\widehat{\psi}, i\phi\rangle| \leq C_{\hat{\delta}}\big(\lambda_\beta^{-1/2}(\beta^{2/3}\|\phi\|_{1,2}^2 + \beta^{-1}\|f\|_2^2) + \hat{\delta}\,\|\tilde{v}_{\mathfrak{D}}'\|_2^2\big). \quad (5.4.16)$$

Since, as in the proof of (4.2.18) (see also [3, equation (6.18)]), we have for $\hat{g}$, introduced in (5.4.5),

$$|\hat{g}(x)| \leq C\,\big(\beta\,(1-x)^2|\widehat{\psi}(x)| + (1-x)^3|\widehat{\psi}'(x)|\big), \quad (5.4.17)$$

we obtain from (4.2.10), (4.2.16), (4.2.17), and (4.6.1) that

$$\beta^{1/3}\lambda_\beta^{-1}\|(U+i\lambda)\hat{g}\|_2 + \|\hat{g}\|_2 \leq C\beta^{1/6}\lambda_\beta^{-5/4}. \quad (5.4.18)$$

Using the fact that $|\lambda| > \beta^{-1/5}$ we can then conclude

$$|\lambda|^{-1}\|(U+i\lambda)\hat{g}\|_2 + \|\hat{g}\|_2 \leq C\,\beta^{-1/4}|\lambda|^{-5/4}. \quad (5.4.19)$$

Step 4: Estimate of $\|\tilde{v}_{\mathfrak{D}}'\|$. From (5.4.8), we obtain, using (5.4.7) and the fact that $U'' \neq 0$

$$\frac{1}{C}\|\tilde{v}_{\mathfrak{D}}'\|_2^2 \leq \Re\langle(U'')^{-1}\tilde{v}_{\mathfrak{D}}, -f+\phi''(1)\hat{g}\rangle \\ -\Re\langle((U'')^{-1})'\tilde{v}_{\mathfrak{D}}, \tilde{v}_{\mathfrak{D}}'\rangle + \beta\mu\,\|\tilde{v}_{\mathfrak{D}}\|_2^2 - \beta\,\Re\langle\phi''(1)\widehat{\psi}, i\phi\rangle. \quad (5.4.20)$$

Next, we obtain from (5.4.20) and (5.4.16) (for sufficiently small $\hat{\delta}$) that

$$\|\tilde{v}_{\mathfrak{D}}'\|_2^2 \leq C\big[\|\tilde{v}_{\mathfrak{D}}\|_2\big(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2\big) \\ + (\mu_{\beta,+}+1)\|\tilde{v}_{\mathfrak{D}}\|_2^2 + \lambda_\beta^{-1/2}\big(\beta^{2/3}\|\phi\|_{1,2}^2 + \beta^{-1}\|f\|_2^2\big)\big], \quad (5.4.21)$$

where

$$\mu_{\beta,+} := \max(\mu\beta, 0).$$

We now substitute (5.4.14) and (5.4.18) into (5.4.21) to obtain

$$\|\tilde{v}_{\mathfrak{D}}'\|_2^2 \leq C\big(\|\tilde{v}_{\mathfrak{D}}\|_2\big[\|f\|_2 + \lambda_\beta^{-3/4}\beta^{1/2}\|\phi\|_{1,2}\big] \\ + \lambda_\beta^{-1/2}(\beta^{2/3}\|\phi\|_{1,2}^2 + \beta^{-1}\|f\|_2^2) + (\mu_{\beta,+}+1)\,\|\tilde{v}_{\mathfrak{D}}\|_2^2\big). \quad (5.4.22)$$

Hence, for $|\lambda| > \beta^{-1/5}$ we can conclude that

$$\|\tilde{v}_{\mathfrak{D}}'\|_2^2 \leq C\,\big[\|\tilde{v}_{\mathfrak{D}}\|_2(\|f\|_2 + \beta^{1/4}|\lambda|^{-3/4}\|\phi\|_{1,2}) \\ + (\mu_{\beta,+}+1)\|\tilde{v}_{\mathfrak{D}}\|_2^2 + |\lambda|^{-1/2}\beta^{1/2}\|\phi\|_{1,2}^2 + |\lambda|^{-1/2}\beta^{-7/6}\|f\|_2^2\big]. \\ (5.4.23)$$

Step 5: Estimate of $\|\tilde{v}_{\mathfrak{D}}\|$. Given that $\nu < \nu_0$, we may apply (3.3.10), (3.1.3a), and Hardy's inequality (2.6.13), to (5.4.7) to obtain

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\big(\|\phi'\|_2 + \beta^{1/6}|\phi(x_\nu)| + \beta^{-2/3}[\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2]\big). \tag{5.4.24}$$

Using again (5.4.18) and (5.4.14) we obtain

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq \\ C\big(\|\phi'\|_2 + \beta^{1/6}|\phi(x_\nu)| + \beta^{-2/3}\big[\|f\|_2 + \lambda_\beta^{-3/4}\big(\beta^{1/2}\|\phi\|_{1,2} + \beta^{-1/3}\|f\|_2\big)\big]\big),$$

which implies for any $\nu < \nu_0$

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\big(\|\phi\|_{1,2} + \beta^{1/6}|\phi(x_\nu)| + \beta^{-2/3}\|f\|_2\big). \tag{5.4.25}$$

Step 6: Estimate of $\|g_{\mathfrak{D}}\|_2$. Substituting (5.4.25) into (5.4.22) yields that

$$\|\tilde{v}'_{\mathfrak{D}}\|_2 \leq C\,\big[(\beta^{1/3}\lambda_\beta^{-1/4} + 1)\|\phi\|_{1,2} + \beta^{1/6}|\phi(x_\nu)| \\ + (1 + \mu_{\beta,+}^{1/2}\beta^{-2/3})\|f\|_2 + \mu_{\beta,+}^{1/2}(\beta^{1/6}|\phi(x_\nu| + \|\phi'\|_2))\big]. \tag{5.4.26}$$

For $|\lambda| \geq \beta^{-1/5}$ and $\mu \leq \beta^{-2/5}$ we conclude from (5.4.26) that

$$\|\tilde{v}'_{\mathfrak{D}}\|_2 \leq C\,\big[(\beta^{1/4}|\lambda|^{-1/4} + 1)\|\phi\|_{1,2} + \beta^{1/6}|\phi(x_\nu)| + \|f\|_2 \\ + \mu_{\beta,+}^{1/2}(\beta^{1/6}|\phi(x_\nu|) + \|\phi'\|_2)\big].$$

Furthermore, as

$$(U''\phi)'' = -U''\big(\tilde{v}_{\mathfrak{D}} + \alpha^2\phi + |\phi''(1)|\,\hat{\psi}\big) + 2U^{(3)}\phi' + U^{(4)}\phi,$$

we obtain from (5.4.25), the boundedness of α, (5.4.14), and (4.6.5), that

$$\|(U''\phi)''\|_2 \leq C\lambda_\beta^{\frac{1}{4}}(\beta^{1/6}\|\phi\|_{1,2} + \beta^{-2/3}\|f\|_2). \tag{5.4.27}$$

For $|\lambda| \geq \beta^{-1/5}$ the above inequality implies

$$\|(U''\phi)''\|_2 \leq C\,|\lambda|^{1/4}\beta^{1/4}(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2). \tag{5.4.28}$$

By (5.4.4b) it holds that

$$\|g_{\mathfrak{D}}\| \leq C\big(\|(U + i\lambda)f\|_2 + |\phi''(1)|\,\|(U + i\lambda)\hat{g}\| + \|((U''\phi)'')\| + \|\tilde{v}'_{\mathfrak{D}}\| + \|\tilde{v}_{\mathfrak{D}}\|\big). \tag{5.4.29}$$

We now substitute into (5.4.29) the estimates (5.4.15) and (5.4.19), (5.4.28), (5.4.26), and (5.4.23), to obtain that

$$\|g_{\mathfrak{D}}\|_2 \leq C\big[(1 + |\lambda|)\,\|f\|_2 + \mu_{\beta,+}^{1/2}(\beta^{1/6}|\phi(x_\nu)| + \|\phi\|_{1,2}) \\ + \beta^{1/4}(|\lambda|^{-1/4} + |\lambda|^{1/4})\,\|\phi\|_{1,2}\big]. \tag{5.4.30}$$

Step 7: Estimate the contribution of the boundary term $\mathcal{A}_{\lambda,\alpha}^{-1}([U+i\lambda]\phi''(1)\widehat{\psi})$. We continue as in the proof of [3, Lemma 8.8]. We first write, in view of (5.4.2)

$$\phi = \phi_{\mathfrak{D}} + \check{\phi}, \tag{5.4.31a}$$

where

$$\phi_{\mathfrak{D}} = \mathcal{A}_{\lambda,\alpha}^{-1} v_{\mathfrak{D}}; \quad \check{\phi} = -\mathcal{A}_{\lambda,\alpha}^{-1}\big([U+i\lambda]\phi''(1)\widehat{\psi}\big). \tag{5.4.31b}$$

Note here that

$$\check{\phi} = -\phi''(1)\phi_{\lambda,\beta,\alpha}. \tag{5.4.31c}$$

By (5.2.5) and (5.4.15) we have

$$\|\check{\phi}\|_{1,2} \le C\,|\nu|^{-1}\beta^{-1/4}|\lambda|^{-1/4}\big(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2\big). \tag{5.4.32}$$

Consequently, as $|\nu| > \beta^{-1/5+\delta}$, we obtain for sufficiently large β_0

$$\|\check{\phi}\|_{1,2} \le C\,|\nu|^{-1}\beta^{-1/4}|\lambda|^{-1/4}\big(\beta^{-5/6}\|f\|_2 + \|\phi_{\mathfrak{D}}\|_{1,2}\big). \tag{5.4.33}$$

Furthermore, by (5.2.6) and (5.4.31c), we have

$$|\check{\phi}(x_\nu)| \le C[|\lambda|\beta]^{-3/4}|\phi''(1)|.$$

Hence, by (5.4.15) we obtain

$$|\check{\phi}(x_\nu)| \le C[|\lambda|\beta]^{-1/4}\big(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2\big),$$

and, therefore, by equation (5.4.33) and Poincaré's inequality we may conclude for $\nu \ge \beta^{-1/5+\delta}$ and β_0 large enough

$$|\check{\phi}(x_\nu)| \le C[|\lambda|\beta]^{-1/4}\big(\|\phi_{\mathfrak{D}}'\|_2 + \beta^{-5/6}\|f\|_2\big). \tag{5.4.34}$$

Step 8: Estimate $\phi_{\mathfrak{D}}$. Substituting the above into (5.4.30) yields, with the aid of (5.4.31), (5.4.33), and (5.4.34)

$$\begin{aligned}\|g_{\mathfrak{D}}\|_2 \le C\big[&(1+|\lambda|)\|f\|_2 + \mu_{\beta,+}^{1/2}(\beta^{1/6}|\phi_{\mathfrak{D}}(x_\nu)| + \|\phi_{\mathfrak{D}}\|_{1,2})\\ &+ \beta^{1/4}(|\lambda|^{-1/4} + |\lambda|^{1/4})\|\phi_{\mathfrak{D}}\|_{1,2}\big].\end{aligned} \tag{5.4.35}$$

By either (2.6.81) or (2.11.1) (for $|\mu|$ large) applied to the pair $(v_{\mathfrak{D}}, \phi_{\mathfrak{D}})$ together with (5.4.3), it holds for any $1 < q < 2$ that

$$|\phi_{\mathfrak{D}}(x_\nu)| \le C\,\big[\|v_{\mathfrak{D}}\|_{1,q} + \nu^{-1/2}\|v_{\mathfrak{D}}\|_2 + \nu^{-1}\|v_{\mathfrak{D}}\|_1\big]. \tag{5.4.36}$$

We now estimate $\|\phi_{\mathfrak{D}}\|_{1,2}$ by applying (2.6.1) and (2.6.2) to the pair $(v_{\mathfrak{D}}, \phi_{\mathfrak{D}})$. We then conclude that there exists $\mu_0 > 0$ such that for all $|\mu| \le \mu_0$ and $0 < \nu \le \nu_0$ it holds that

$$\|\phi_{\mathfrak{D}}\|_{1,2} \le C\,\nu^{-1} n_{\mathfrak{D}}, \tag{5.4.37}$$

where

$$n_D := (\|v'_{\mathfrak{D}}\|_q + \nu^{-1/2}\|v_{\mathfrak{D}}\|_2 + \nu^{-1}\|v_{\mathfrak{D}}\|_1).$$

For $|\mu| > \mu_0$ we use (2.11.1) to obtain for $0 < \nu \leq \nu_0$

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\|(1-x)^{1/2}v_{\mathfrak{D}}\|_1 \leq C\,\nu^{-1}n_{\mathfrak{D}}. \tag{5.4.38}$$

We can then substitute (5.4.36) and (5.4.37) or (5.4.38) into (5.4.35) to obtain for any $1 < q < 2$

$$\|g_{\mathfrak{D}}\|_2 \leq C\,\Big([1+|\lambda|]\|f\|_2 + [\mu_{\beta,+}^{1/2}[\beta^{1/6}+\nu^{-1}] + \beta^{1/4}\nu^{-1}(|\lambda|^{-1/4}+|\lambda|^{1/4})]\,n_{\mathfrak{D}}\Big). \tag{5.4.39}$$

Step 9: Estimate $n_{\mathfrak{D}}$. By either (3.1.3a) for $-\beta^{-1/3} \leq \mu < \beta^{-2/5-\delta}$ or (3.1.84) for $\mu < -\beta^{-1/3}$ we have,

$$\|v_{\mathfrak{D}}\|_2 \leq \frac{C}{\beta^{\frac{2}{3}} + |\mu|\beta}\|g_{\mathfrak{D}}\|_2, \tag{5.4.40}$$

whereas by (3.1.3b), which holds for $\mu \leq C\beta^{-1/3}$ and β_0 large, we have for all $\beta > \beta_0$

$$\|v'_{\mathfrak{D}}\|_q \leq C_q\,\beta^{-\frac{2+q}{6q}}\|g_{\mathfrak{D}}\|_2. \tag{5.4.41}$$

Furthermore, by (3.3.1) it holds that

$$\|v_{\mathfrak{D}}\|_1 \leq C_q\,\beta^{-5/6}\|g_{\mathfrak{D}}\|_2. \tag{5.4.42}$$

Substituting (5.4.40), (5.4.41), and (5.4.42) into (5.4.39) yields, for $\delta \leq 1/30$, β_0 large enough, and q satisfying

$$1 < q < \frac{4}{4-15\delta}, \tag{5.4.43}$$

we obtain the existence of $C > 0$ such that for all $\beta \geq \beta_0$

$$\|g_{\mathfrak{D}}\|_2 \leq C(1+|\lambda|)\|f\|_2. \tag{5.4.44}$$

We now use (5.4.40) to obtain

$$\|v_{\mathfrak{D}}\|_2 \leq C\,\frac{1+|\lambda|}{\beta^{\frac{2}{3}}+|\mu|\beta}\|f\|_2 \leq \hat{C}\,\beta^{-2/3}\|f\|_2, \tag{5.4.45}$$

which is valid for all $\mu < \beta^{-\frac{2}{5}-\delta}$.

We next use (5.4.41) and (5.4.39) to obtain

$$\|v'_{\mathfrak{D}}\|_q \leq C\beta^{-\frac{2+q}{6q}}(1+|\lambda|)\|f\|_2. \tag{5.4.46}$$

Finally, use of (5.4.42) and (5.4.39) yields

$$\|v_{\mathfrak{D}}\|_1 \leq C\beta^{-5/6}(1+|\lambda|)\|f\|_2. \tag{5.4.47}$$

For $-\mu_0 \leq \mu < \beta^{-\frac{2}{5}-\delta}$ we have by (5.4.45), (5.4.46), and (5.4.47) that the dominant term is the one involving $\|v'_{\mathfrak{D}}\|_q$ and hence

$$n_{\mathfrak{D}} \leq C\,\beta^{-\frac{2+q}{6q}}\,\|f\|_2. \tag{5.4.48}$$

Step 10: Prove (5.4.1). By (5.4.37) we obtain, for $-\mu_0 \leq \mu \leq \beta^{-\frac{2}{5}-\delta}$,

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\nu^{-1}\,\beta^{-\frac{2+q}{6q}}\,\|f\|_2. \tag{5.4.49}$$

For $\mu < -\mu_0$ we use (2.11.1) and the first inequality in (5.4.45) to obtain

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\,\|v_{\mathfrak{D}}\|_2 \leq \hat{C}\,\beta^{-1}\|f\|_2. \tag{5.4.50}$$

Combining (5.4.33) with (5.4.49) and (5.4.50) yields

$$\|\phi\|_{1,2} \leq C\nu^{-1}\,\beta^{-\frac{2+q}{6q}}\,\|f\|_2. \qquad \blacksquare$$

5.5 Resolvent estimates for $|\Im\lambda| = \mathcal{O}(\beta^{-1/5})$ and large $\beta^{1/10}\alpha$

In the previous section, we considered the case where $\Im\lambda \gg \beta^{-1/5}$, where the inverse estimates derived in Section 2.6 for the Rayleigh operator $\mathcal{A}_{\lambda,\alpha}$ become effective for all $\alpha \geq 0$. Here, we use the estimates obtained in Section 2.5 assuming $\alpha^2 \gg |\Im\lambda|$. Since we consider here $|\Im\lambda| = \mathcal{O}(\beta^{-1/5})$ we need to consider $\alpha^2 \gg \beta^{-1/5}$.

Proposition 5.5.1. *Let $U \in C^4([0,1])$ satisfy* (2.1.3) *and $U'''(0) = 0$. Let further $0 < \delta < 1/15$ and α_0 denote positive constants. There exist $C > 0$, $\beta_0 > 0$, and $a_0 > 0$ such that for all $\beta \geq \beta_0$, it holds that*

$$\sup_{\substack{a_0\beta^{-1/10+\delta/2}\leq\alpha\leq\alpha_0\\ \Re\lambda\leq\beta^{-1/3-\delta}\\ |\Im\lambda|\leq\beta^{-1/5+\delta}}} \Big[\big\|(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\big\| + \Big\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\Big\|\Big] \leq C\,\beta^{-1/2+\delta}. \tag{5.5.1}$$

Proof. Step 1. With $g_{\mathfrak{D}}$ given by (5.4.4b), we prove that

$$\begin{aligned}\|g_{\mathfrak{D}}\|_2 \leq C\big[&(1+|\lambda|)\,\|f\|_2 + (\lambda_\beta^{-1/4}\beta^{1/3} + \beta^{1/3-\delta/2} + \lambda_\beta^{1/4}\beta^{1/6})\|\phi\|_{1,2}\\ &+ \beta^{1/2-\delta/2}|\phi(x_\nu)|\big].\end{aligned} \tag{5.5.2}$$

Let $\phi \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})$, $f = \mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\,\phi$ and $v_{\mathfrak{D}} \in H^2(0,1)$ defined by (5.4.2). As before we write

$$(\mathcal{L}^{\mathfrak{N}}_\beta - \beta\lambda)v_{\mathfrak{D}} = g_{\mathfrak{D}},$$

Let $\tilde{v}_{\mathfrak{D}}$ be given by (5.4.6). For the convenience of the reader we repeat here (5.4.25)

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\big(\|\phi\|_{1,2} + \beta^{1/6}|\phi(x_\nu)| + \beta^{-2/3}\|f\|_2\big),$$

and (5.4.26), which reads

$$\|\tilde{v}_{\mathfrak{D}}'\|_2 \leq C\,\big[(\beta^{1/3}\lambda_\beta^{-1/4}+1)\|\phi\|_{1,2}+\beta^{1/6}|\phi(x_\nu)| + (1+\mu_{\beta,+}^{1/2}\beta^{-2/3})\|f\|_2+\mu_{\beta,+}^{1/2}(\beta^{1/6}|\phi(x_\nu|+\|\phi\|_{1,2}))\big].$$

For $\mu<\beta^{-1/3-\delta}$ with $\delta<1/10$, we then obtain using Poincaré's inequality

$$\|\tilde{v}_{\mathfrak{D}}'\|_2 \leq C\,\big[(\beta^{1/3}\lambda_\beta^{-1/4}+\beta^{1/3-\delta/2})\|\phi\|_{1,2}+\|f\|_2+\beta^{1/2-\delta/2}|\phi(x_\nu)|\big]. \tag{5.5.3}$$

We next estimate $\|g_{\mathfrak{D}}\|_2$, beginning by repeating (5.4.29), which states

$$\|g_{\mathfrak{D}}\| \leq C(\|(U+i\lambda)f\|_2+|\phi''(1)|\,\|(U+i\lambda)\hat{g}\|+\|(U''\phi)''\|+\|\tilde{v}_{\mathfrak{D}}'\|+\|\tilde{v}_{\mathfrak{D}}\|).$$

By (5.4.14), (5.4.18), (5.4.27), (5.5.3), and (5.4.25) it holds, that

$$\|g_{\mathfrak{D}}\|_2 \leq C\big[(1+|\lambda|)\,\|f\|_2 + (\lambda_\beta^{-1/4}\beta^{1/3}+\beta^{1/3-\delta/2}+\lambda_\beta^{1/4}\beta^{1/6})\|\phi\|_{1,2}+(\beta^{1/2-\delta/2}|\phi(x_\nu)|)\big],$$

which is precisely (5.5.2).

Step 2: We estimate $\|v_{\mathfrak{D}}\|_{1,q}$ and $\|v_{\mathfrak{D}}\|_1$. By (5.5.2) it holds for $\mu\geq -1$ that

$$\|g_{\mathfrak{D}}\|_2 \leq C\,\big[\|f\|_2+\beta^{1/3}\|\phi\|_{1,2}+(\beta^{1/2-\delta/2}|\phi(x_\nu)|)\big]. \tag{5.5.4}$$

By (3.1.3b) we have, for any $1<q<2$ and given that $|\nu|<\beta^{-1/5+\delta}$ and $\delta<1/5$,

$$\|v_{\mathfrak{D}}\|_{1,q} \leq C\beta^{-\frac{2+q}{6q}}\|g_{\mathfrak{D}}\|_2.$$

Hence, by (5.5.4) we obtain for $\mu\geq -1$,

$$\|v_{\mathfrak{D}}\|_{1,q} \leq C([\beta^{-\frac{2+q}{6q}}\|f\|_2+\beta^{-\frac{2-q}{6q}}\|\phi\|_{1,2}+\beta^{\delta_1(q)}|\phi(x_\nu)|]), \tag{5.5.5}$$

where $\delta_1(q)=(q-1)/3q-\delta/2$.

Furthermore, we have by (3.3.1) and (5.5.4), for $\mu\geq -1$,

$$\|v_{\mathfrak{D}}\|_1 \leq C\beta^{-5/6}\|g_{\mathfrak{D}}\|_2 \leq \hat{C}\big([\beta^{-5/6}\|f\|_2+\beta^{-1/2}\|\phi\|_{1,2}+\beta^{-1/3-\delta/2}|\phi(x_\nu)|]\big). \tag{5.5.6}$$

For $\mu\leq -1$ we use (3.1.84) to obtain

$$\|v_{\mathfrak{D}}\|_1 \leq \|v_{\mathfrak{D}}\|_2 \leq \frac{C}{|\lambda|\beta}\|g_{\mathfrak{D}}\|_2.$$

We then employ (5.5.2) which implies, for $\delta\leq 1/10$,

$$\|g_{\mathfrak{D}}\|_2 \leq C\big[(1+|\lambda|)\,\|f\|_2+(\beta^{1/3-\delta/2}+|\lambda|^{1/4}\beta^{1/4})\|\phi\|_{1,2}+(\beta^{1/2-\delta/2}|\phi(x_\nu)|)\big]$$

and hence,

$$\|v_{\mathfrak{D}}\|_1 \leq C\big[\beta^{-1}\|f\|_2 + \beta^{-2/3-\delta/2}\|\phi\|_{1,2} + \beta^{-1/2-\delta/2}|\phi(x_\nu)|\big]. \tag{5.5.7}$$

Combining (5.5.6) and (5.5.7) yields

$$\|v_{\mathfrak{D}}\|_1 \leq C\,\big([\beta^{-5/6}\|f\|_2 + \beta^{-1/2}\|\phi\|_{1,2} + \beta^{-1/3-\delta/2}|\phi(x_\nu)|]\big). \tag{5.5.8}$$

Step 3. We prove (5.5.1). We continue as in the proof of Proposition 5.4.1. Recall from (5.4.2) that

$$v_{\mathfrak{D}} = \mathcal{A}_{\lambda,\alpha}\phi + (U + i\lambda)\phi''(1)\widehat{\psi}.$$

Set

$$\phi = \phi_{\mathfrak{D}} + \check{\phi}, \tag{5.5.9}$$

where

$$\phi_{\mathfrak{D}} = \mathcal{A}_{\lambda,\alpha}^{-1}v_{\mathfrak{D}} \quad \text{and} \quad \check{\phi} = -\mathcal{A}_{\lambda,\alpha}^{-1}\big([U + i\lambda]\phi''(1)\widehat{\psi}\big).$$

Step 3a: We estimate $\|\check{\phi}\|_{1,2}$. Note that by (5.2.4), we have

$$\check{\phi} = -\phi''(1)\phi_{\lambda,\beta,\alpha}. \tag{5.5.10}$$

By (5.2.24) there exist $C>0$ and $\lambda_0>0$, and for any $a_0>\|U\|_2^{-1}(1+\delta)^{1/2}$, $\beta(a_0)>0$ such that, for $-\lambda_0 < \mu$, $|\nu| \leq \beta^{-1/5+\delta}$, $\alpha \geq a_0\beta^{-1/10+\delta/2}$, and $\beta \geq \beta(a_0)$, we have

$$\|\check{\phi}\|_{1,2} \leq Ca_0^{-2}\beta^{-1/3}\lambda_\beta^{-1/2}|\phi''(1)|. \tag{5.5.11}$$

For $\mu \leq -\lambda_0 < 0$ we obtain from (2.11.3) applied to the pair $(\check{\phi}, (U + i\lambda)\phi''(1)\widehat{\psi})$

$$\|\check{\phi}'\|_2^2 \leq C\,|\langle\check{\phi}, \widehat{\psi}\rangle\, \phi''(1)| \leq C\,\|\check{\phi}'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_1\,|\phi''(1)|,$$

and hence by (5.2.2) (with $s = 1/2$) for the first inequality, we conclude for the second inequality that there exists $\beta(a_0) > 0$ such that for $\beta \geq \beta(a_0)$

$$\|\check{\phi}'\|_2 \leq C\,\lambda_\beta^{-3/4}\beta^{-1/2}|\phi''(1)| \leq \hat{C}\,a_0^{-2}\beta^{-1/3}\lambda_\beta^{-1/2}|\phi''(1)|.$$

Hence, (5.5.11) holds true for any $\mu < \beta^{-1/3+\delta}$ and $|\nu| \leq \beta^{-1/5+\delta}$. From (5.4.14) we then deduce

$$\|\check{\phi}\|_{1,2} \leq Ca_0^{-2}\beta^{-1/3}(\beta^{1/3}\|\phi\|_{1,2} + \beta^{-1/2}\|f\|_2) \leq \hat{C}(\beta^{-5/6}\|f\|_2 + a_0^{-2}\|\phi\|_{1,2}).$$

Hence, for sufficiently large a_0 we conclude with the aid of (5.5.9)

$$\|\check{\phi}\|_{1,2} \leq C\,\big(\beta^{-5/6}\|f\|_2 + a_0^{-2}\|\phi_{\mathfrak{D}}\|_{1,2}\big). \tag{5.5.12}$$

Step 3b: We estimate $|\check{\phi}(x_\nu)|$. Using (5.5.10), the decomposition

$$\phi_{\lambda,\beta,\alpha} = (\phi_{\lambda,\beta,\alpha} - c_\|(\lambda,\beta,\alpha)U) + c_\|(\lambda,\beta,\alpha)U,$$

where

$$c_\|(\lambda,\beta,\alpha) = \frac{\langle U, \phi_{\lambda,\beta,\alpha}\rangle}{\|U\|_2^2},$$

and Hölder's inequality, we may write

$$|\check{\phi}(x_\nu)| \le C\big(|\nu|\,|c_\|(\lambda,\beta,\alpha)| + |\nu|^{1/2}\big\|\phi_{\lambda,\beta,\alpha} - c_\|(\lambda,\beta,\alpha)U\big\|_{1,2}\big)\,|\phi''(1)|. \quad (5.5.13)$$

Then, we obtain by (5.2.24) and the fact that $\alpha \ge a_0\beta^{-1/10+\delta/2}$

$$|\check{\phi}(x_\nu)| \le$$
$$C\Big[|\nu|^{1/2}\lambda_\beta^{-3/4}\beta^{-1/2} + \frac{|\mu|\nu^{1/2} + |\nu|}{a_0^2\beta^{-1/5+\delta} + |\mu|}\big(\lambda_\beta^{-1}\beta^{-2/3} + \lambda_\beta^{-1/2}|\lambda|\beta^{-1/3}\big)\Big]|\phi''(1)|. \quad (5.5.14)$$

Using (5.4.14) and the fact that $|\lambda| \le \beta^{-1/3}\lambda_\beta$ we obtain

$$|\check{\phi}(x_\nu)| \le C\big(a_0^{-2}\beta^{-1/3}\lambda_\beta + |\nu|^{1/2}\lambda_\beta^{-1/4}\beta^{-1/6}\big)\big(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2\big). \quad (5.5.15)$$

We now consider three different cases.

- For $-\beta^{-1/5+\delta} < \mu < \beta^{-1/3+\delta}$, we have $\lambda_\beta \lesssim \beta^\delta$ and since $|\nu| \le |\lambda|$, we deduce from (5.5.15)

$$|\check{\phi}(x_\nu)| \le C\beta^{-1/5+\delta}\big(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2\big). \quad (5.5.16)$$

- For $-\beta^{-1/10+\delta/2} < \mu < -\beta^{-1/5+\delta}$ we have, since $|\nu| \le |\mu|$,

$$\frac{|\mu|\,|\nu|^{1/2} + |\nu|}{a_0^2\beta^{-1/5+\delta} + |\mu|}|\lambda| \le C(|\mu|\,|\nu|^{1/2} + |\nu|) \le \hat{C}\beta^{-1/5+\delta},$$

  and hence, as $\lambda_\beta \sim |\lambda|\beta^{1/3}$, we can conclude (5.5.16) in this case as well.

- Finally, for $\mu \le -\beta^{-1/10+\delta/2}$, we use (2.11.2) with $v = (U + i\lambda)\hat{\psi}$, (5.4.31c), and (5.4.14) to obtain that

$$\|\check{\phi}'\|_2 \le C|\mu|^{-1}\|(1-x)^{1/2}\hat{\psi}\|_1|\phi''(1)| \le \hat{C}|\lambda|^{-5/4}\beta^{-1/4}\big(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2\big),$$

  which implies

$$|\check{\phi}(x_\nu)| \le C|\nu|^{1/2}\|\check{\phi}'\|_2 \le \hat{C}\beta^{-9/40-\delta/8}\big(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2\big). \quad (5.5.17)$$

Combining (5.5.17) with (5.5.16) which holds in the two first cases yields for all $\mu < \beta^{-1/3-\delta}$

$$|\check{\phi}(x_\nu)| \leq C\beta^{-1/5+\delta}\big(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2\big). \tag{5.5.18}$$

Then, by (5.5.9) and (5.5.18) it holds that

$$|\check{\phi}(x_\nu)| \leq C\beta^{-1/5+\delta}\big(\beta^{-5/6}\|f\|_2 + \|\phi_{\mathfrak{D}}\|_{1,2}\big). \tag{5.5.19}$$

Step 3c: We prove (5.5.1) *for* $\mu > -\beta^{-\delta}$. From (5.5.5) and (5.5.19) we get, for $\delta \leq 1/15$ and $\mu > -1$, that

$$\|v_{\mathfrak{D}}\|_{1,q} \leq C\,\big(\beta^{-\frac{2+q}{6q}}\|f\|_2 + \beta^{-\frac{2-q}{6q}}\|\phi\|_{1,2} + \beta^{\delta_1(q)}|\phi_{\mathfrak{D}}(x_\nu)|\big).$$

Using (5.5.12) we then obtain that

$$\|v_{\mathfrak{D}}\|_{1,q} \leq C\big(\beta^{-\frac{2+q}{6q}}\|f\|_2 + \beta^{-\frac{2-q}{6q}}\|\phi_{\mathfrak{D}}\|_{1,2} + \beta^{\delta_1(q)}|\phi_{\mathfrak{D}}(x_\nu)|\big).$$

As

$$-\frac{2-q}{6q} + \frac{1}{6} = \frac{1}{3} - \frac{1}{3q} = \delta_1(q) + \delta/2,$$

we can finally conclude, for $\mu > -1$, that

$$\|v_{\mathfrak{D}}\|_{1,q} \leq C\,\big(\beta^{-\frac{2+q}{6q}}\|f\|_2 + \beta^{-\frac{2-q}{6q}}[\|\phi_{\mathfrak{D}}\|_{1,2} + \beta^{1/6}|\phi_{\mathfrak{D}}(x_\nu)|]\big). \tag{5.5.20}$$

From (5.5.8), (5.5.12), and (5.5.19) we obtain

$$\|v_{\mathfrak{D}}\|_1 \leq C\,\big(\beta^{-5/6}\|f\|_2 + \beta^{-1/2}\|\phi_{\mathfrak{D}}\|_{1,2} + \beta^{-1/3-\delta/2}|\phi_{\mathfrak{D}}(x_\nu)|\big). \tag{5.5.21}$$

As in the proof of (5.5.13) we then write

$$|\phi_{\mathfrak{D}}(x_\nu)| \leq C\,\big(|\nu|\,|c_\parallel^{\mathfrak{D}}| + |\nu|^{1/2}\|\phi_{\mathfrak{D}} - c_\parallel^{\mathfrak{D}}U\|_{1,2}\big), \tag{5.5.22}$$

where

$$c_\parallel^{\mathfrak{D}} = \langle U, \phi_{\mathfrak{D}}\rangle / \|U\|_2^2.$$

We then conclude from (2.5.3a) applied to the pair $(\phi_{\mathfrak{D}}, v_{\mathfrak{D}})$ that for all $-\beta^{-\delta} < \mu$ it holds that

$$|\phi_{\mathfrak{D}}(x_\nu)| \leq C\,\big(\|v_{\mathfrak{D}}\|_1 + |\lambda|N_1(v_{\mathfrak{D}},\lambda) + |\nu|^{1/2}\|v_{\mathfrak{D}}\|_{1,q}\big), \tag{5.5.23}$$

where $N_1(v_{\mathfrak{D}},\lambda)$ is defined in (2.5.2b).

By (2.5.38) (and the fact that $v_{\mathfrak{D}}(1) = 0$) we then obtain that

$$|\phi_{\mathfrak{D}}(x_\nu)| \leq C\,\big(\|v_{\mathfrak{D}}\|_1 + \beta^{-\delta}\|v_{\mathfrak{D}}\|_{1,q}\big). \tag{5.5.24}$$

Substituting (5.5.20) and (5.5.21) into (5.5.24) then yields, for $\mu > -\beta^{-\delta}$, sufficiently large β_0, and $1 < q < (1-3\delta)^{-1}$

$$|\phi_{\mathfrak{D}}(x_\nu)| \leq C\beta^{-\delta}\left(\beta^{-\frac{2+q}{6q}}\|f\|_2 + \beta^{-\frac{2-q}{6q}}\|\phi_{\mathfrak{D}}\|_{1,2}\right). \tag{5.5.25}$$

Substituting (5.5.25) into (5.5.20) then leads to

$$\|v_{\mathfrak{D}}\|_{1,q} \leq C\left(\beta^{-\frac{2+q}{6q}}\|f\|_2 + \beta^{-\frac{2-q}{6q}}\|\phi_{\mathfrak{D}}\|_{1,2}\right). \tag{5.5.26}$$

Similarly, by substituting (5.5.25) into (5.5.21) we obtain that for $\mu > -\beta^{-\delta}$ and $1 < q < (1-3\delta)^{-1}$

$$\|v_{\mathfrak{D}}\|_1 \leq C(\beta^{-5/6}\|f\|_2 + \beta^{-1/2}\|\phi_{\mathfrak{D}}\|_{1,2}). \tag{5.5.27}$$

For sufficiently large a_0 and $|\nu| \leq \beta^{-1/5+\delta}$ we have

$$\frac{1 + C|\lambda|^2\log|\lambda|^{-1}}{|\alpha^2\|U\|_2^2 + i\lambda|} \leq \widetilde{C}\frac{1}{|\alpha^2\|U\|_2^2 - \nu|} \leq \widehat{C}\beta^{1/5-\delta},$$

and (recall that $|\nu| < |\alpha^2\|U\|_2^2 - \nu|$)

$$\frac{|\lambda|(1 + C|\lambda|^2\log|\lambda|^{-1})}{|\alpha^2\|U\|_2^2 + i\lambda|} \leq \widetilde{C}\frac{|\nu| + |\mu|}{|\alpha^2\|U\|_2^2 - \nu| + |\mu|} \leq \widehat{C}.$$

Hence, we obtain from (2.5.3a) and (2.5.38) that

$$|c_\|^{\mathfrak{D}}| \leq C\left[\beta^{1/5-\delta}\|v_{\mathfrak{D}}\|_1 + \|v_{\mathfrak{D}}\|_{1,q}\right],$$

and from (2.5.3b) we obtain that

$$\|\phi_{\mathfrak{D}} - c_\|^{\mathfrak{D}}U\|_{1,2} \leq C[\|v_{\mathfrak{D}}\|_1 + \|v_{\mathfrak{D}}\|_{1,q}].$$

Consequently, by (5.5.26) and (5.5.27) it holds that

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C(\|v_{\mathfrak{D}}\|_{1,q} + \beta^{1/5}\|v_{\mathfrak{D}}\|_1) \leq \widehat{C}\left(\beta^{-\frac{2+q}{6q}}\|f\|_2 + \beta^{-\frac{2-q}{6q}}\|\phi_{\mathfrak{D}}\|_{1,2}\right).$$

It follows that, for sufficiently large β_0,

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\beta^{-\frac{2+q}{6q}}\|f\|_2. \tag{5.5.28}$$

Combining (5.5.28) with (5.5.12) gives (note that $5/6 > (2+q)/(6q)$)

$$\|\check{\phi}\|_{1,2} \leq C\,\beta^{-\frac{2+q}{6q}}\|f\|_2. \tag{5.5.29}$$

As $\phi = \phi_{\mathfrak{D}} + \check{\phi}$, we can deduce from (5.5.28) and (5.5.29) that for any $\delta \in (0, 1/15)$ and $q \in (1, (1-3\delta)^{-1})$ and $\mu > -\beta^{-\delta}$ we have

$$\|\phi\|_{1,2} \leq C\,\beta^{-\frac{2+q}{6q}}\|f\|_2 \leq C\beta^{1/2-\delta}. \tag{5.5.30}$$

Step 3d. We prove (5.5.1) *for* $\mu \leq -\beta^{-\delta}$. For $\mu \leq -\beta^{-\delta}$ we use (2.11.2), Sobolev embeddings, and Poincaré's inequality to obtain that

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq \frac{C}{|\mu|}\|\phi_{\mathfrak{D}}\|_\infty^{1/2}\|v_{\mathfrak{D}}\|_1^{1/2} \leq \hat{C}\,\beta^\delta\|\phi_{\mathfrak{D}}'\|_2^{1/2}\|v_{\mathfrak{D}}\|_1^{1/2},$$

which implies

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\beta^{2\delta}\|v_{\mathfrak{D}}\|_1. \tag{5.5.31}$$

Consequently, by (5.5.8), (5.5.31), and Sobolev embeddings we obtain that

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\beta^{2\delta}\left(\beta^{-5/6}\|f\|_2 + \beta^{-1/3-\delta/2}\|\phi\|_{1,2}\right).$$

Making use of (5.5.12) we establish that for sufficiently large a_0 and β_0

$$\|\phi\|_{1,2} \leq C\,\beta^{2\delta-5/6}\|f\|_2,$$

which, together with (5.5.30), yields (5.5.1) for $\delta < 1/15$. ■

5.6 Resolvent estimates for intermediate α

In this section, we provide inverse estimates for $\mathcal{B}_{\lambda,\alpha,\beta}$ for $1 \ll \alpha \ll \beta^{1/3}$. Let $\mathfrak{z}_\alpha$ be given by (4.7.2). Since $\|\mathfrak{z}_\alpha'\|_2 \ll \beta^{1/6}$ in this section we may conclude by (4.1.3) that $\mathfrak{z}_\alpha \in \mathfrak{U}_1$. Consequently, we may still use (4.2.4) in this section to estimate $\phi''(1)$. Furthermore, we can use the fact that $\alpha \gg 1$ to obtain a much simpler proof than in the previous section (which is valid only for bounded values of α).

Proposition 5.6.1. *Let* $U \in C^4([0,1])$ *satisfy* (2.1.3) *and* $\nu_2 < U(0)$ *denote a positive constant. There exist* $C > 0$, $\Upsilon > 0$, $\beta_0 > 0$, $\alpha_0 > 0$, *and* $\alpha_1 > 0$ *such that for all* $\beta \geq \beta_0$, *it holds that*

$$\sup_{\substack{\alpha_0\leq\alpha\leq\alpha_1\beta^{1/3}\\ \Re\lambda\leq\Upsilon\beta^{-1/3}\\ |\Im\lambda|\leq\nu_2}} \left\|(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})^{-1}\right\| + \left\|\frac{d}{dx}(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})^{-1}\right\| \leq C\beta^{-5/6}. \tag{5.6.1}$$

Proof. Let $f \in L^2(0,1)$, $\phi \in D(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})$ satisfy

$$\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}}\phi = f.$$

We first recall the definition of $\tilde{v}_{\mathfrak{D}}$ from (5.4.6)

$$\tilde{v}_{\mathfrak{D}} = -\phi'' + \alpha^2\phi + \phi''(1)\hat{\psi}, \tag{5.6.2}$$

and rewrite (5.4.7) in the form

$$(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)\tilde{v}_{\mathfrak{D}} = f + i\beta U''\phi + \phi''(1)\hat{g}, \tag{5.6.3}$$

where $\hat{g}$ is given by (5.4.5).

By (3.3.1) (which is applicable for $|\nu| \leq \nu_2 < U(0)$) it holds that

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(f + \phi''(1)\hat{g})\|_1 \leq C\beta^{-5/6}\|(f + \phi''(1)\hat{g})\|_2.$$

Furthermore, by (3.3.23) applied with f replaced by $U''\phi$ it holds that

$$\Big\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U''\phi)+i\,\frac{U''(x_\nu)\phi(x_\nu)}{\beta(U-\nu+i\,\mathfrak{m})}\Big\|_1 \leq C\beta^{-1}\|U''\phi\|_{1,2} \leq \widehat{C}\beta^{-1}\|\phi\|_{1,2},$$

where

$$\mathfrak{m} = -\max(-\mu, x_\nu^{2/3}\beta^{-1/3}).$$

Hence,

$$\Big\|\tilde{v}_{\mathfrak{D}} - \frac{U''(x_\nu)\phi(x_\nu)}{(U-\nu+i\,\mathfrak{m})}\Big\|_1 \leq C\,\big(\beta^{-5/6}\|f\|_2 + \|\phi\|_{1,2} + \beta^{-5/6}|\phi''(1)|\,\|\hat{g}\|_2\big). \quad (5.6.4)$$

We next use (4.2.4) together with (5.4.13) to obtain, as in (5.4.14), that

$$|\phi''(1)| \leq C\,\lambda_\beta^{1/2}\,\big(\beta^{1/3}\big[\|\phi\|_{1,2} + |\phi(x_\nu)|\log\beta\big] + \beta^{-1/2}\|f\|_2\big). \quad (5.6.5)$$

For $\alpha \leq \alpha_1\beta^{1/3}$ we may then use (5.6.5), which, combined with (5.4.18), yields, as

$$|\phi(x_\nu)| \leq |\nu|^{1/2}\|\phi'\|_2, \quad (5.6.6)$$

$$|\phi''(1)|\,\|\hat{g}\|_2 \leq C\lambda_\beta^{-3/4}\big(\beta^{1/2}[1 + |\nu|^{1/2}\log\beta]\|\phi\|_{1,2} + \beta^{-1/3}\|f\|_2\big). \quad (5.6.7)$$

Consequently, by substituting (5.6.7) into (5.6.4), we obtain

$$\Big\|\tilde{v}_{\mathfrak{D}} - \frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\,\mathfrak{m}}\Big\|_1 \leq C\,\big(\beta^{-5/6}\|f\|_2 + \|\phi\|_{1,2}\big). \quad (5.6.8)$$

Taking the scalar product of (5.6.2) with ϕ, and integrating by parts gives

$$\Big\langle\phi, \tilde{v}_{\mathfrak{D}} - \frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\,\mathfrak{m}}\Big\rangle = \|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 + \phi''(1)\langle\phi,\widehat{\psi}\rangle - \Big\langle\phi, \frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\,\mathfrak{m}}\Big\rangle. \quad (5.6.9)$$

Using (5.6.8) we then conclude

$$\Big|\Big\langle\phi, \tilde{v}_{\mathfrak{D}} - \frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\,\mathfrak{m}}\Big\rangle\Big| \leq C\,\big(\beta^{-5/6}\|f\|_2 + \|\phi\|_{1,2}\big)\,\|\phi\|_\infty. \quad (5.6.10)$$

We next use (5.2.2), with $s = 1/2$, together with (5.6.5) and a Sobolev inequality to obtain that

$$|\phi''(1)\langle\phi,\widehat{\psi}\rangle| \leq C\,\lambda_\beta^{-1/4}\,\big(\beta^{-1/6}[\|\phi\|_{1,2} + |\phi(x_\nu)|\log\beta] + \beta^{-1}\|f\|_2\big)\|\phi'\|_2.$$

Using (5.6.6), as $\lambda_\beta^{-1/4}\nu^{1/2}\log\beta \leq 1$ for $\beta \geq \beta_0$ with sufficiently large β_0, we can conclude that

$$|\phi''(1)\langle\phi,\widehat{\psi}\rangle| \leq C\,\big(\beta^{-1/6}\|\phi\|_{1,2}^2 + \beta^{-11/6}\|f\|_2^2\big). \quad (5.6.11)$$

For the last term on the right-hand side of (5.6.9) we write

$$\Big\langle \phi, \frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle = \Big\langle \phi, \frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle_{L^2(0,x_\nu/2)} + \Big\langle \phi, \frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle_{L^2(x_\nu/2,1)}.$$

For the first term on the right-hand side we have, since $|\nu| \leq \nu_2$,

$$\Big|\Big\langle \phi, \frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle_{L^2(0,x_\nu/2)}\Big| \leq C\|\phi\|_2\|\phi\|_\infty. \tag{5.6.12}$$

For the second term we use integration by parts to obtain

$$\begin{aligned}\Big\langle \phi, \frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle_{L^2(x_\nu/2,1)} &= U''(x_\nu)\phi(x_\nu)\frac{\phi}{U'}\log(U-\nu+i\mathfrak{m})\Big|_{x=x_\nu/2} \\ &\quad - U''(x_\nu)\phi(x_\nu)\Big\langle\Big(\frac{\phi}{U'}\Big)', \log(U-\nu+i\mathfrak{m})\Big\rangle_{L^2(x_\nu/2,1)},\end{aligned}$$

from which we readily obtain

$$\Big|\Big\langle \phi, \frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle_{L^2(x_\nu/2,1)}\Big| \leq C\|\phi\|_{1,2}|\phi(x_\nu)| \leq C\|\phi\|_{1,2}\|\phi\|_\infty.$$

In conjunction with (5.6.12) the above inequality yields

$$\Big|\Big\langle \phi, \frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle\Big| \leq C\|\phi\|_{1,2}\|\phi\|_\infty.$$

Substituting the above, together with (5.6.11) and (5.6.10) into (5.6.9) yields

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq C\left(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2\right)\|\phi\|_\infty + \beta^{-1/6}\|\phi\|_{1,2}^2.$$

Since $\phi(1) = 0$ we have $\|\phi\|_\infty^2 \leq 2\|\phi'\|_2\|\phi\|_2$ and hence, for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\|\phi\|_\infty^2 \leq \epsilon\|\phi'\|_2^2 + C_\epsilon\|\phi\|^2.$$

By choosing sufficiently small ϵ and sufficiently large β_0 we can then conclude, with the aid of Poincaré's inequality,

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq C\left(\beta^{-5/3}\|f\|_2^2 + \|\phi\|_2^2\right).$$

We then obtain for sufficiently large α_0 and β_0 the existence of $C > 0$ such that, under the conditions of the proposition,

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq C\beta^{-5/3}\,\|f\|_2^2,$$

from which (5.6.1) readily follows. ■

5.7 Resolvent estimates for large α

For $\alpha \gtrsim \beta^{1/3}$, we can no longer use the estimates of the previous section, relying on (4.2.4). In the following we thus establish estimates for the inverse of the Orr–Sommerfeld operator, relying on (4.7.3), which is valid for $\alpha \gtrsim \beta^{1/3}$.

Proposition 5.7.1. *Let $U \in C^4([0,1])$ satisfying (2.1.3), and α_2 denote a positive constant. For any $\Upsilon < \sqrt{-U''(0)}/2$ and any $\hat{\Upsilon} > 0$ there exist $C > 0$ and $\beta_0 > 0$ such that for all $\beta \geq \beta_0$ it holds that*

$$\sup_{\substack{\alpha_2\beta^{1/3}\leq\alpha \\ \Re\lambda\leq\beta^{-1/3}[\hat{\mu}_m-\hat{\Upsilon}-\alpha^2\beta^{-2/3}/2]}} \Big(\|(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})^{-1}\| + \Big\|\frac{d}{dx}(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})^{-1}\Big\|\Big) \leq C\beta^{-5/6}. \tag{5.7.1}$$

Proof. Let $\mathfrak{z} = \mathfrak{z}_\alpha$ be given by (4.7.1)–(4.7.2). Let $f \in L^2(0,1)$, $\phi \in D(\mathcal{B}_{\lambda,\alpha}^{\mathfrak{N},\mathfrak{D}})$ satisfy

$$\mathcal{B}_{\lambda,\alpha}^{\mathfrak{N},\mathfrak{D}}\phi = f.$$

An integration by parts yields $\langle \mathfrak{z}_\alpha, -\phi'' + \alpha^2\phi\rangle = 0$, and hence we may conclude that $-\phi'' + \alpha^2\phi \in D(\mathcal{L}_\beta^{\mathfrak{z}_\alpha})$. Furthermore, it holds that

$$(\mathcal{L}_\beta^{\mathfrak{z}_\alpha} - \beta\lambda)(-\phi'' + \alpha^2\phi) = f + i\beta U''\phi.$$

By (4.7.3) we then have

$$\| -\phi'' + \theta^2\beta^{2/3}\phi\|_2 \leq C\left(\beta^{1/2}\|\phi\|_2 + \beta^{-1/2}\|f\|_2\right),$$

where $\theta = \alpha\beta^{-1/3}$.

Hence,

$$\|\phi'\|_2^2 + \theta^2\beta^{2/3}\|\phi\|_2^2 = \langle -\phi'' + \theta^2\beta^{2/3}\phi, \phi\rangle \leq C\left(\beta^{1/2}\|\phi\|_2^2 + \beta^{-1/2}\|f\|_2\|\phi\|_2\right). \tag{5.7.2}$$

As $\theta \geq \alpha_2$, we obtain that for sufficiently large β_0,

$$\|\phi'\|_2 \leq C\beta^{-5/6}\|f\|_2.$$

With the aid of Poincaré's inequality we then obtain (5.7.1). ■

Remark 5.7.2. An improved version of (5.7.1) can be obtained by introducing the effect of $|U(0) - \nu|$ from (4.7.3)

$$\sup_{\substack{\alpha_2\beta^{1/3}\leq\alpha \\ \Re\lambda\leq\Upsilon\beta^{-1/2}}} \|(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})^{-1}\| + \Big\|\frac{d}{dx}(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})^{-1}\Big\| \leq C\frac{\beta^{-5/6}}{1 + |U(0) - \nu|\,\beta^{1/6}}. \tag{5.7.3}$$

5.8 Resolvent estimates for small α

We continue by considering for some positive $\hat{\alpha}_0$ and $0 < \hat{\nu}_0 < U(0)$, the zone

$$0 \le \alpha \le \hat{\alpha}_0 \beta^{-1/6}, \quad |\Im\lambda| \le \hat{\nu}_0, \quad \Re\lambda \le \beta^{-1/2}. \tag{5.8.1}$$

We begin by considering the case $\alpha = 0$ and then obtain estimates of $(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}$ for $0 < \alpha \le \hat{\alpha}_0 \beta^{-1/6}$ by treating it as a perturbation of $(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta})^{-1}$. More precisely, we introduce the set

$$\mathfrak{W}(\beta, \hat{\nu}_0) := \left\{\lambda \in \mathbb{C} : |\Im\lambda| \le \hat{\nu}_0 \text{ and } \mu \le \beta^{-1/2}\right\}$$

and prove the following proposition.

Proposition 5.8.1. *Let $U \in C^4([0,1])$ satisfy (2.1.3). There exist $C > 0$, $\hat{\nu}_0 > 0$ and $\beta_0 > 0$ such that for all $\beta \ge \beta_0$ and $\lambda \in \mathfrak{W}(\beta, \hat{\nu}_0)$ it holds that*

$$\|(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta})^{-1}\| + \left\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta})^{-1}\right\| + \lambda_\beta^{-1/2}\left\|\frac{d^2}{dx^2}(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta})^{-1}\right\| \le C\,\beta^{-2/3}. \tag{5.8.2}$$

Proof. Step 1. Preliminaries. Let $(\phi, f) \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta}) \times L^2(0,1)$ satisfy $\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta}\phi = f$. Setting $\alpha = 0$ in (5.4.2) yields

$$v_{\mathfrak{D}} = -(U + i\lambda)\phi'' + U''\phi + (U + i\lambda)\phi''(1)\hat{\psi}. \tag{5.8.3}$$

Note that $v_{\mathfrak{D}}(1) = 0$ and hence

$$(\mathcal{L}^{\mathfrak{N}}_\beta - \beta\lambda)v_{\mathfrak{D}} = g_{\mathfrak{D}}, \tag{5.8.4}$$

where $g_{\mathfrak{D}}$ is given by (5.4.4b), which we recall here for the benefit of the reader in the equivalent form

$$g_{\mathfrak{D}} + (U + i\lambda)f = (U + i\lambda)\phi''(1)\hat{g} - (U''\phi)'' - 2U'\tilde{v}'_{\mathfrak{D}} - U''\tilde{v}_{\mathfrak{D}}. \tag{5.8.5}$$

In (5.8.5), $\tilde{v}_{\mathfrak{D}}$ is given by setting $\alpha = 0$ in (5.4.6), i.e.,

$$\tilde{v}_{\mathfrak{D}} = -\phi'' + \phi''(1)\hat{\psi}. \tag{5.8.6}$$

Step 2: We estimate $|\phi''(1)|$. Let $(\phi, f) \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta}) \times L^2(0,1)$ satisfy $\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta}\phi = f$. An integration by parts yields

$$\begin{aligned}\|(U'')^{-1/2}\phi^{(3)}\|_2^2 = -\Re\langle (U'')^{-1}\phi'', \mathcal{B}_{\lambda,0,\beta}\phi\rangle - \frac{1}{U''(1)}\Re(\bar{\phi}''(1)\phi^{(3)}(1)) \\ - \Re\langle [(U'')^{-1}]'\phi'', \phi^{(3)}\rangle + \mu\beta\|(U'')^{-1/2}\phi''\|_2^2.\end{aligned} \tag{5.8.7}$$

To estimate the second term on the right-hand side of (5.8.7) we use the identity (which is obtained via an integration by parts of the balance $\mathcal{B}_{0,0,\beta}\phi = f$)

$$\phi^{(3)}(1) = -\int_0^1 f(x)\,dx. \tag{5.8.8}$$

Hence,

$$\|\phi^{(3)}\|_2^2 \le C\left(|\phi''(1)|\,\|f\|_2 + \beta^{1/2}\|\phi''\|_2^2 + \|f\|_2\|\phi''\|_2 + \|\phi^{(3)}\|_2\|\phi''\|_2\right),$$

which implies

$$\|\phi^{(3)}\|_2^2 \le \hat{C}\left(|\phi''(1)|\,\|f\|_2 + \beta^{1/2}\|\phi''\|_2^2 + \|f\|_2\|\phi''\|_2\right). \tag{5.8.9}$$

Sobolev embeddings yield

$$|\phi''(1)|^2 \le \left(\|\phi^{(3)}\|_2 + \|\phi''\|_2\right)\|\phi''\|_2. \tag{5.8.10}$$

Combining (5.8.9) and (5.8.10) leads to

$$|\phi''(1)| \le C(\beta^{1/8}\|\phi''\|_2 + \|f\|_2^{1/4}\|\phi''\|_2^{3/4} + \|f\|_2^{1/3}\|\phi''\|_2^{2/3}). \tag{5.8.11}$$

By (5.4.6) and the left inequality of (4.6.5) it holds that

$$\|\phi''\|_2 \le C\lambda_\beta^{-1/4}\beta^{-1/6}|\phi''(1)| + \|\tilde{v}_{\mathfrak{D}}\|_2. \tag{5.8.12}$$

Using (5.8.11) we then obtain for β_0 large enough

$$\|\phi''\|_2 \le 2\|\tilde{v}_{\mathfrak{D}}\|_2 + C\lambda_\beta^{-3/4}\beta^{-1/2}\|f\|_2. \tag{5.8.13}$$

By (5.4.18) and (5.4.24) (note that (5.4.24) results from a straightforward application of (3.3.10) and (3.1.3) to (5.4.7)) it holds that

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \le C\left(\|\phi'\|_2 + \beta^{1/6}|\phi(x_\nu)| + \beta^{-2/3}\|f\|_2 + \beta^{-1/2}|\phi''(1)|\right). \tag{5.8.14}$$

Since

$$|\phi(x_\nu)| = |\phi(x_\nu) - \phi(1)| \le |\nu|^{1/2}\|\phi'\|_2 \le |\lambda|^{1/2}\|\phi'\|_2, \tag{5.8.15}$$

we obtain from (5.8.13), (5.8.14), with the aid of (5.8.11) that

$$\|\phi''\|_2 \le C\left(\lambda_\beta^{1/2}\|\phi'\|_2 + [\lambda_\beta^{-3/4}\beta^{-1/2} + \beta^{-2/3}]\|f\|_2\right). \tag{5.8.16}$$

We now substitute (5.8.16) into (5.8.11) to obtain

$$|\phi''(1)| \le C\left(\beta^{1/6}\lambda_\beta^{1/2}\|\phi'\|_2 + \beta^{-1/3}\|f\|_2\right). \tag{5.8.17}$$

Step 3: Given some $\mu_0 > 0$, we estimate $\tilde{v}_{\mathfrak{D}}$ and $\tilde{v}'_{\mathfrak{D}}$ under the additional assumption $\mu \geq -\mu_0$. We begin by recalling (5.4.22), which is still valid in the present case and reads

$$\|\tilde{v}'_{\mathfrak{D}}\|_2^2 \leq C\,\Big(\|\tilde{v}_{\mathfrak{D}}\|_2\big[\|f\|_2 + \lambda_\beta^{-3/4}\beta^{1/2}\|\phi\|_{1,2}\big] + \lambda_\beta^{-1/2}(\beta^{2/3}\|\phi\|_{1,2}^2 + \beta^{-1}\|f\|_2^2) + (\mu_{\beta,+}+1)\,\|\tilde{v}_{\mathfrak{D}}\|_2^2\Big).$$

Observing that $\mu_{\beta,+} \leq \beta^{1/2}$, where $\mu_{\beta,+}$ is given by (3.1.26), we conclude that

$$\|\tilde{v}'_{\mathfrak{D}}\|_2^2 \leq C\,\Big(\|\tilde{v}_{\mathfrak{D}}\|_2\big[\|f\|_2 + \lambda_\beta^{-3/4}\beta^{1/2}\|\phi\|_{1,2}\big] + \lambda_\beta^{-1/2}(\beta^{2/3}\|\phi\|_{1,2}^2 + \beta^{-1}\|f\|_2^2) + \beta^{1/2}\,\|\tilde{v}_{\mathfrak{D}}\|_2^2\Big). \tag{5.8.18}$$

By (5.8.14), (5.8.15), and (5.8.17) it holds that

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\,\big(\beta^{-2/3}\|f\|_2 + \lambda_\beta^{1/2}\|\phi'\|_2\big). \tag{5.8.19}$$

Substituting (5.8.19) into (5.8.18) then yields that for given $\mu_0 > 0$ there exists $C > 0$ such that for $-\mu_0 \leq \mu \leq \beta^{-1/2}$ it holds that

$$\|\tilde{v}'_{\mathfrak{D}}\|_2 \leq C\,\big(\beta^{-1/4}\|f\|_2 + \beta^{5/12}\|\phi\|_{1,2}\big). \tag{5.8.20}$$

Step 4: We estimate $\|v_{\mathfrak{D}}\|_\infty$ under the assumptions of Step 3. We begin by estimating the L^2-norm of $g_{\mathfrak{D}} + (U + i\lambda)f$ using (5.8.5). By (5.4.14), (5.4.18), and (5.4.27), it holds that

$$\|(U + i\lambda)\phi''(1)\hat{g}\| + \|(U''\phi)''\| \leq C\,\big(\beta^{1/4}\|\phi\|_{1,2} + \beta^{-7/12}\|f\|_2\big). \tag{5.8.21}$$

Substituting (5.8.21) together with (5.8.16), (5.8.19), and (5.8.20) into (5.8.5) yields

$$\|g_{\mathfrak{D}} + (U + i\lambda)f\|_2 \leq C\,\big(\beta^{-1/4}\|f\|_2 + \beta^{5/12}\|\phi\|_{1,2}\big). \tag{5.8.22}$$

Since, by a Sobolev embedding, we have

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(U - \nu)f\|_\infty \leq \Big\|\frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(U - \nu)f\Big\|_2^{1/2}\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(U - \nu)f\|_2^{1/2},$$

we can conclude from (3.3.10) and (3.3.11) that

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(U - \nu)f\|_\infty \leq C\beta^{-3/4}\,\|f\|_2. \tag{5.8.23}$$

Furthermore, by (3.1.3) (for $-\Upsilon\beta^{-1/3} \leq \mu \leq \beta^{-1/2}$) or (3.1.84) and (3.1.85) (for $-\mu_0 \leq \mu \leq -\Upsilon\beta^{-1/3}$) we obtain

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}i\mu f\|_\infty \leq C\beta^{-3/4}\|f\|_2. \tag{5.8.24}$$

Consequently, by (5.8.23) and (5.8.24) we may infer that for $\mu \geq -\mu_0$

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(U + i\lambda) f\|_\infty \leq C\beta^{-3/4} \|f\|_2. \tag{5.8.25}$$

By (5.8.22) and (3.1.3a) it holds that

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(g_{\mathfrak{D}} + [U + i\lambda] f)\|_2 \leq C \left(\beta^{-11/12}\|f\|_2 + \beta^{-1/4}\|\phi\|_{1,2}\right).$$

Furthermore, (5.8.22) and (3.1.3b) with $p = 2$ yield

$$\left\|\frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(g_{\mathfrak{D}} + [U + i\lambda] f)\right\|_2 \leq C \left(\beta^{-7/12}\|f\|_2 + \beta^{1/12}\|\phi\|_{1,2}\right).$$

Hence, by Sobolev's embeddings

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(g_{\mathfrak{D}} + [U + i\lambda] f)\|_\infty \leq C \left(\beta^{-3/4}\|f\|_2 + \beta^{-1/12}\|\phi\|_{1,2}\right). \tag{5.8.26}$$

In view of (5.8.4) we may combine (5.8.26) with (5.8.25) to obtain, for $\mu > -\mu_0$ that

$$\|v_{\mathfrak{D}}\|_\infty \leq C \left(\beta^{-3/4}\|f\|_2 + \beta^{-1/12}\|\phi\|_{1,2}\right). \tag{5.8.27}$$

Step 5. We prove (5.8.2). Recall from (5.8.1) that $\mu \leq \beta^{-1/2}$ and $|\nu| \leq \hat{\nu}_0 < U(0)$.

Step 5a. With $\mu_0 > 0$, we prove (5.8.2) for $|\nu| \leq \hat{\nu}_0$ and for μ satisfying

$$\mu \in (-\mu_0, -e^{-\beta^{1/24}}) \cup (e^{-\beta^{1/24}}, \beta^{-1/2}). \tag{5.8.28}$$

Set

$$v = \mathcal{A}_{\lambda,0}\phi = -(U + i\lambda)\phi'' + U''\phi,$$

and note from (5.4.2) (with $\alpha = 0$) that

$$v = v_{\mathfrak{D}} - \phi''(1)(U + i\lambda)\hat{\psi}.$$

An integration by parts yields

$$\int_0^1 v\,dx = 0,$$

and hence by (2.4.1) and (2.4.23) it holds that

$$\|\phi'\|_2 \leq C \left(\|v_{\mathfrak{D}}\|_\infty[1 + \log|\mu|^{-1}] + |\phi''(1)|\|\hat{\psi}\|_1\right).$$

By (5.2.2), (5.8.17), and (5.8.27), we obtain for $|\mu| < \mu_0$ that

$$\|\phi'\|_2 \leq C(\beta^{-1/24}\|\phi'\|_2 + \beta^{-2/3}\|f\|_2).$$

For sufficiently large β_0 we obtain for $\beta \geq \beta_0$

$$\|\phi\|_{1,2} \leq C\,\beta^{-2/3}\|f\|_2. \tag{5.8.29}$$

To obtain an estimate for $\|\phi''\|_2$ we use (5.8.16) to obtain

$$\|\phi''\|_2 \leq C\lambda_\beta^{1/2}\beta^{-2/3}\|f\|_2. \tag{5.8.30}$$

Step 5b. With $0 < \hat{\nu}_0 < U(0)$, we prove (5.8.2) for $|\nu| \leq \hat{\nu}_0$ and $|\mu| \leq e^{-\beta^{1/24}}$.

Here, we write for some $0 < \tilde{\mu} \leq 1/2$

$$\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda+\tilde{\mu}\beta^{-1/2},0,\beta}\phi = -\beta^{1/2}\tilde{\mu}\phi'' + f. \tag{5.8.31}$$

Note that $\lambda + \tilde{\mu}\beta^{-1/2}$ meets the assumptions of Step 5a, and hence, we can use (5.8.30) to obtain

$$\|\phi''\|_2 \leq C(\tilde{\mu}\lambda_\beta^{1/2}\beta^{-1/6}\|\phi''\|_2 + \beta^{-2/3}\|f\|_2).$$

For sufficiently small $\tilde{\mu}$ and sufficiently large β_0 we obtain (5.8.30) once again. Consequently,

$$\|-\beta^{1/2}\tilde{\mu}\phi'' + f\|_2 \leq C\|f\|_2.$$

Hence, we can apply (5.8.29) once again to (5.8.31) to establish (5.8.29) for $|\mu| \leq e^{-\beta^{1/24}}$.

Step 5c. With $0 < \hat{\nu}_0 < U(0)$, we prove that there exists $\mu_0 > 0$ such that (5.8.2) holds for $\mu \leq -\mu_0$ and $|\nu| < \hat{\nu}_0$. Since $\mu < 0$, we have, after two integrations by parts,

$$\Re\langle\phi, \mathcal{B}_{\lambda,0,\beta}\phi\rangle = \|\phi''\|_2^2 + |\mu|\beta\,\|\phi'\|_2^2 + \beta\Im\langle U'\phi, \phi'\rangle. \tag{5.8.32}$$

Consequently, using Poincaré's inequality, we obtain

$$\|\phi'\|_2 \leq \frac{C}{|\mu|}(\beta^{-1}\|f\|_2 + \|\phi'\|_2).$$

For sufficiently large μ_0 and β_0 we can then conclude

$$\|\phi'\|_2 \leq \frac{C}{|\lambda|\beta}\|f\|_2.$$

Using (5.8.16) completes the proof of (5.8.2). ■

Using a perturbation argument we now obtain the following proposition.

Proposition 5.8.2. *Let* $0 < \hat{\nu}_0 < U(0)$*. Under the conditions of Proposition* 5.8.1 *there exist* $C > 0$*,* $\hat{\alpha}_0 > 0$*, and* $\beta_0 > 0$ *such that for all* $\beta \geq \beta_0$ *it holds that*

$$\sup_{\substack{\alpha<\hat{\alpha}_0\beta^{-1/6}\\ |\nu|<\hat{\nu}_0\\ \mu<\beta^{-1/2}}} \|(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\| + \left\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\right\| \leq C\beta^{-2/3}. \tag{5.8.33}$$

Proof. Let $(\phi, f) \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}) \times L^2(0,1)$ satisfy $\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\phi = f$. We then write

$$\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta}\phi = -\alpha^2\Big(-\frac{d^2}{dx^2} + i\beta(U + i\lambda)\Big)\phi + f. \tag{5.8.34}$$

By (5.8.2) we obtain that

$$\|\phi''\|_2 \le C\,[\hat{\alpha}_0^2\lambda_\beta^{1/2}(\beta^{-1}\|\phi''\|_2 + \|\phi\|_2) + \lambda_\beta^{1/2}\beta^{-2/3}\|f\|_2].$$

Hence, for sufficiently large β we obtain that

$$\|\phi''\|_2 \le C\,\big[\hat{\alpha}_0^2\beta^{1/6}\|\phi\|_2 + \beta^{-1/2}\|f\|_2\big].$$

We can thus conclude that

$$\Big\|\Big(-\frac{d^2}{dx^2} + i\beta(U + i\lambda)\Big)\phi\Big\|_2 \le C\,\big(\beta\|\phi\|_2 + \beta^{-1/2}\|f\|_2\big).$$

By (5.8.2) and (5.8.34) we then obtain

$$\|\phi\|_{1,2} \le C\,\big(\hat{\alpha}_0^2\|\phi\|_2 + \beta^{-2/3}\|f\|_2\big).$$

For sufficiently small $\hat{\alpha}_0$ we may now conclude (5.8.33). ∎

5.9 Some auxiliary results

This section is devoted to the proof of two auxiliary results which will become useful in the next two sections.

Lemma 5.9.1. *Let $U \in C^4(0,1)$ satisfy (2.1.3). Let further κ_0 and ν_1 denote positive constants. There exist positive β_0, Υ, α_1, and C such that, for $\lambda = \mu + i\nu$, where ν and μ satisfy $\nu_1 < \nu < U(0) + \kappa_0\beta^{-1/2}$ and $\mu < \Upsilon\beta^{-1/2}$, $\beta > \beta_0$, $0 \le \alpha \le \alpha_1\beta^{1/3}$, and any $(\phi, f) \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}) \times L^2(0,1)$ satisfying $\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\phi = f$, it holds that*

$$\begin{aligned}|\phi''(1)| \le C\,\Big(&\beta^{-1/3}[\beta^{-1/4} + x_\nu]^{-5/6}\|f\|_2 \\ &+ \beta^{1/2}\Big[\beta^{-1/4} + \frac{x_\nu}{\log(1 + x_\nu\beta^{1/4})}\Big]^{-1}|\phi(x_\nu)| \\ &+ \beta^{1/2}[\beta^{-1/4} + x_\nu]^{-1/2}\|\phi'\|_2\Big),\end{aligned} \tag{5.9.1}$$

where x_ν is defined by (2.4.5).

Proof. Consider first the case $U(0) - \kappa_1\beta^{-1/2} < \nu < U(0) + \kappa_0\beta^{-1/2}$ for some $\kappa_1 > 0$. In this case we have $x_\nu \le C\beta^{-1/4}$. As in the proof of (5.4.14) we use the

$\mathcal{L}(L^2, L^\infty)$ estimate of (4.3.3) applied to $f = i\beta U''\phi(x_\nu)$ and (4.3.4) applied to $i\beta U''(\phi - \phi(x_\nu))$. We obtain

$$|\phi''(1)| \le C\left(\beta^{-1/8}\|f\|_2 + \beta^{3/4}|\phi(x_\nu)| + \beta^{5/8}\|\phi'\|_2\right).$$

For $\nu_1 < \nu < U(0) - \kappa_1\beta^{-1/2}$ with sufficiently large κ_1 and β_0, we use (4.2.4) and (4.2.7), applied to the pair $(\phi'' - \alpha^2\phi, i\beta U''\phi + f)$ using the decomposition

$$i\beta U''\phi = i\beta U''(\phi - \phi(x_\nu)) + i\beta U''\phi(x_\nu)$$

and Hardy's inequality, to obtain

$$\begin{aligned}&|\phi''(1)|\\ &\le C\left(\beta^{-1/3}x_\nu^{-5/6}\|f\|_2 + \beta^{1/2}x_\nu^{-1}\log(1 + x_\nu\beta^{1/4})|\phi(x_\nu)| + \beta^{1/2}x_\nu^{-1/2}\|\phi'\|_2\right).\end{aligned}$$

Combining the above pair of inequalities yields (5.9.1). ∎

Lemma 5.9.2. *Let $U \in C^4(0,1)$ satisfy (2.1.3) and κ_0, Υ and ν_1 denote positive constants. Let further*

$$\check{x}_\mu(\Upsilon, \beta) = \begin{cases} \min(\Upsilon\mu_{\beta,+}^{-1/2}, \beta^{-1/8}) & \mu > 0, \\ \beta^{-1/8} & \mu < 0, \end{cases} \tag{5.9.2}$$

where $\mu_{\beta,+}$ is defined by (3.1.26). Suppose that $\lambda = \mu + i\nu$, where $\beta^{-1} < |\mu|$, $\mu < \Upsilon\beta^{-1/2}$, $\nu_1 < \nu < U(0) + \kappa_0\beta^{-1/2}$ and

$$x_\nu < \check{x}_\mu(\Upsilon, \beta), \tag{5.9.3}$$

where x_ν is defined by (2.4.5). Then, there exist positive Υ_0, β_0, and C such that for all $\beta > \beta_0$ and $\Upsilon < \Upsilon_0$ it holds that

$$\begin{aligned}\|\tilde{v}_{\mathfrak{D}}\|_2 \le C\Big[&\Upsilon^{-5/2}\beta^{-1/4}\|f\|_2 + (\Upsilon^{-3/8}\mu_{+,\beta}^{3/4} + \Upsilon^{3/2}\beta^{3/16})|\phi(x_\nu)| \\ &+ (\mu_{+,\beta}^{1/2} + \Upsilon^2\beta^{1/8})\|\phi'\|_2\Big]\end{aligned} \tag{5.9.4}$$

in which $\tilde{v}_{\mathfrak{D}}$ is given by (5.4.6).

Proof. Let $\delta := \delta(\beta, \Upsilon) \in (0, 1/4)$ be much greater than $\beta^{-1/4}$. More precisely, we introduce for sufficiently small Υ and $\beta \ge \beta_0(\Upsilon)$ with $\beta_0(\Upsilon)$ large enough

$$\delta(\beta, \Upsilon) := \begin{cases} \min(\Upsilon^{1/4}\mu_{\beta,+}^{-1/2}, \Upsilon^{-1}\beta^{-1/8}) & \mu > 0, \\ \Upsilon^{-1}\beta^{-1/8} & \mu < 0. \end{cases} \tag{5.9.5}$$

Recall the definition of $\chi \in C_0^\infty(\mathbb{R}, [0,1])$ from (2.6.20)

$$\chi(x) = \begin{cases} 1 & x < 1/2, \\ 0 & x > 3/4. \end{cases}$$

We further set $\chi_\delta(x) = \chi(x/\delta)$, and $\tilde{\chi}_\delta = 1 - \chi_\delta$. Note that χ_δ is supported in $(0, 3\delta/4)$ and that $\tilde{\chi}_\delta$ is supported in $[\delta/2, +\infty)$.

Step 1: We estimate $\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}$. Using (5.4.7) we now write

$$(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)(\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}) = f_\delta + i\beta U'' \tilde{\chi}_\delta \,\phi, \tag{5.9.6a}$$

where

$$f_\delta = 2\delta^{-1}\tilde{\chi}'(\cdot/\delta)\tilde{v}'_{\mathfrak{D}} + \delta^{-2}\tilde{\chi}''(\cdot/\delta)\tilde{v}_{\mathfrak{D}} + \tilde{\chi}_\delta\,[-f + \phi''(1)\hat{g}]. \tag{5.9.6b}$$

Setting $\gamma = 2^{-1}\beta^{-1/4}\delta^{-1}$ and $v = \tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}$ in (3.2.11) yields

$$\begin{aligned}\beta\| \,|U-\nu|^{1/2}\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2^2 &\leq \|\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2\,\|f_\delta\|_2 \\ &+ C\beta\,\|\,|U-\nu|^{1/2}\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2(\|\phi'\|_2 + \|(U-\nu)^{-1/2}\tilde{\chi}_\delta\|\,|\phi(x_\nu)|),\end{aligned} \tag{5.9.7}$$

where we have used the identities $\tilde{\chi}_\delta\chi_\gamma = \tilde{\chi}_\delta$ and $\tilde{\chi}_\delta\chi'_\gamma = 0$, and the inequality (relying on Hardy's inequality)

$$\begin{aligned}&|\langle\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}, \tilde{\chi}_\delta[(\phi - \phi(x_\nu)) + \phi(x_\nu)]\rangle| \\ &\qquad \leq C\||U-\nu|^{1/2}\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2(\|\phi'\|_2 + \|(U-\nu)^{-1/2}\tilde{\chi}_\delta\|\,|\phi(x_\nu)|).\end{aligned}$$

Since by (5.9.3) and (5.9.5) there exist positive C and $\widehat{C}$ such that

$$(U(0)-\nu)_+ \leq Cx_\nu^2 \leq \widehat{C}\Upsilon^{3/2}\delta^2,$$

we have, for sufficiently small Υ, the existence of $\widetilde{C} > 0$ such that

$$|U-\nu|^{1/2}\tilde{\chi}_\delta \geq \frac{1}{\widetilde{C}}\delta\,\tilde{\chi}_\delta. \tag{5.9.8}$$

Hence, by (5.9.7) and (5.9.8),

$$\|\,|U-\nu|^{1/2}\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2^2 \leq \beta^{-1}\|\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2\,\|f_\delta\|_2 + C(\|\phi'\|_2^2 + \delta^{-1}\,|\phi(x_\nu)|^2), \tag{5.9.9}$$

which implies, using again (5.9.8),

$$\|\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2 \leq C\,\big((\delta^2\beta)^{-1}\,\|f_\delta\|_2 + \delta^{-1}\|\phi'\|_2 + \delta^{-3/2}|\phi(x_\nu)|\big). \tag{5.9.10}$$

Consequently, by (5.9.6b) and (5.4.19)

$$\begin{aligned}\|\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2 \leq C\big([\delta^2\beta]^{-1}\big(\delta^{-1}\|\mathbf{1}_{[\delta/2,\delta]}\tilde{v}'_{\mathfrak{D}}\|_2 + \delta^{-2}\|\mathbf{1}_{[\delta/2,\delta]}\tilde{v}_{\mathfrak{D}}\|_2 + \|f\|_2\big) \\ + \delta^{-3/2}|\phi(x_\nu)| + \delta^{-1}\|\phi'\|_2 + \delta^{-2}\beta^{-5/4}|\phi''(1)|\big).\end{aligned} \tag{5.9.11}$$

Substituting (5.10.13) into (5.9.11) yields in view of (5.9.5) (note that $\delta^2\mu_{\beta,+} \leq \Upsilon^{1/2}$)

$$\begin{aligned}\|\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2^2 \leq C\big([\delta^2\beta]^{-2}\big(\delta^{-2}\|\mathbf{1}_{[\delta/2,\delta]}\tilde{v}'_{\mathfrak{D}}\|_2^2 + \delta^{-4}\|\mathbf{1}_{[\delta/2,\delta]}\tilde{v}_{\mathfrak{D}}\|_2^2\big) \\ + [\delta^2\beta]^{-2}\|f\|_2^2 + \delta^{-3}|\phi(x_\nu)|^2 + \delta^{-2}\|\phi'\|_2^2\big).\end{aligned} \tag{5.9.12}$$

Step 2: We estimate $\chi_{2\delta}\tilde{v}_{\mathfrak{D}}$. Taking the inner product of (5.4.7) with $\chi_{2\delta}^2(U'')^{-1}\tilde{v}_{\mathfrak{D}}$ we obtain (see also (5.4.8)) that

$$\begin{aligned}&\Re\langle\chi_{2\delta}^2(U'')^{-1}\tilde{v}_{\mathfrak{D}},(\mathcal{L}_\beta^{\mathfrak{N}}-\beta\lambda)\tilde{v}_{\mathfrak{D}}-i\beta U''\phi\rangle\\&\quad=\|[\chi_{2\delta}(U'')^{-1/2}\tilde{v}_{\mathfrak{D}}]'\|_2^2-\|[\chi_{2\delta}(U'')^{-1/2}]'\tilde{v}_{\mathfrak{D}}\|_2^2-\beta\mu\,\|\chi_{2\delta}(U'')^{-1/2}\tilde{v}_{\mathfrak{D}}\|_2^2\\&\qquad+\beta\Re\langle\chi_{2\delta}^2\phi''(1)\widehat{\psi},i\phi\rangle.\end{aligned}\tag{5.9.13}$$

As $|U^{(3)}\chi_{2\delta}|\leq C\delta$ (given that $U^{(3)}(0)=0$) we can conclude that

$$\|[\chi_{2\delta}(U'')^{-1/2}\tilde{v}_{\mathfrak{D}}]'\|_2^2\geq\frac{1}{C}\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2-C\delta^2\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2.\tag{5.9.14}$$

Furthermore, as $|(\chi_{2\delta})'|\leq C\delta^{-1}\tilde{\chi}_\delta$, we obtain, using again the fact that $U^{(3)}(0)=0$,

$$\|[\chi_{2\delta}(U'')^{-1/2}]'\tilde{v}_{\mathfrak{D}}\|_2^2\leq C(\delta^2\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2+\delta^{-2}\|\tilde{\chi}_\delta\tilde{v}_{\mathfrak{D}}\|_2^2).$$

Substituting the above, together with (5.9.14) into (5.9.13), recalling that by (5.2.2)

$$\beta|\langle\chi_{2\delta}^2\phi''(1)\widehat{\psi},i\phi\rangle|\leq\beta|\phi''(1)|\,\|\phi'\|_2\|(1-x)^3\widehat{\psi}\|_1\leq C\beta^{-1}|\phi''(1)|\,\|\phi'\|_2,$$

and that by (5.4.17), (5.2.2), (5.2.3), (4.2.10), and (4.2.17)

$$|\langle\chi_{2\delta}(U'')^{-1}\tilde{v}_{\mathfrak{D}},\phi''(1)\tilde{\chi}_{2\delta}\hat{g}\rangle|\leq C\beta^{-3/4}\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2|\phi''(1)|$$

yields

$$\begin{aligned}\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2\leq C\big[&\Upsilon^{-1}\delta^2\|f\|_2^2+\beta^{-3/4}|\phi''(1)|\,\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2\\&+(\Upsilon\delta^{-2}+\mu_{\beta,+})\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2+\delta^{-2}\|\tilde{\chi}_\delta\tilde{v}_{\mathfrak{D}}\|_2^2+\beta^{-1}|\phi''(1)|\|\phi'\|_2\big].\end{aligned}$$

By Poincaré's inequality we have

$$\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2\leq C\delta^2\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2.\tag{5.9.15}$$

Hence, in view of (5.9.5) and (5.9.15), we obtain for sufficiently small Υ

$$\begin{aligned}&\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2+\delta^2\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2\leq\\&C\big[\Upsilon^{-1}\delta^4\|f\|_2^2+\|\tilde{\chi}_\delta\tilde{v}_{\mathfrak{D}}\|_2^2+\beta^{-1}\delta^2|\phi''(1)|(\|\phi'\|_2+\delta^2\beta^{-1/2}|\phi''(1)|)\big].\end{aligned}\tag{5.9.16}$$

Substituting (5.9.1) into (5.9.16) yields, in view of (5.9.5)

$$\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2+\delta^2\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2\leq C\big(\Upsilon^{-1}\delta^4\|f\|_2^2+(\beta^{-1/4}\delta^2+\delta^4)\|\phi'\|_2^2+\|\tilde{\chi}_\delta\tilde{v}_{\mathfrak{D}}\|_2^2\big).\tag{5.9.17}$$

Combining (5.9.12) with (5.9.17), and (5.9.5) we obtain

$$\begin{aligned}\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2+\delta^2\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2\leq C\big[&\Upsilon^{-1}\delta^4\|f\|_2^2+\delta^{-6}\beta^{-2}\|\mathbf{1}_{[\delta/2,\delta]}\tilde{v}_{\mathfrak{D}}'\|_2^2\\&+\delta^{-8}\beta^{-2}\|\mathbf{1}_{[\delta/2,\delta]}\tilde{v}_{\mathfrak{D}}\|_2^2+\delta^{-3}|\phi(x_\nu)|^2+\delta^{-2}\|\phi'\|_2^2\big].\end{aligned}\tag{5.9.18}$$

As $\mathbf{1}_{[\delta/2,\delta]} \leq \chi_{2\delta}$, and $\delta^{-8}\beta^{-2} \leq \Upsilon^{-2}\beta^2\mu_+^4 + \Upsilon^8\beta^{-1}$, where $\mu_+ = \max(\mu,0)$, we obtain

$$\begin{aligned}&\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2 + \delta^2\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2 \\ &\quad\leq C\big[\Upsilon^{-1}\delta^4\|f\|_2^2 + (\Upsilon^{-2}\beta^2\mu_+^4 + \Upsilon^8\beta^{-1})\delta^2\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2 \\ &\qquad + (\Upsilon^{-2}\beta^2\mu_+^4 + \Upsilon^8\beta^{-1})\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2 + \delta^{-3}|\phi(x_\nu)|^2 + \delta^{-2}\|\phi'\|_2^2\big].\end{aligned}$$

For sufficiently small Υ and β_0^{-1} we then conclude (as $\Upsilon^{-2}\beta^2\mu_+^4 \leq \Upsilon^2$) that

$$\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2 + \delta^2\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2 \leq C\big[\Upsilon^{-1}\delta^4\|f\|_2^2 + \delta^{-3}|\phi(x_\nu)|^2 + \delta^{-2}\|\phi'\|_2^2\big]. \tag{5.9.19}$$

Similarly, by (5.9.12) and (5.9.5) (note that $\delta^{-4}\beta^{-2} \leq (\Upsilon^{-2}\beta^2\mu_+^4 + \Upsilon^8\beta^{-1})\delta^4$), it holds that

$$\begin{aligned}\|\tilde{\chi}_\delta\tilde{v}_{\mathfrak{D}}\|_2^2 \leq C\big(&(\Upsilon^{-2}\beta^2\mu_+^4 + \Upsilon^8\beta^{-1})\delta^2\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2 \\ &+(\Upsilon^{-2}\beta^{-2}\mu_+^4+\Upsilon^8\beta^{-1})\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2+(\Upsilon^{-2}\beta^2\mu_+^4+\Upsilon^8\beta^{-1})\delta^4\|f\|_2^2 \\ &+\delta^{-3}|\phi(x_\nu)|^2 + \delta^{-2}\|\phi'\|_2^2\big).\end{aligned} \tag{5.9.20}$$

Since $\chi_{2\delta} + \tilde{\chi}_\delta > \frac{1}{C}$ we obtain from (5.9.19) and (5.9.20) (as $\Upsilon^{-1}\delta^4 \leq C\Upsilon^{-5}\beta^{-1/2}$ by (5.9.5) and since $|\mu| > \beta^{-1}$) that for Υ and β_0^{-1} small enough

$$\begin{aligned}\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\big[&\Upsilon^{-5/2}\beta^{-1/4}\|f\|_2 + (\Upsilon^{-3/8}\mu_{+,\beta}^{3/4} + \Upsilon^{3/2}\beta^{3/16})|\phi(x_\nu)| \\ &+ (\mu_{+,\beta}^{1/2} + \Upsilon^2\beta^{1/8})\|\phi'\|_2\big],\end{aligned}$$

which is precisely (5.9.4). ∎

5.10 Resolvent estimates for $|\Im\lambda - U(0)| = \mathcal{O}(\beta^{-1/2})$

We consider, for given positive $\kappa_0, \alpha_1, \lambda = \mu + i\nu$, and some positive Υ, the zone

$$\mathcal{E}(\alpha_1,\beta_0,\Upsilon,\kappa_0) := \left\{\begin{matrix}(\lambda,\alpha,\beta) \in \mathbb{C}\times\mathbb{R}_+^2,\ 0 \leq \alpha \leq \alpha_1\beta^{1/3}, \beta \geq \beta_0 \\ \mu < \Upsilon\beta^{-1/2},\ U(0) - \kappa_0\beta^{-1/2} \leq \nu \leq U(0) + \kappa_0\beta^{-1/2}\end{matrix}\right\}. \tag{5.10.1}$$

Proposition 5.10.1. *Let $U \in C^4([0,1])$ satisfy (2.1.3) and $U^{(3)}(0) = 0$. Let further $\alpha_1 > 0$ and $\kappa_0 > 0$. Then, there exist positive Υ, β_0 and C, such that for $(\lambda,\alpha,\beta) \in \mathcal{E}(\alpha_1,\beta_0,\Upsilon,\kappa_0)$, it holds*

$$\max(1,|\mu\beta|^{1/4})\Big(\|(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})^{-1}\| + \Big\|\frac{d}{dx}(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})^{-1}\Big\|\Big) \leq C\beta^{-3/8}. \tag{5.10.2}$$

Proof. Step 1: Preliminaries. We follow the same outlines as in the proof of Proposition 5.4.1. Nevertheless, given that $|\nu - U(0)| \sim \mathcal{O}(\beta^{-1/2})$, we need to address here the quadratic behavior of $U(x) - U(0)$ in the vicinity of $x = 0$ (see Section 3.2 for instance).

Let $\tilde{v}_{\mathfrak{D}}$ be given by (5.4.6). For the convenience of the reader, we repeat (5.4.8), which reads

$$\begin{aligned}\Re\langle (U'')^{-1}\tilde{v}_{\mathfrak{D}}, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)\tilde{v}_{\mathfrak{D}} + i\beta U''\phi\rangle \\ = \|(U'')^{-1/2}\tilde{v}_{\mathfrak{D}}'\|_2^2 + \Re\langle \big((U'')^{-1}\big)'\tilde{v}_{\mathfrak{D}}, \tilde{v}_{\mathfrak{D}}'\rangle \\ - \beta\mu\|(U'')^{-1/2}\tilde{v}_{\mathfrak{D}}\|_2^2 + \beta\Re\langle\phi''(1)\hat{\psi}, i\phi\rangle,\end{aligned} \tag{5.10.3}$$

where $\hat{\psi} = \hat{\psi}_{\lambda,\beta}$ is introduced in (5.2.1) and $\phi \in D(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})$ satisfies for $f \in L^2(0,1)$

$$\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}}\phi = f.$$

We begin by estimating the last term on the right-hand side of (5.10.3). For technical reasons we distinguish between the case $\mu < -\mu_0$ (for some sufficiently small $\mu_0 > 0$) and $\mu > -\mu_0$.

Step 2: We estimate $\Re\langle\phi''(1)\hat{\psi}, i\phi\rangle$ for $\mu \geq -\mu_0$. As in Step 2 of the proof of Proposition 5.4.1 we write

$$\phi(x) = \int_x^1 (\xi - x)\phi''(\xi)\,d\xi = \phi''(1)w + \int_x^1 (\xi - x)[\phi''(\xi) - \phi''(1)\hat{\psi}(\xi)]\,d\xi,$$

where

$$w(x) = \int_x^1 (\xi - x)\hat{\psi}(\xi)\,d\xi.$$

Then we write

$$\Re\langle\phi''(1)\hat{\psi}, i\phi\rangle = -|\phi''(1)|^2\Im\langle\hat{\psi}, w\rangle + \Re\langle\phi''(1)\hat{\psi}, i(\phi - \phi''(1)w)\rangle. \tag{5.10.4}$$

For the first term on the right-hand side we write, using the fact that $w'' = \hat{\psi}$, and integration by parts,

$$\Im\langle\hat{\psi}, w\rangle = \Im\langle w'', w\rangle = -\Im\{\bar{w}'(0)w(0)\}. \tag{5.10.5}$$

We now use [3, Proposition A.1] to obtain the following improvement of (5.2.2):

$$\hat{\psi}(x) = e^{-\beta^{1/2}(-\lambda)^{1/2}(1-x)} + \hat{\psi}_1(x), \tag{5.10.6}$$

where

$$\|\hat{\psi}_1\|_1 + \beta^{1/2}\|(1-x)\hat{\psi}_1\|_1 \leq C\beta^{-1}. \tag{5.10.7}$$

Next, we write, using (5.10.1), (5.10.6), (5.10.7)

$$\begin{aligned}\bar{w}'(0) &= -\int_0^1 \overline{\widehat{\psi}}(\xi)\,d\xi = -\int_0^1 [e^{-\beta^{1/2}(-\bar\lambda)^{1/2}(1-\xi)} + \overline{\widehat{\psi}}_1(\xi)]\,d\xi \\ &= -\frac{1}{(-\beta\bar\lambda)^{1/2}} + \mathcal{O}(\beta^{-1}) \\ &= -\frac{e^{-i\pi/4}\beta^{-1/2}}{[U(0)]^{1/2}}[1+\mathcal{O}(|\mu|^{1/2})] + \mathcal{O}(\beta^{-1}). \end{aligned} \tag{5.10.8}$$

To obtain (5.10.8) we used the identities

$$\int_0^1 e^{-\beta^{1/2}(-\bar\lambda)^{1/2}(1-x)}dx = \beta^{-1/2}(-\bar\lambda)^{-1/2}(1-e^{-\beta^{1/2}(-\lambda)^{1/2}})$$

and

$$\beta^{-1/2}(-[\mu - i\nu])^{-1/2} = e^{-i\pi/4}\beta^{-1/2}(U(0)+[\nu - U(0) + i\mu])^{-1/2},$$

which shows the exponentially small behavior of $e^{-\beta^{1/2}(-\lambda)^{1/2}}$ if $\mu_0 > 0$ is chosen small enough.

Furthermore, it holds that

$$w(0) = \int_0^1 \xi\widehat{\psi}(\xi)\,d\xi = -w'(0) - \int_0^1 (1-\xi)\widehat{\psi}(\xi)\,d\xi,$$

which implies by (5.2.2), (5.10.6), and (5.10.7)

$$w(0) = -w'(0) - \frac{i\beta^{-1}}{U(0)}[1+\mathcal{O}(|\mu|^{1/2})] + \mathcal{O}(\beta^{-3/2}). \tag{5.10.9}$$

Combining (5.10.8) and (5.10.9) yields

$$\Im\{\bar{w}'(0)w(0)\} = \frac{\beta^{-3/2}}{[U(0)]^{3/2}\sqrt{2}}[1+\mathcal{O}(|\mu|^{1/2})] + \mathcal{O}(\beta^{-2}). \tag{5.10.10}$$

Substituting (5.10.10) into (5.10.5) yields

$$\beta\Re\langle\widehat{\psi}, iw\rangle = \frac{\beta^{-1/2}}{[U(0)]^{3/2}\sqrt{2}}[1+\mathcal{O}(|\mu|^{1/2})] + \mathcal{O}(\beta^{-1}).$$

For sufficiently large β_0 and sufficiently small μ_0, $\beta\Re\langle\widehat{\psi}, iw\rangle$ is positive and hence, by (5.10.4) we can conclude that

$$\Re\langle\phi''(1)\widehat{\psi}, i\phi\rangle \ge \Re\langle\phi''(1)\widehat{\psi}, i(\phi - \phi''(1)w)\rangle. \tag{5.10.11}$$

As in the proof of Proposition 5.4.1 (Step 2) we now apply (5.4.11), recalling that $\alpha \leq \alpha_1 \beta^{1/3}$, to obtain (note that $\lambda_\beta \geq \frac{1}{2}|U(0)|\beta^{1/3}$ by (5.10.1))

$$\begin{aligned} \beta\,\Re\langle \phi''(1)\widehat{\psi}, i\phi\rangle &\geq -C\,\beta^{-1/6}\lambda_\beta^{-7/4}\,|\phi''(1)|(\|\tilde{v}'_{\mathfrak{D}}\|_2 + \beta^{2/3}\|\phi'\|_2) \\ &\geq -\widehat{C}\,\beta^{-3/4}|\phi''(1)|(\|\tilde{v}'_{\mathfrak{D}}\|_2 + \beta^{2/3}\|\phi'\|_2). \end{aligned} \tag{5.10.12}$$

Let x_ν be defined by (2.4.5). By the assumption on ν it holds that

$$x_\nu \leq C\beta^{-1/4}.$$

Hence, by (5.9.1)

$$|\phi''(1)| \leq C\left(\beta^{-1/8}\|f\|_2 + \beta^{3/4}|\phi(x_\nu)| + \beta^{5/8}\|\phi'\|_2\right), \tag{5.10.13}$$

from which we conclude

$$\beta\,\Re\langle \phi''(1)\widehat{\psi}, i\phi\rangle \geq -C\,(\|\phi\|_{1,2} + \beta^{-7/8}\|f\|_2)(\|\tilde{v}'_{\mathfrak{D}}\|_2 + \beta^{2/3}\|\phi'\|_2). \tag{5.10.14}$$

Step 3: We estimate $\tilde{v}_{\mathfrak{D}}$ and $\tilde{v}'_{\mathfrak{D}}$. Here, we follow the Steps 4 and 5 in the proof of Proposition 5.4.1 with (5.4.15) replaced by (5.10.13).

By (3.2.1a), (5.4.7), (5.4.19), and (5.10.13), we obtain (compare with (5.4.24)) that

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C(\beta^{1/4}\|\phi'\|_2 + \beta^{3/8}|\phi(x_\nu)| + \beta^{-1/2}\|f\|_2). \tag{5.10.15}$$

Substituting (5.10.15), together with (5.10.13), (5.10.14), and (5.4.19), into (5.4.20) yields

$$\begin{aligned} &\frac{1}{C}\|\tilde{v}'_{\mathfrak{D}}\|_2^2 \\ &\quad\leq (\beta^{1/4}\|\phi'\|_2 + \beta^{3/8}|\phi(x_\nu)| + \beta^{-1/2}\|f\|_2)(\|f\|_2 + \beta^{3/8}\|\phi'\|_2 + \beta^{1/2}|\phi(x_\nu)|) \\ &\qquad + (1+\mu_{\beta,+})(\beta^{1/4}\|\phi'\|_2 + \beta^{3/8}|\phi(x_\nu)| + \beta^{-1/2}\|f\|_2)^2 \\ &\qquad + (\|\phi\|_{1,2} + \beta^{-7/8}\|f\|_2)(\|\tilde{v}'_{\mathfrak{D}}\|_2 + \beta^{2/3}\|\phi'\|_2). \end{aligned}$$

Hence,

$$\|\tilde{v}'_{\mathfrak{D}}\|_2 \leq C\left((\beta^{1/8} + \mu_{\beta,+}^{1/2})(\beta^{1/4}\|\phi'\|_2 + \beta^{3/8}|\phi(x_\nu)|) + \beta^{-1/8}\|f\|_2\right). \tag{5.10.16}$$

Recall that

$$\mu_{\beta,+} = \beta\mu_+ = \beta\max(\mu, 0).$$

Since (5.10.16) is unsatisfactory, given that the coefficient of $\beta^{1/4}\|\phi'\|_2 + \beta^{3/8}|\phi(x_\nu)|$ is not necessarily small, as will become clear in the sequel, we obtain an improved estimate in the next step.

Step 4. For $|\mu| > \beta^{-1}$ we prove under the assumptions of the proposition that

$$\|\phi'' - \phi''(1)\widehat{\psi}\|_2 \leq C\big[\Upsilon^{-5/2}\beta^{-1/4}\|f\|_2 + (\Upsilon^{-3/8}\mu_{+,\beta}^{3/4} + \beta^{1/3})|\phi(x_\nu)| + (\mu_{+,\beta}^{1/2} + \Upsilon^2\beta^{1/8})\|\phi'\|_2\big]. \tag{5.10.17}$$

Using the definition of $\tilde{v}_{\mathfrak{D}}$ given in (5.4.6), an integration by parts yields

$$\langle -\phi'' + \phi''(1)\widehat{\psi}, \tilde{v}_{\mathfrak{D}}\rangle = \|\phi'' - \phi''(1)\widehat{\psi}\|_2^2 + \alpha^2\|\phi'\|_2^2 + \langle \phi''(1)\widehat{\psi}, \alpha^2\phi\rangle. \tag{5.10.18}$$

By (5.2.2) for $s = 1/2$ and (5.10.13) it holds that

$$\begin{aligned} |\langle \phi''(1)\widehat{\psi}, \alpha^2\phi\rangle| &\leq \alpha^2|\phi''(1)|\,\|(1-x)^{1/2}\widehat{\psi}\|_1\,\|\phi'\|_2 \\ &\leq C\alpha^2\left(\beta^{-7/8}\|f\|_2 + |\phi(x_\nu)| + \beta^{-1/8}\|\phi'\|_2\right)\|\phi'\|_2. \end{aligned}$$

Substituting the above into (5.10.18) yields for sufficiently large β_0

$$\begin{aligned} &\|\phi'' - \phi''(1)\widehat{\psi}\|_2^2 + \alpha^2\|\phi'\|_2^2 \\ &\leq \|-\phi''+\phi''(1)\widehat{\psi}\|_2\|\tilde{v}_{\mathfrak{D}}\|_2 + C\alpha^2\big(\beta^{-7/8}\|f\|_2 + |\phi(x_\nu)| + \beta^{-1/8}\|\phi'\|_2\big)\|\phi'\|_2. \end{aligned}$$

For sufficiently large β_0 we then obtain that

$$\|\phi'' - \phi''(1)\widehat{\psi}\|_2^2 \leq C\left[\|\tilde{v}_{\mathfrak{D}}\|_2^2 + \alpha^2(\beta^{-7/4}\|f\|_2^2 + |\phi(x_\nu)|^2)\right]. \tag{5.10.19}$$

We now obtain (5.10.17) from (5.9.4) and the fact that $\alpha \leq \alpha_1\beta^{1/3}$.

Step 5. We estimate $\|v_{\mathfrak{D}}\|_\infty$ under the assumption of the proposition and the additional conditions $|\mu| > \beta^{-1}$ and $\mu \geq -\mu_0$.

Let $v_{\mathfrak{D}}$ be given by (5.4.2). For the convenience of the reader we recall here (5.4.4)

$$(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)v_{\mathfrak{D}} = g_{\mathfrak{D}}, \tag{5.10.20a}$$

where (after reordering)

$$\begin{aligned} g_{\mathfrak{D}} = (U + i\lambda)(-f + \phi''(1)\hat{g}) - 2U'\tilde{v}_{\mathfrak{D}}' - U''\phi''(1)\widehat{\psi} \\ + U''([\phi''(1)\widehat{\psi} - \phi''] - \tilde{v}_{\mathfrak{D}}) - 2U^{(3)}\phi' - U^{(4)}\phi. \end{aligned} \tag{5.10.20b}$$

Next, we obtain a bound for $\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}g_{\mathfrak{D}}\|_\infty$ by separately estimating the contribution of each of the six terms on the right-hand side of (5.10.20b).

To obtain the L^∞ estimates we repeatedly use the following Sobolev embedding inequality

$$\|v_{\mathfrak{D}}\|_\infty \leq \|v_{\mathfrak{D}}\|_2^{1/2}\,\|v_{\mathfrak{D}}'\|_2^{1/2}. \tag{5.10.21}$$

Writing $(U + i\lambda) = (U - \nu) + i\mu$ we obtain by (3.1.84)–(3.1.85) (for $\mu < 0$), (3.2.1a) (for $\mu > 0$), and (3.3.35), that

$$\begin{aligned}&\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U + i\lambda)(-f + \phi''(1)\hat{g})\|_\infty \\ &\qquad \leq C\big(\beta^{-3/4} + |\mu|^{1/4}\beta^{-3/4} + \mu_+\beta^{-3/8}\big)\big(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2\big).\end{aligned}$$

Since $-\mu_0 < \mu < \Upsilon\beta^{-1/2}$, we obtain

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U + i\lambda)(-f + \phi''(1)\hat{g})\|_\infty \leq C\beta^{-3/4}(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2). \tag{5.10.22a}$$

Using (3.2.1a) and (3.2.1b) we obtain, recalling that, for $|\nu - U(0)| \leq \kappa_0\beta^{-1/2}$, we have $x_\nu \leq C\beta^{-1/4}$,

$$\begin{aligned}&\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U'\tilde{v}'_{\mathfrak{D}})\|_\infty \\ &\quad = \|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U'(x_\nu)) + (U'(x) - U'(x_\nu))\tilde{v}'_{\mathfrak{D}}\|_\infty \\ &\quad \leq C\,\big(x_\nu\beta^{-3/8} + \beta^{-5/8}\big)\|\tilde{v}'_{\mathfrak{D}}\|_2 \leq \hat{C}\beta^{-5/8}\|\tilde{v}'_{\mathfrak{D}}\|_2.\end{aligned} \tag{5.10.22b}$$

By (4.6.3) and (3.2.1a) we have

$$\begin{aligned}&\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(\phi''(1)U''\hat{\psi})\|_\infty \\ &\quad = \|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}\phi''(1)[U''(1) + (U''(x) - U''(1))]\hat{\psi}\|_\infty \\ &\quad \leq C\,\big(\beta^{-1} + \beta^{-3/8}\|(1-x)\hat{\psi}\|_2\big)\,|\phi''(1)|.\end{aligned}$$

By (4.2.10), (4.2.16), and (4.6.1) it holds that

$$\|(1-x)^k\,\hat{\psi}\|_2 \leq C\,\lambda_\beta^{-(1+2k)/4}\beta^{-(1+2k)/6}. \tag{5.10.22c}$$

Using (5.10.22c) with $k = 1$ yields

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(\phi''(1)U''\hat{\psi})\|_\infty \leq C\beta^{-1}|\phi''(1)|. \tag{5.10.22d}$$

For the next term we use (3.2.1a) and (3.2.1b) to obtain that

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}U''([\phi''(1)\hat{\psi} - \phi''] - \tilde{v}_{\mathfrak{D}})\|_\infty \leq C\beta^{-3/8}\,\big(\|\phi''(1)\hat{\psi} - \phi''\|_2 + \|\tilde{v}_{\mathfrak{D}}\|_2\big).$$

We then use (5.9.4) and (5.10.17) to obtain that

$$\begin{aligned}&\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}U''([\phi''(1)\hat{\psi} - \phi''] - \tilde{v}_{\mathfrak{D}})\|_\infty \\ &\quad \leq C\beta^{-3/8}\big[\Upsilon^{-5/2}\beta^{-1/4}\|f\|_2 + (\Upsilon^{-3/8}\mu_{+,\beta}^{3/4} + \beta^{1/3})|\phi(x_\nu)| \\ &\qquad\qquad + (\mu_{+,\beta}^{1/2} + \Upsilon^2\beta^{1/8})\|\phi'\|_2\big].\end{aligned} \tag{5.10.22e}$$

As by (5.4.6)

$$\phi''(1)\hat{\psi} - \phi'' - \tilde{v}_{\mathfrak{D}} = \alpha^2\phi,$$

we may also write

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}U''([\phi''(1)\hat{\psi} - \phi''] - \tilde{v}_{\mathfrak{D}})\|_\infty = \|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}U''\alpha^2\phi\|_\infty$$
$$\leq \|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}U''\alpha^2[\phi(x_\nu)]\|_\infty + \|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}U''\alpha^2[\phi - \phi(x_\nu)]\|_\infty .$$

Then we use (3.2.1a) and (3.2.1b) together with Hardy's inequality for the second term to obtain,

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}U''\alpha^2\phi\|_\infty \leq C\alpha^2\beta^{-1/2}\big(|\phi(x_\nu)| + \beta^{-1/8}\|\phi'\|_2\big). \quad (5.10.22\text{f})$$

In the sequel we use (5.10.22e) for $\alpha \geq \beta^{1/8}$ and (5.10.22f) for $\alpha < \beta^{1/8}$.

We estimate the next term as in (5.10.22a) (as $|U^{(3)}(x)| \leq Cx$):

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(2U^{(3)}\phi')\|_\infty \leq C\,\beta^{-5/8}\|\phi'\|_2. \quad (5.10.22\text{g})$$

Finally, we estimate the last term as in (5.10.22e):

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(2U^{(4)}\phi)\|_\infty \leq C\beta^{-1/2}(|\phi(x_\nu)| + \beta^{-1/8}\|\phi'\|_2). \quad (5.10.22\text{h})$$

Combining (5.10.22a)–(5.10.22h) then yields

$$\|v_{\mathfrak{D}}\|_\infty \leq C\big[\beta^{-3/4}\gamma(\alpha,\beta)\|f\|_2 + \beta^{-3/4}|\phi''(1)|\{\|\hat{g}\|_2 + \beta^{-1/4}\} + \beta^{-5/8}\|\tilde{v}_{\mathfrak{D}}'\|_2$$
$$+ ([\beta\Upsilon]^{-3/8}\mu_{+,\beta}^{3/4} + \beta^{-1/24})|\phi(x_\nu)| + (\beta^{-3/8}\mu_{+,\beta}^{1/2} + \beta^{-1/4})\|\phi'\|_2\big], \quad (5.10.23)$$

where

$$\gamma(\alpha,\beta) = \begin{cases} 1 & \alpha < \beta^{1/8}, \\ \Upsilon^{-5/2}\beta^{1/8} & \alpha \geq \beta^{1/8}. \end{cases}$$

Substituting (5.10.13), (5.10.16), (5.4.19), and (5.9.4) into (5.10.23), yields, with the aid of (4.6.5)

$$\|v_{\mathfrak{D}}\|_\infty \lesssim \beta^{-3/4}\gamma(\alpha,\beta)\|f\|_2$$
$$+ ([\beta\Upsilon]^{-3/8}\mu_{+,\beta}^{3/4} + \beta^{-1/24})|\phi(x_\nu)| + (\beta^{-3/8}\mu_{+,\beta}^{1/2} + \beta^{-1/4})\|\phi'\|_2.$$

By the assumption on μ_+, we may finally conclude

$$\|v_{\mathfrak{D}}\|_\infty \leq C\big(\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2 + [\beta^{-1/4} + \mu_+^{1/2}\beta^{1/8}]\|\phi\|_{1,2} + [\beta^{-1/24}$$
$$+ \mu_+^{3/8}\beta^{3/16}]|\phi(x_\nu)|\big). \quad (5.10.24)$$

Step 6. We prove (5.10.2) in the case $\mu > -\mu_0$ and $\alpha \leq \alpha_0\beta^{1/8}$.

Step 6a. Preliminaries. We continue as in the proof of [3, Lemma 8.8]. We first write, as in (5.4.31)

$$\phi = \phi_{\mathfrak{D}} + \breve{\phi}, \tag{5.10.25a}$$

where

$$\phi_{\mathfrak{D}} = \mathcal{A}_{\lambda,\alpha}^{-1} v_{\mathfrak{D}}, \quad \breve{\phi} = -\mathcal{A}_{\lambda,\alpha}^{-1}\big([U + i\lambda]\phi''(1)\widehat{\psi}\big) = -\phi''(1)\phi_{\lambda,\alpha,\beta}. \tag{5.10.25b}$$

By Propositions 2.8.1 and 2.9.1, there exists (sufficiently small) $C > 0$ so that we can use (2.8.47), for $|\mu| \le C\, x_\nu^2$, and (2.9.14) for $|\mu| \ge C x_\nu^2$, both holding for sufficiently small μ_0. Hence, we can conclude that for any pair $(\tilde{v}, \tilde{\phi}) \in W^{1,p}(0,1) \times D(\mathcal{A}_{\lambda,\alpha})$ satisfying $\tilde{v} = \mathcal{A}_{\lambda,\alpha}\tilde{\phi}$,

$$|\tilde{\phi}(x_\nu)|^2 \le C\Big(|\mu|^{1/2}\Big|\Big\langle\tilde{\phi}, \frac{\tilde{v}}{U + i\lambda}\Big\rangle\Big| + x_\nu\Big\|(1-x)^{1/2}\frac{\tilde{v}}{U + i\lambda}\Big\|_1^2\Big). \tag{5.10.26}$$

We apply the above inequality to the pair $(-(U + i\lambda)\phi''(1)\widehat{\psi}, \breve{\phi})$ to obtain

$$|\breve{\phi}(x_\nu)|^2 \le C|\phi''(1)|\,(|\mu|^{1/2}|\langle\breve{\phi}, \widehat{\psi}\rangle| + x_\nu|\phi''(1)|\|(1-x)^{1/2}\widehat{\psi}|_1^2).$$

Given that $\breve{\phi}(1) = 0$ we write

$$|\langle\phi, \widehat{\psi}\rangle| \le \|\phi'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_1 \le C\,[|\lambda|\beta]^{-3/4}\|\phi'\|_2, \tag{5.10.27}$$

to obtain, with the aid of (5.2.2) and (5.4.31),

$$|\langle\breve{\phi}, \widehat{\psi}\rangle| \le \|\breve{\phi}'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_1 \le C\,[|\lambda|\beta]^{-3/4}\|\breve{\phi}'\|_2, \tag{5.10.28}$$

and consequently it holds that

$$|\breve{\phi}(x_\nu)|^2 \le \widehat{C}\,(|\mu|^{1/2}\beta^{-3/4}\|\breve{\phi}'\|_2|\phi''(1)| + x_\nu\beta^{-3/2}|\phi''(1)|^2). \tag{5.10.29}$$

Combining (5.10.29) with (5.10.13) yields

$$\begin{aligned}|\breve{\phi}(x_\nu)|^2 \le \widehat{C}\,\big(|\mu|^{1/2}\|\breve{\phi}'\|_2 + x_\nu\big(\beta^{-7/8}\|f\|_2 + |\phi(x_\nu)| + \beta^{-1/8}\|\phi'\|_2\big)\big)\\ \times\big(\beta^{-7/8}\|f\|_2 + |\phi(x_\nu)| + \beta^{-1/8}\|\phi'\|_2\big).\end{aligned}$$

Since

$$|\mu|^{1/2}\|\breve{\phi}'\|_2|\phi(x_\nu)| \le C(\delta^{-2}|\mu|\|\breve{\phi}'\|_2^2 + \delta^2|\phi(x_\nu)|^2)$$

for any $\delta > 0$ and $x_\nu < C\beta^{-1/4}$, we can conclude that there exists $C > 0$ such that for any $\delta > 0$

$$\begin{aligned}|\breve{\phi}(x_\nu)| \le C\,\big(\delta^{-1}(|\mu|^{1/2} + \beta^{-1/8})[\|\breve{\phi}'\|_2 + \|\phi_{\mathfrak{D}}'\|_2]\\ + (\delta + \beta^{-1/8})[|\breve{\phi}(x_\nu)| + |\phi_{\mathfrak{D}}(x_\nu)|] + \delta^{-1}\beta^{-7/8}\|f\|_2\big).\end{aligned} \tag{5.10.30}$$

By applying to the pair $(\phi_{\mathfrak{D}}, v_{\mathfrak{D}})$ (2.8.65) for $U(0)-\kappa_0\beta^{-1/2} \le \nu \le U(0)-\kappa_0|\mu|$, (2.9.29) for $U(0)-\kappa_0\min(|\mu|,\beta^{-1/2}) \le \nu \le U(0)+\kappa_0\min(|\mu|,\beta^{-1/2})$, and (2.10.12) for $U(0)+\kappa_0|\mu| \le \nu \le U(0)+\kappa_0\beta^{-1/2}$, we obtain that

$$|\phi_{\mathfrak{D}}(x_\nu)| \le C\Big(1+\log\frac{\max(x_\nu,|\mu|^{1/2})}{|\mu|^{1/2}}\Big)\|v_{\mathfrak{D}}\|_\infty .$$

With the aid of (5.10.24) we then obtain

$$\begin{aligned}|\phi_{\mathfrak{D}}(x_\nu)| &\le C\left(1+\log\frac{\max(x_\nu,|\mu|^{1/2})}{|\mu|^{1/2}}\right)\\ &\times\big(\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2+[\beta^{-1/4}+\mu_+^{1/2}\beta^{1/8}]\|\phi\|_{1,2}+[\beta^{-1/24}+\mu_+^{3/8}\beta^{3/16}]|\phi(x_\nu)|\big),\end{aligned}$$

(where μ_+ is given by (3.3.20)), from which we conclude by (5.10.25) that

$$\begin{aligned}\mu_+|\phi_{\mathfrak{D}}(x_\nu)| \le C\Big(1+\log\frac{\max(x_\nu,|\mu|^{1/2})}{|\mu|^{1/2}}\Big)&\\ \times\big(\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2+(\beta^{-1/4}+\mu_+^{1/2}\beta^{1/8})[\|\check{\phi}'\|_2+\|\phi_{\mathfrak{D}}'\|_2]&\\ +(\beta^{-1/24}+\mu_+^{3/8}\beta^{3/16})[|\check{\phi}(x_\nu)|+|\phi_{\mathfrak{D}}(x_\nu)|]\big).&\end{aligned} \tag{5.10.31}$$

Combining (5.10.30) and (5.10.31) yields,

$$\begin{aligned}|\check{\phi}(x_\nu)|+|\phi_{\mathfrak{D}}(x_\nu)| \le C\left(1+\log\frac{\max(x_\nu,|\mu|^{1/2})}{|\mu|^{1/2}}\right)&\\ \times\big([\beta^{-1/24}+\mu_+^{3/8}\beta^{3/16}+\delta][|\check{\phi}(x_\nu)|+|\phi_{\mathfrak{D}}(x_\nu)|]&\\ +[\beta^{-1/4}+\mu_+^{1/2}\beta^{1/8}+\delta^{-1}|\mu|^{1/2}][\|\check{\phi}'\|_2+\|\phi_{\mathfrak{D}}'\|_2]&\\ +\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2\big).&\end{aligned} \tag{5.10.32}$$

Step 6b. We prove the existence of $\Upsilon>0$, $\mu_0>0$, and $\beta_0>0$ such that (5.10.2) holds for $\Upsilon^{-1}\beta^{-1}<\mu<\Upsilon\beta^{-1/2}$ or $-\mu_0<\mu<-\Upsilon^{-1}\beta^{-1}$, with $\beta\ge\beta_0$.

Let

$$\delta=\hat{\delta}\Big/\Big(1+\log\frac{\max(x_\nu,|\mu|^{1/2})}{|\mu|^{1/2}}\Big),$$

where $\hat{\delta}>0$ is independent of β.

Note that for $\beta^{-1}<|\mu|<x_\nu^2$, we have, for any $s>0$,

$$\Big(\frac{|\mu|^{1/2}}{x_\nu}\Big)^s\Big(1+\log\frac{\max(x_\nu,|\mu|^{1/2})}{|\mu|^{1/2}}\Big)\le C_s. \tag{5.10.33}$$

The above inequality implies (with $s=1$), since $x_\nu\le C\beta^{-1/4}$, the existence of $\beta_0>0$ such that for all $\beta>\beta_0$,

$$\delta^{-1}|\mu|^{1/2}\le C\beta^{-1/4}. \tag{5.10.34a}$$

Furthermore, (5.10.34) with $s = 1/2$ leads to

$$\Big(1 + \log \frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big)\, \mu_+^{3/8}\beta^{3/16} \leq C\mu_+^{1/8} x_\nu^{1/2}\beta^{3/16} \leq \widehat{C}\,\Upsilon^{1/8}, \tag{5.10.34b}$$

and with $s = 1/12$ to

$$\Big(1 + \log \frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big)\, \beta^{-1/24} \leq C_s\beta^{-1/24}(x_\nu/|\mu|^{1/2})^s \leq \widehat{C}\,\beta^{-1/48}. \tag{5.10.34c}$$

Hence, we obtain from (5.10.32) and (5.10.34) that, for sufficiently small Υ, $\hat{\delta}$, and μ_0 and sufficiently large β_0,

$$\begin{aligned}|\check{\phi}(x_\nu)| + |\phi_{\mathfrak{D}}(x_\nu)| &\leq C\Big(1 + \log \frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big) \\ &\times \big(\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2 + [\beta^{-1/4} + \mu_+^{1/2}\beta^{1/8} + |\mu|^{1/2}][\|\check{\phi}'\|_2 + \|\phi_{\mathfrak{D}}'\|_2]\big).\end{aligned} \tag{5.10.35}$$

Next, we apply (2.8.1b), for $U(0) - \kappa_0\beta^{-1/2} \leq \nu \leq U(0) - \kappa_0|\mu|$, (2.9.1) for $U(0) - \kappa_0 \min(|\mu|, \beta^{-1/2}) \leq \nu \leq U(0) + \kappa_0 \min(|\mu|, \beta^{-1/2})$, and (2.10.1) for $U(0) + \kappa_0|\mu| \leq \nu \leq U(0) + \kappa_0\beta^{-1/2}$, for $p = +\infty$, to the pair $(\phi_{\mathfrak{D}}, v_{\mathfrak{D}})$ to obtain that, for $|\mu| > \Upsilon^{-1}\beta^{-1}$,

$$\Big(1 + \log \frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big)\|\phi_{\mathfrak{D}}'\|_2 \leq C|\mu|^{-1/4}\|v_{\mathfrak{D}}\|_\infty. \tag{5.10.36}$$

Note that while applying (2.8.1b) we have, since $x_\nu^2 > \frac{1}{C}|\mu|$ in this case, that

$$\frac{[\log \frac{x_\nu}{|\mu|^{1/2}}]^2}{x_\nu^{1/2}} \leq \widehat{C}\,|\mu|^{-1/4}.$$

Then (5.10.24) and (5.10.36) yield with the aid of (5.10.25a),

$$\begin{aligned}\|\phi_{\mathfrak{D}}'\|_2 \leq C\Big(1 + \log \frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big)^{-1}\Big(&|\mu|^{-1/4}\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2 \\ &+ (|\mu|^{-1/4}\beta^{-1/4} + \mu_+^{1/4}\beta^{1/8})[\|\check{\phi}'\|_2 + \|\phi_{\mathfrak{D}}'\|_2] \\ &+ (\beta^{-1/24}|\mu|^{-1/4} + \mu_+^{1/8}\beta^{3/16})[|\check{\phi}(x_\nu)| + |\phi_{\mathfrak{D}}(x_\nu)|]\Big).\end{aligned} \tag{5.10.37}$$

Substituting (5.10.35) into (5.10.37) yields

$$\begin{aligned}\|\phi_{\mathfrak{D}}'\|_2 \leq C\big(&|\mu|^{-1/4}\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2 \\ &+ [|\mu|^{-1/4}\beta^{-1/4} + \mu_+^{1/4}\beta^{1/8} + |\mu|^{1/4}][\|\check{\phi}'\|_2 + \|\phi_{\mathfrak{D}}'\|_2]\big).\end{aligned} \tag{5.10.38}$$

We now use (2.9.13) for the pair $(\check{\phi}, \phi''(1)(U + i\lambda)\widehat{\psi})$, together with (5.10.28) to obtain that

$$\|\check{\phi}'\|_2 \leq C\,[\lambda|\beta|]^{-3/4}|\phi''(1)|. \tag{5.10.39}$$

Using (5.10.13), we deduce from (5.10.39) for sufficiently large β_0

$$\|\check{\phi}'\|_2 \leq C([|\check{\phi}(x_\nu)| + |\phi_{\mathfrak{D}}(x_\nu)|] + \beta^{-1/8}[\|\check{\phi}'\|_2 + \|\phi'_{\mathfrak{D}}\|_2] + \beta^{-7/8}\|f\|_2). \tag{5.10.40}$$

We then obtain from (5.10.35)

$$\begin{aligned}\|\check{\phi}'\|_2 &\leq C\Big(1 + \log\frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big)\\ &\times \big(\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2 + [\beta^{-1/4} + \mu_+^{1/2}\beta^{1/8} + |\mu|^{1/2}][\|\check{\phi}'\|_2 + \|\phi'_{\mathfrak{D}}\|_2]\big).\end{aligned} \tag{5.10.41}$$

To obtain the coefficient of $\|f\|_2$, we set $s = 1/2$ in (5.10.33) to conclude for $|\mu| < x_\nu^2$

$$\Big(1 + \log\frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big) \leq C x_\nu^{1/2}|\mu|^{-1/4} \leq \widetilde{C}\beta^{-1/4}|\mu|^{-1/4} \leq \widehat{C}|\mu|^{-1/4}.$$

Hence, combining (5.10.41) with (5.10.38) yields

$$\begin{aligned}\|\check{\phi}'\|_2 + \|\phi'_{\mathfrak{D}}\|_2 &\leq C\bigg(|\mu|^{-1/4}\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2\\ &+ [\beta^{-1/4} + \mu_+^{1/2}\beta^{1/8} + |\mu|^{1/2}]\Big(1 + \log\frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big)[\|\check{\phi}'\|_2 + \|\phi'_{\mathfrak{D}}\|_2]\bigg).\end{aligned}$$

Hence, with the aid of (5.10.34), we obtain that there exist $\Upsilon > 0$ and $\beta_0 > 0$ (so that $\Upsilon + \beta_0^{-1}$ is small enough) such that for either $\Upsilon^{-1}\beta^{-1} \leq \mu \leq \Upsilon\beta^{-1/2}$ or $-\mu_0 \leq \mu \leq -\Upsilon^{-1}\beta^{-1}$ we have

$$\|\phi'\|_2 \leq C\,|\mu|^{-1/4}\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2 \leq \widehat{C}\gamma(\alpha,\beta)\beta^{-1/2}\|f\|_2. \tag{5.10.42}$$

Combined with Poincaré's inequality (5.10.42) yields (5.10.2).

In the next step we use a shifting argument and hence it is necessary to obtain first an estimate for $\|\phi'' - \alpha^2\phi\|_2$. By (5.10.35) and the first inequality of (5.10.42), we obtain that

$$|\phi(x_\nu)| \leq |\check{\phi}(x_\nu)| + |\phi_{\mathfrak{D}}(x_\nu)| \leq C\gamma(\alpha,\beta)\beta^{-3/4}\log\beta\,\|f\|_2. \tag{5.10.43}$$

Substituting the first inequality of (5.10.42) and (5.10.43) into (5.10.15) yields

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\,\gamma(\alpha,\beta)\big(\beta^{-3/8}\log\beta + |\mu|^{-1/4}\beta^{-1/2}\big)\,\|f\|_2. \tag{5.10.44}$$

Consequently, by (5.4.6), (5.10.13), and (4.6.5) we obtain, from (5.10.42), (5.10.43) and (5.10.44),

$$\|\phi'' - \alpha^2\phi\|_2 \leq C\gamma(\alpha,\beta)\big(\beta^{-1/4}\log\beta + |\mu|^{-1/4}\beta^{-3/8}\big)\,\|f\|_2. \tag{5.10.45}$$

Hence, we have proven, under the additional condition that $\alpha \leq \beta^{1/8}$, that there exist $\Upsilon > 0$ and $\beta_0 > 0$ such that for all $\beta > \beta_0$ and either $\Upsilon^{-1}\beta^{-1} \leq \mu \leq \Upsilon\beta^{-1/2}$ or $-\mu_0 \leq \mu \leq -\Upsilon^{-1}\beta^{-1}$,

$$\|\phi'' - \alpha^2\phi\|_2 \leq C[\beta^{-1/4}\log\beta + |\mu|^{-1/4}\beta^{-3/8}]\|f\|_2. \tag{5.10.46}$$

For $\alpha \geq \beta^{1/8}$ (5.10.45) is deficient (in this case $\gamma(\alpha,\beta) = \Upsilon^{-5/2}\beta^{1/8}$), hence we use (4.5.4) (with $v = \phi'' - \alpha^2\phi$ and f replaced by $f + i\beta U''\phi$) instead of (5.10.13), to obtain that

$$|\phi''(1)| \leq C\beta^{7/16}(\|\phi\|_2 + \beta^{-1}\|f\|_2).$$

Then, with the aid of (5.10.44) and (4.6.5) we establish (5.10.46) for $\alpha \geq \beta^{1/8}$ as well.

Step 6c. We prove (5.10.2) for $|\mu| < \Upsilon^{-1}\beta^{-1}$, where $\Upsilon > 0$ has been determined in the previous step.

Here, we use a shifting argument. We begin by writing

$$\mathcal{B}_{\lambda+2\Upsilon^{-1}\beta^{-1},\alpha}\phi = f + 2\Upsilon^{-1}(\phi'' - \alpha^2\phi), \tag{5.10.47}$$

and observe that $\hat{\lambda} := \lambda + 2\Upsilon^{-1}\beta^{-1}$ satisfies the assumptions of Step 6b. We then have by (5.10.46), with λ replaced by $\hat{\lambda}$,

$$\|\phi'' - \alpha^2\phi\|_2 \leq C(\beta^{-1/8} + \hat{\mu}^{-1/4}\beta^{-3/8})[\|\phi'' - \alpha^2\phi\|_2 + \|f\|_2].$$

Consequently,

$$\|\phi'' - \alpha^2\phi\|_2 \leq C\beta^{-1/8}\|f\|_2.$$

We now apply (5.10.42) to (5.10.47) to obtain, with the aid of the above inequality,

$$\|\phi'\|_2 \leq C\gamma(\alpha,\beta)\beta^{-1/2}(\|f\|_2 + \|\phi'' - \alpha^2\phi\|_2) \leq \hat{C}\gamma(\alpha,\beta)\beta^{-1/2}\|f\|_2.$$

Combining the above with Poincaré's inequality yields (5.10.2).

Step 7: The case $\mu \leq -\mu_0$. Here, we use (4.4.3), applied to the pair $(\phi'' - \alpha^2\phi, f + i\beta U''\phi)$, and (1.1.7b) to obtain that

$$|\phi''(1)| \leq C(\beta^{1/2}\|\phi\|_2 + \beta^{-1/2}\|f\|_2). \tag{5.10.48}$$

We then use (2.11.1) for the pair $(\check{\phi}, \phi''(1)(U + i\lambda)\hat{\psi})$, together with (5.2.2) for $s = 1/2$ to conclude

$$\|\check{\phi}'\|_2 \leq C\,|\lambda\beta|^{-3/4}|\phi''(1)|. \tag{5.10.49}$$

From (5.10.48) and (5.10.49), we obtain, with the aid of Poincaré's inequality, for sufficiently large β_0

$$\|\check{\phi}'\|_2 \leq C\left(\beta^{-1/4}\|\phi'_{\mathfrak{D}}\|_2 + \beta^{-5/4}\|f\|_2\right). \tag{5.10.50}$$

To estimate $\|\phi'_{\mathfrak{D}}\|_2$ we apply (2.11.1) to the pair $(\phi_{\mathfrak{D}}, v_{\mathfrak{D}})$ to obtain

$$\|\phi'_{\mathfrak{D}}\|_2 \leq C\, \|v_{\mathfrak{D}}\|_2. \tag{5.10.51}$$

We use (5.4.7), (5.10.48), (5.4.19), and (3.1.85), applied to the pair $(\tilde{v}_{\mathfrak{D}}, i\beta U''\phi - f + \phi''(1)\hat{g})$, to obtain, that

$$\|\tilde{v}'_{\mathfrak{D}}\|_2 \leq C\, |\mu|^{-1/2}\big(\beta^{1/2}\|\phi'\|_2 + \beta^{-1/2}\|f\|_2\big). \tag{5.10.52}$$

By (3.1.84) applied to the pair $(v_{\mathfrak{D}}, g_{\mathfrak{D}})$ and (5.4.4) it holds that

$$\|v_{\mathfrak{D}}\|_2 \leq \frac{C}{\beta|\mu|}\big((1+|\mu|)(\|f\|_2+|\phi''(1)|\,\|\hat{g}\|_2)+\|\tilde{v}'_{\mathfrak{D}}\|_2+\|\tilde{v}_{\mathfrak{D}}\|_2+\|\phi''\|_2+\|\phi\|_{1,2}\big). \tag{5.10.53}$$

To bound $\|\phi''\|_2$ we use the identity

$$\|\phi''\|_2^2 + \alpha^2\|\phi'\|_2^2 = \langle \phi'', \phi'' - \alpha^2\phi\rangle, \tag{5.10.54}$$

to obtain, with the aid of (5.4.6),

$$\|\phi''\|_2 \leq \|\tilde{v}_{\mathfrak{D}}\|_2 + |\phi''(1)|\,\|\widehat{\psi}\|_2. \tag{5.10.55}$$

Consequently, we obtain from the substitution of (5.4.19), (5.10.52), (5.10.55), and (4.6.5), into (5.10.53)

$$\|v_{\mathfrak{D}}\|_2 \leq C\big(\beta^{-1/2}\|\phi'\|_2 + \beta^{-1}\|f\|_2\big). \tag{5.10.56}$$

Next, we apply (5.10.51) and (5.10.56) to obtain

$$\|\phi'_{\mathfrak{D}}\|_2 \leq C\,\|v_{\mathfrak{D}}\|_2 \leq \hat{C}\big(\beta^{-1/2}\|\phi'\|_2 + \beta^{-1}\|f\|_2\big).$$

Combining the above inequality with (5.10.50) yields

$$\|\phi'\|_2 \leq C\,\beta^{-1}\|f\|_2. \tag{5.10.57}$$

The above inequality, combined with Poincaré's inequality, completes the proof of (5.11.2). ■

5.11 Resolvent estimates for $\beta^{-1/2} \ll U(0) - \nu < U(0) - U(1/2)$

We now consider the case where $\beta^{-1/4} \ll x_\nu < 1/2$. More precisely, given some positive Υ and $\nu_1 < U(1/2)$, we consider for suitable $\nu_2 > 0$ and β_0 the zone

$$\mathcal{C}_1(\nu_1,\nu_2,\Upsilon,\alpha_1,\beta_0) = \left\{ \begin{array}{l} (\lambda,\alpha,\beta) \in \mathbb{C}\times\mathbb{R}_+^2, \beta \geq \beta_0, \mu < \Upsilon\beta^{-1/2} \\ \nu_1 \leq \nu \leq U(0) - \nu_2\beta^{-1/2}, 0 \leq \alpha \leq \alpha_1\beta^{1/3} \end{array} \right\}, \tag{5.11.1}$$

for some sufficiently small $\alpha_1 > 0$.

Proposition 5.11.1. *Let $U \in C^4([0,1])$ satisfy (2.1.3). Let further $\nu_1 < U(1/2)$ denote a positive constant. Then, there exist $\Upsilon > 0$, $\alpha_1 > 0$, $\beta_0 > 0$, $\nu_2 > 0$, and $C > 0$, such that for $(\lambda, \alpha, \beta) \in \mathcal{C}_1(\nu_1, \nu_2, \Upsilon, \alpha_1, \beta_0)$ it holds that*

$$\|(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})^{-1}\| + \Big\|\frac{d}{dx}(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})^{-1}\Big\| \leq C\,\beta^{-1/2}\log\beta. \tag{5.11.2}$$

Proof. We refer to the notation introduced in (5.4.2)–(5.4.7) for $v_{\mathfrak{D}}$, $\tilde{v}_{\mathfrak{D}}$ and $g_{\mathfrak{D}}$.

Step 1. We estimate $\tilde{v}_{\mathfrak{D}}$ and $\tilde{v}_{\mathfrak{D}}'$ in L^2 for $\mu > -\mu_0$, for some, sufficiently small, $\mu_0 > 0$.

For the convenience of the reader we repeat here once again (5.4.8)

$$\begin{aligned}&\Re\langle (U'')^{-1}\tilde{v}_{\mathfrak{D}}, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)\tilde{v}_{\mathfrak{D}} + i\beta U''\phi\rangle \\ &\quad = \|(U'')^{-1/2}\tilde{v}_{\mathfrak{D}}'\|_2^2 + \Re\langle ((U'')^{-1})'\tilde{v}_{\mathfrak{D}}, \tilde{v}_{\mathfrak{D}}'\rangle - \beta\mu\,\|\tilde{v}_{\mathfrak{D}}\|_2^2 + \beta\Re\langle \phi''(1)\widehat{\psi}, i\phi\rangle.\end{aligned}$$

As in (5.10.12) we obtain that

$$\beta\,\Re\langle \phi''(1)\widehat{\psi}, i\phi\rangle \geq -C\beta^{-3/4}|\phi''(1)|(\|\tilde{v}_{\mathfrak{D}}'\|_2 + \beta^{2/3}\|\phi'\|_2). \tag{5.11.3}$$

Let x_ν be defined by (2.4.5). Since by assumption we have $x_\nu \geq C\nu_2^{1/2}\beta^{-1/4}$ it holds by (5.9.1) that, for sufficiently large ν_2

$$\begin{aligned}|\phi''(1)| \leq C\big(&\beta^{-1/3}x_\nu^{-5/6}\|f\|_2 \\ &+ \beta^{1/2}x_\nu^{-1}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)| + \beta^{1/2}x_\nu^{-1/2}\|\phi'\|_2\big).\end{aligned} \tag{5.11.4}$$

For $\alpha > \beta^{1/6}$ we use (4.5.4) for the pair $(\phi'' - \alpha^2\phi, f + i\beta U''\phi)$ to obtain, with the aid of Poincaré's inequality, that

$$|\phi''(1)| \leq C\alpha^{-1/2}(\beta^{1/2}\|\phi'\|_2 + \beta^{-1/2}\|f\|_2) \tag{5.11.5}$$

By substituting (5.11.4) into (5.11.3), we get

$$\begin{aligned}\beta\,\Re\langle \phi''(1)\widehat{\psi}, i\phi\rangle \geq -C\,\big(&\beta^{-1/4}[x_\nu^{-1/2}\|\phi\|_{1,2} + x_\nu^{-1}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)|] \\ &+ \beta^{-13/12}x_\nu^{-5/6}\|f\|_2\big)\big(\|\tilde{v}_{\mathfrak{D}}'\|_2 + \beta^{2/3}\|\phi'\|_2\big).\end{aligned} \tag{5.11.6}$$

Next, by (5.4.7), (5.4.19), (5.11.4), (3.1.3a), and (3.3.10), we obtain, for the parameter range set in (5.11.1), that (compare with (5.10.15))

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\,\big(x_\nu^{-1}\|\phi'\|_2 + \beta^{1/6}x_\nu^{-5/6}|\phi(x_\nu)| + [\beta x_\nu]^{-2/3}\|f\|_2\big). \tag{5.11.7}$$

Substituting (5.11.7) together with (5.11.6), (5.4.19), and (5.11.4) into (5.4.20) yields, with the aid of (5.4.7)

$$
\begin{aligned}
&\frac{1}{C}\|\tilde{v}'_{\mathfrak{D}}\|_2^2 \\
&\quad \leq \big(x_\nu^{-1}\|\phi'\|_2 + \beta^{1/6}x_\nu^{-5/6}|\phi(x_\nu)| + [\beta x_\nu]^{-2/3}\|f\|_2\big) \\
&\qquad \times \big(\|f\|_2 + \beta^{1/4}x_\nu^{-1/2}\|\phi'\|_2 + \beta^{1/4}x_\nu^{-1}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)|\big) \\
&\qquad + \max(1,\mu\beta)\big(x_\nu^{-1}\|\phi'\|_2 + \beta^{1/6}x_\nu^{-5/6}|\phi(x_\nu)| + [\beta x_\nu]^{-2/3}\|f\|_2\big)^2 \\
&\qquad + \big(\beta^{-1/4}[x_\nu^{-1/2}\|\phi\|_{1,2} + x_\nu^{-1}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)|] + \beta^{-13/12}x_\nu^{-5/6}\|f\|_2\big) \\
&\qquad \times \big(\|\tilde{v}'_{\mathfrak{D}}\|_2 + \beta^{2/3}\|\phi'\|_2\big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|\tilde{v}'_{\mathfrak{D}}\|_2 \leq C\,\big[&(\beta^{1/8} + \mu_+^{1/2}\beta^{1/2})(x_\nu^{-1}\|\phi'\|_2 + \beta^{1/6}x_\nu^{-5/6}|\phi(x_\nu)|) \\
&+ \beta^{5/24}x_\nu^{-1/4}\|\phi'\|_2 + \beta^{-1/8}\|f\|_2\big].
\end{aligned} \tag{5.11.8}
$$

Step 2. We prove that

$$
\begin{aligned}
&\|\phi'' - \phi''(1)\widehat{\psi}\|_2 \\
&\leq C\,\big(\|\tilde{v}_{\mathfrak{D}}\|_2 + \beta^{-11/12}x_\nu^{-5/6}\|f\|_2 + \beta^{-1/12}x_\nu^{-1}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)|\big).
\end{aligned} \tag{5.11.9}
$$

Step 2.1: Prove (5.11.9) *in the case* $0 \leq \alpha \leq \beta^{1/6}$. We write, as in Step 5 of the proof of Proposition 5.10.1 and with the aid of (5.11.4) and (5.2.2) (with $s = 1/2$)

$$
\begin{aligned}
|\langle\phi''(1)\widehat{\psi},\alpha^2\phi\rangle| &\leq \alpha^2|\phi''(1)|\,\|(1-x)^{1/2}\widehat{\psi}\|_1\,\|\phi'\|_2 \\
&\leq C\alpha^2\,\big(\beta^{-13/12}x_\nu^{-5/6}\|f\|_2 + \beta^{-1/4}[x_\nu^{-1}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)| \\
&\qquad + x_\nu^{-1/2}\|\phi'\|_2]\big)\|\phi'\|_2.
\end{aligned} \tag{5.11.10}
$$

Substituting (5.11.10) into (5.10.18) yields

$$
\begin{aligned}
&\|\phi'' - \phi''(1)\widehat{\psi}\|_2^2 + \alpha^2\|\phi'\|_2^2 \\
&\leq \|-\phi'' + \phi''(1)\widehat{\psi}\|_2\,\|\tilde{v}_{\mathfrak{D}}\|_2 + C\alpha^2\big(\beta^{-13/12}x_\nu^{-5/6}\|f\|_2 \\
&\quad + \beta^{-1/4}[x_\nu^{-1}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)| + x_\nu^{-1/2}\|\phi'\|_2]\big)\|\phi'\|_2.
\end{aligned}
$$

For sufficiently large β_0 we then obtain, as $x_\nu > \beta^{-1/4}$,

$$
\begin{aligned}
&\|\phi'' - \phi''(1)\widehat{\psi}\|_2 \\
&\quad \leq C\,\big(\|\tilde{v}_{\mathfrak{D}}\|_2 + \alpha(\beta^{-13/12}x_\nu^{-5/6}\|f\|_2 + \beta^{-1/4}x_\nu^{-1}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)|)\big).
\end{aligned}
$$

For $\alpha < \beta^{1/6}$ (5.11.9) is readily verified.

Step 2.2. We prove (5.11.9) *in the case* $\beta^{1/6} \leq \alpha \leq \alpha_1 \beta^{1/3}$. In this case, we obtain, instead of (5.11.10), with the aid of (5.11.5) and again (5.2.2) (with $s = 1/2$) that

$$|\langle \phi''(1)\widehat{\psi}, \alpha^2 \phi\rangle| \leq C\alpha^{3/2} \left(\beta^{-5/4}\|f\|_2 + \beta^{-1/4}\|\phi'\|_2\right)\|\phi'\|_2 .$$

Substituting the above into (5.10.18) yields, as above

$$\|\phi'' - \phi''(1)\widehat{\psi}\|_2 \leq C(\|\tilde{v}_{\mathfrak{D}}\|_2 + \beta^{-7/6}\|f\|_2).$$

Consequently, (5.11.9) is valid also for $\beta^{1/6} \leq \alpha < \alpha_1 \beta^{1/3}$.

Step 3. We estimate $v_{\mathfrak{D}}$ *in* L^∞ *for* $\mu \geq -\mu_0$. Let $v_{\mathfrak{D}}$ be given by (5.4.2). Recall that by (5.10.20) $(\mathcal{L}_\beta - \beta\lambda)v_{\mathfrak{D}} = g_{\mathfrak{D}}$. To obtain an estimate for $\|v_{\mathfrak{D}}\|_\infty$ we begin, as in Step 6 in Proposition 5.10.1, by rewriting (5.10.20b) as the sum of five terms:

$$\begin{aligned} g_{\mathfrak{D}} = {} & [(U-\nu)(-f + \phi''(1)\hat{g})] + [i\mu(-f + \phi''(1)\hat{g})] \\ & - [2(U' - U'(x_\nu))\tilde{v}'_{\mathfrak{D}}] + [-2U'(x_\nu)\tilde{v}'_{\mathfrak{D}} - U''([\phi'' - \phi''(1)\widehat{\psi}] + \tilde{v}_{\mathfrak{D}}) \\ & - 2U^{(3)}\phi' - U^{(4)}\phi] - [U''\phi''(1)\widehat{\psi}]. \qquad (5.11.11) \end{aligned}$$

We separately estimate the contribution of each term on the right-hand side of equation (5.11.11), using the interpolation inequality (5.10.21). For the first term on the right-hand side of (5.11.11), we apply (3.3.35) with f replaced by $-f + \phi''(1)\hat{g}$. For the second term, we use (3.1.3a), and (3.1.3b) with $p = 2$ (both valid for sufficiently large ν_2) for $-\beta^{-1/2} \leq \mu < \Upsilon\beta^{-1/2}$ and (3.1.84)–(3.1.85) for the case $\mu < -\beta^{-1/2}$. For the third term we use (3.3.10) and (3.3.11). For the fourth term we use (3.1.3) to obtain

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U'(x_\nu)\tilde{v}'_{\mathfrak{D}})\|_\infty \leq C\, x_\nu[\beta x_\nu]^{-1/2}\|\tilde{v}'_{\mathfrak{D}}\|_2 .$$

Finally, for the fifth term we use (4.6.3) as in the proof of (5.10.22d). Combining the above yields for $x_\nu \geq C\nu_2^{1/2}\beta^{-1/4}$

$$\begin{aligned} \|v_{\mathfrak{D}}\|_\infty \leq C\, \big[& \beta^{-3/4}(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2) \\ & + [\beta x_\nu]^{-1/2}(\|\phi\|_{1,2} + \|\phi'' - \phi''(1)\widehat{\psi}\|_2 + \|\tilde{v}_{\mathfrak{D}}\|_2) \\ & + \beta^{-1/2}x_\nu^{1/2}\|\tilde{v}'_{\mathfrak{D}}\|_2 + \beta^{-1}|\phi''(1)|\big]. \end{aligned}$$

Using (5.11.9) we then obtain

$$\begin{aligned} \|v_{\mathfrak{D}}\|_\infty \leq C\, \big[& \beta^{-3/4}(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2) \\ & + [\beta x_\nu]^{-1/2}(\|\phi\|_{1,2} + \|\tilde{v}_{\mathfrak{D}}\|_2 + \beta^{-1/12}x_\nu^{-1}\log(\beta^{1/4}x_\nu)\,|\phi(x_\nu)|) \\ & + \beta^{-1/2}x_\nu^{1/2}\|\tilde{v}'_{\mathfrak{D}}\|_2 + \beta^{-1}|\phi''(1)|\big]. \qquad (5.11.12) \end{aligned}$$

For the estimate of $\|\tilde{v}_{\mathfrak{D}}\|_2$ we use a combination of (5.9.4) and (5.11.7).

Let $\check{x}_\mu := \check{x}_\mu(\Upsilon, \beta)$ be defined by (5.9.2). Then, there exists $\Upsilon > 0$ such that

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C \begin{cases} \big[\Upsilon^{-5/2}\beta^{-1/4}\|f\|_2 + (\Upsilon^{-3/8}\mu_{+,\beta}^{3/4} + \Upsilon^{3/2}\beta^{3/16})|\phi(x_\nu)| \\ \qquad + (\mu_{+,\beta}^{1/2} + \Upsilon^2\beta^{1/8})\|\phi'\|_2\big] & x_\nu < \check{x}_\mu, \\ \big(x_\nu^{-1}\|\phi'\|_2 + \beta^{1/6}x_\nu^{-5/6}|\phi(x_\nu)| + [\beta x_\nu]^{-2/3}\|f\|_2\big) & \text{otherwise.} \end{cases} \tag{5.11.13}$$

Thus, we set

$$\gamma(x_\nu, \check{x}_\mu) = \begin{cases} 1 & x_\nu \geq \check{x}_\mu, \\ 0 & x_\nu < \check{x}_\mu. \end{cases}$$

Substituting (5.11.4), (5.11.8), (5.11.13), (4.6.5), and (5.4.18) into (5.11.12) yields

$$\begin{aligned} \|v_{\mathfrak{D}}\|_\infty \leq C \,\big(&\beta^{-5/8}(x_\nu^{1/2} + \Upsilon^{-5/2}\beta^{-1/8}x_\nu^{-1/2})\|f\|_2 \\ &+ [\gamma(x_\nu, \check{x}_\mu)\beta^{-1/3}x_\nu^{-4/3} + \beta^{-5/24}x_\nu^{-1/3} + \mu_+^{1/2}\beta^{1/6}x_\nu^{-1/3}]|\phi(x_\nu)| \\ &+ [\gamma(x_\nu, \check{x}_\mu)\beta^{-1/2}x_\nu^{-3/2} + \beta^{-3/8}x_\nu^{-1/2} + \beta^{-7/24}x_\nu^{1/4} + \mu_+^{1/2}x_\nu^{-1/2}]\|\phi\|_{1,2}\big). \end{aligned} \tag{5.11.14}$$

We now obtain an estimate of $|\phi(x_\nu)|$ (see (5.11.18) below).

Step 4. With, as in (5.10.25)

$$\phi = \phi_{\mathfrak{D}} + \check{\phi}, \tag{5.11.15}$$

$$\phi_{\mathfrak{D}} = \mathcal{A}_{\lambda,\alpha}^{-1} v_{\mathfrak{D}}; \quad \check{\phi} = -\mathcal{A}_{\lambda,\alpha}^{-1}([U + i\lambda]\phi''(1)\widehat{\psi}) \tag{5.11.16}$$

and

$$\begin{aligned} &\mathcal{C}_2(\nu_1, \nu_2, \Upsilon, \alpha_1, \beta_0, \kappa_0) \\ &:= \Big\{(\lambda, \alpha, \beta) \in \mathcal{C}_1 \,\Big|\, \beta^{-2} \leq |\mu| \leq \Upsilon\beta^{-1/2} \text{ or } -\frac{|U(0) - \nu|}{\kappa_0} < \mu < -\Upsilon\beta^{-1/2}\Big\}, \end{aligned} \tag{5.11.17}$$

we prove that there exist $C > 0$, Υ_0, and $\hat{\kappa}_0$ such that, for $0 < \Upsilon \leq \Upsilon_0$ and $\kappa_0 \geq \hat{\kappa}_0$, there exist $\hat{\nu}_2 = \hat{\nu}_2(\Upsilon, \kappa_0)$ and $\beta_0 = \beta_0(\Upsilon, \kappa_0)$ so that for $\nu_2 \geq \hat{\nu}_2$ and $(\lambda, \alpha, \beta) \in \mathcal{C}_2(\nu_1, \nu_2, \Upsilon, \alpha_1, \beta_0, \kappa_0)$ we have

$$\begin{aligned} &|\check{\phi}(x_\nu)| + |\phi_{\mathfrak{D}}(x_\nu)| \\ &\quad \leq C \,\log(|\mu|^{-1/2}x_\nu)\,\big([\beta^{-1/4} + \gamma(x_\nu, \check{x}_\mu)\beta^{-1/2}x_\nu^{-3/2} + \mu_+^{1/2}x_\nu^{-1/2}]\|\phi'\|_2 \\ &\qquad\qquad + \beta^{-5/8}[x_\nu^{1/2} + \Upsilon^{-5/2}\beta^{-1/8}x_\nu^{-1/2}]\|f\|_2\big). \end{aligned} \tag{5.11.18}$$

By (2.8.47) applied to the pair $(\check{\phi}, ([U + i\lambda]\phi''(1)\widehat{\psi}))$, which holds for $(\lambda, \alpha, \beta) \in \mathcal{C}_2(\nu_1, \nu_2, \Upsilon, \alpha_1, \beta_0, \kappa_0)$ when $\nu_2 \geq \Upsilon\kappa_0$, (5.11.4), and (5.2.2) for $s = 1/2$, we obtain that

$$\begin{aligned}|\check{\phi}(x_\nu)| &\leq C\, x_\nu^{1/2} |\phi''(1)| \|(1-x)^{1/2} \widehat{\psi}\|_1 \\ &\leq \widehat{C}\, \big(\beta^{-1/4} \|\phi'\|_2 + \beta^{-1/4} x_\nu^{-1/2} \log(\beta^{1/4} x_\nu) |\phi(x_\nu)| \\ &\qquad + \beta^{-13/12} x_\nu^{-1/3} \|f\|_2\big). \end{aligned} \tag{5.11.19}$$

By (5.11.14) and (2.8.65) applied to the pair $(\phi_{\mathfrak{D}}, v_{\mathfrak{D}})$, we obtain that

$$\begin{aligned}|\phi_{\mathfrak{D}}(x_\nu)| \leq C\, \log\Big(\frac{x_\nu}{|\mu|^{1/2}}\Big)\big(&\beta^{-5/8}(x_\nu^{1/2} + \Upsilon^{-5/2}\beta^{-1/8} x_\nu^{-1/2})\|f\|_2 \\ &+ [\gamma(x_\nu, \check{x}_\mu)\beta^{-1/3} x_\nu^{-4/3} + \beta^{-5/24} x_\nu^{-1/3} + \mu_+^{1/2}\beta^{1/6} x_\nu^{-1/3}]|\phi(x_\nu)| \\ &+ [\gamma(x_\nu, \check{x}_\mu)\beta^{-1/2} x_\nu^{-3/2} + \beta^{-3/8} x_\nu^{-1/2} + \beta^{-7/24} x_\nu^{1/4} \\ &\quad + \mu_+^{1/2} x_\nu^{-1/2}]\|\phi\|_{1,2}\big).\end{aligned}$$

Combining the above yields, using the fact that $\beta^{-1/4} < x_\nu < 1$ and that $|\mu| < \mu_0$

$$\begin{aligned}&|\check{\phi}(x_\nu)| + |\phi_{\mathfrak{D}}(x_\nu)| \\ &\quad \leq C\, \log(|\mu|^{-1/2} x_\nu)\big([\beta^{-1/4} + \gamma(x_\nu, \check{x}_\mu)\beta^{-1/2} x_\nu^{-3/2} + \mu_+^{1/2} x_\nu^{-1/2}]\|\phi'\|_2 \\ &\qquad\qquad + \big[\beta^{-1/4} x_\nu^{-1/2} \log(\beta^{1/4} x_\nu) + \gamma(x_\nu, \check{x}_\mu)\beta^{-1/3} x_\nu^{-4/3} \\ &\qquad\qquad + \beta^{-5/24} x_\nu^{-1/3} + \mu_+^{1/2}\beta^{1/6} x_\nu^{-1/3}\big]|\phi(x_\nu)| \\ &\qquad\qquad + \beta^{-5/8}[x_\nu^{1/2} + \Upsilon^{-5/2}\beta^{-1/8} x_\nu^{-1/2}]\|f\|_2\big).\end{aligned} \tag{5.11.20}$$

We proceed by showing that the coefficient of $|\phi(x_\nu)|$ on the right-hand side of (5.11.20) can be made arbitrarily small by choosing ν_2 sufficiently large. We separately bound each term in the brackets.

For the second term we first observe that for $x_\nu > \Upsilon\mu_+^{-1/2}\beta^{-1/2}$, we have

$$\begin{aligned}\log(|\mu|^{-1/2} x_\nu)\, \beta^{-1/3} x_\nu^{-4/3} &\leq \log(\Upsilon^{-1} x_\nu^2 \beta^{1/2})[\beta^{1/2} x_\nu^2]^{-2/3} \\ &\leq [\log(x_\nu^2\beta^{1/2}) + \log(\Upsilon^{-1})][\beta^{1/2} x_\nu^2]^{-2/3} \leq C[\nu_2^{-1/3} + \log\Upsilon^{-1}\nu_2^{-2/3}].\end{aligned}$$

Furthermore, for $\beta_0(\kappa_0)$ large enough and $(\lambda, \alpha, \beta) \in \mathcal{C}_2(\nu_1, \nu_2, \Upsilon, \alpha_1, \beta_0, \kappa_0)$, it holds that

$$|\mu|^{-1/2} x_\nu \leq \beta. \tag{5.11.21}$$

Hence, for $x_\nu > \beta^{-1/8}$, we have

$$\log(|\mu|^{-1/2} x_\nu)\beta^{-1/3} x_\nu^{-4/3} \leq \beta^{-1/6} \log(|\mu|^{-1/2} x_\nu) \leq C\beta^{-1/6} \log\beta.$$

Combining the pair of estimates, we obtain

$$\log(|\mu|^{-1/2} x_\nu)\gamma(x_\nu, \check{x}_\mu)\beta^{-1/3} x_\nu^{-4/3} \leq \widetilde{C}\big(\nu_2^{-1/3} + \log\Upsilon^{-1}\nu_2^{-2/3} + \beta^{-1/6}\log\beta\big). \tag{5.11.22}$$

For the third term, we write, assuming that $\nu_2\Upsilon \geq 1$ (which implies $\mu_+^{-1/2}x_\nu \geq \frac{1}{C}\nu_2^{1/2}\Upsilon^{1/2}$ for $(\lambda,\alpha,\beta) \in \mathcal{C}_2(\nu_1,\nu_2,\Upsilon,\alpha_1,\beta_0,\kappa_0)$),

$$\begin{aligned}\log(|\mu|^{-1/2}x_\nu)\mu_+^{1/2}\beta^{1/6}x_\nu^{-1/3} &= \log(|\mu|^{-1/2}x_\nu)[\mu_+^{-1/2}x_\nu]^{-1/3}\mu_+^{1/3}\beta^{1/6}\\ &\leq C\mu_+^{1/3}\beta^{1/6} \leq \hat{C}\Upsilon^{1/3}.\end{aligned}$$

For the first term we obtain with the aid of (5.11.21)

$$\log(|\mu|^{-1/2}x_\nu)\beta^{-1/4}x_\nu^{-1/2}\log(\beta^{1/4}x_\nu) \leq C\beta^{-1/8}\log^2\beta.$$

We can now conclude (5.11.18).

Step 5. We prove that there exist $C>0$, Υ_0, and $\hat{\kappa}_0$ such that, for $0<\Upsilon\leq\Upsilon_0$ and $\kappa_0 \geq \hat{\kappa}_0$, we have for some $\widehat{\nu}_2 = \widehat{\nu}_2(\Upsilon,\kappa_0)$ and $\beta_0 = \beta_0(\Upsilon,\kappa_0)$ that, for $\nu_2 \geq \widehat{\nu}_2$ and $(\lambda,\alpha,\beta) \in \mathcal{C}_2(\nu_1,\nu_2,\Upsilon,\alpha_1,\beta_0,\kappa_0)$, (5.11.2) holds true

We first use (2.8.1b) (which holds $(\lambda,\alpha,\beta) \in \mathcal{C}_2$) applied to the pair $(\phi_{\mathfrak{D}}, v_{\mathfrak{D}})$ and (5.11.14) to obtain that

$$\begin{aligned}\|\phi_{\mathfrak{D}}'\|_2 \leq C\ \log(|\mu|^{-1/2}x_\nu)\big(&\beta^{-5/8}(1+\Upsilon^{-5/2}\beta^{-1/8}x_\nu^{-1})\|f\|_2\\ &+[\gamma(x_\nu,\check{x}_\mu)\beta^{-1/3}x_\nu^{-11/6}+\beta^{-5/24}x_\nu^{-5/6}+\mu_+^{1/2}\beta^{1/6}x_\nu^{-5/6}]|\phi(x_\nu)|\\ &+[\gamma(x_\nu,\check{x}_\mu)\beta^{-1/2}x_\nu^{-2}+\beta^{-3/8}x_\nu^{-1}+\beta^{-7/24}x_\nu^{-1/4}+\mu_+^{1/2}x_\nu^{-1}]\|\phi\|_{1,2}\big).\end{aligned} \tag{5.11.23}$$

Using (5.11.22) (which holds for $(\lambda,\alpha,\beta) \in \mathcal{C}_2$), and the fact that $\beta^{-1/4} < x_\nu$, we write

$$\begin{aligned}&\log(|\mu|^{-1/2}x_\nu)\gamma(x_\nu,\check{x}_\mu)\beta^{-1/2}x_\nu^{-2}\\ &\quad\leq \widetilde{C}\beta^{-1/6}x_\nu^{-2/3}\big(\nu_2^{-1/3}+\log\Upsilon^{-1}\nu_2^{-2/3}+\beta^{-1/6}\log\beta\big)\\ &\quad\leq \widetilde{C}\big(\nu_2^{-2/3}+\log\Upsilon^{-1}\nu_2^{-1}+\beta^{-1/6}\log\beta\big).\end{aligned} \tag{5.11.24}$$

Substituting (5.11.18) and (5.11.24) into (5.11.23), we obtain the existence of $C>0$ and $\beta_0>0$ such that for all $\beta>\beta_0$ we have

$$\begin{aligned}&\|\phi_{\mathfrak{D}}'\|_2 \leq C\big(\log(|\mu|^{-1/2}x_\nu)\beta^{-1/2}\|f\|_2\\ &+\big[\beta^{-1/6}x_\nu^{-2/3}+\beta^{-1/8}\log\beta+\nu_2^{-1/3}+\log\Upsilon^{-1}\nu_2^{-1}+\Upsilon^{1/4}\big]\,[\|\check{\phi}'\|_2+\|\phi_{\mathfrak{D}}'\|_2]\big).\end{aligned} \tag{5.11.25}$$

Note that in the proof we repeatedly use the inequalities

$$(|\mu|^{-1/2}x_\nu)^{-1/2}\log(|\mu|^{-1/2}x_\nu) \leq 1 \tag{5.11.26}$$

(which holds since $|\mu|^{-1/2}x_\nu > 1$ for sufficiently large ν_2), $\frac{1}{\hat{C}}\nu_2^{1/2}\beta^{-1/4} < x_\nu < 1$ (which holds for some $\widehat{C} > 0$), $\mu_+ \leq \Upsilon\beta^{-1/2}$, and (5.11.21).

We now apply (2.8.46) (which holds for $(\lambda, \alpha, \beta) \in \mathcal{C}_2$ with sufficiently large ν_2) to the pair $(\check{\phi}, ([U + i\lambda]\phi''(1)\widehat{\psi}))$, (5.2.2) for $s = 1/2$, and (5.11.4) as in (5.11.19) to obtain that

$$\|\check{\phi}'\|_2 \leq C\left(\beta^{-1/4}x_\nu^{-1/2}\|\phi'\|_2 + \beta^{-1/4}x_\nu^{-1}|\phi(x_\nu)| + \beta^{-13/12}x_\nu^{-5/6}\|f\|_2\right). \tag{5.11.27}$$

By (5.11.21), (5.11.18), (5.11.24), and (5.11.26) we have

$$\begin{aligned}|\phi(x_\nu)| \leq\;& \\ C\Big(\{\beta^{-1/4}\log\beta &+ x_\nu^{1/2}[\beta^{-1/6}\log\beta + \nu_2^{-2/3} + \nu_2^{-1}\log\Upsilon^{-1}] + \Upsilon^{1/4}\beta^{-1/8}\}\|\phi'\|_2 \\ &+ \log(|\mu|^{-1/2}x_\nu)\beta^{-5/8}\Upsilon^{-5/2}\|f\|_2\Big).\end{aligned}$$

Hence, for $s \geq 1/2$, we have

$$\begin{aligned}(\beta^{-1/4}x_\nu^{-1})^s|\phi(x_\nu)| \leq C\Big[&(\nu_2^{-2/3} + \Upsilon^{1/4})\beta^{-1/8}\|\phi'\|_2 \\ &+ \beta^{-5/8}\,\log\beta\,\Upsilon^{-5/2}\nu_2^{-1/2}\|f\|_2\Big].\end{aligned} \tag{5.11.28}$$

By (5.11.27) and (5.11.28) for $s = 1$, we obtain

$$\|\check{\phi}'\|_2 \leq C(\beta^{-1/2}\log\beta\|f\|_2 + \beta^{-1/8}\|\phi'\|_2). \tag{5.11.29}$$

Combining (5.11.29) with (5.11.25) yields if $(\lambda, \alpha, \beta) \in \mathcal{C}_2(\nu_1, \nu_2, \alpha_1, \beta_0, \Upsilon, \kappa_0,)$ that there exist positive β_0 and C such that for all $\beta > \beta_0$ (the reader is referred for a precise description of the parameter space to the statement of step 4)

$$\|\phi'\|_2 \leq \|\check{\phi}'\|_2 + \|\phi'_{\mathfrak{D}}\|_2 \leq C\beta^{-1/2}\log\beta\|f\|_2. \tag{5.11.30}$$

Together with Poincaré's inequality, (5.11.30) implies (5.11.2) under the assumptions of Step 4.

To prove (5.11.2) in the next step for $|\mu| < \beta^{-2}$, we need to obtain an estimate of $\|\phi'' - \alpha^2\phi\|_2$ under the present condition on μ. By (5.11.28) for $s = 5/6$ and (5.11.30) we obtain under (5.11.17) that for some $C > 0$ (here and in the sequel the explicit dependence of $C = C(\kappa_0, \Upsilon)$ on (κ_0, Υ) is of little concern)

$$x_\nu^{-5/6}|\phi(x_\nu)| \leq C\beta^{-5/12}\,\log\beta\,\|f\|_2. \tag{5.11.31}$$

Substituting (5.11.30) and (5.11.31) into (5.11.7) yields that

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\beta^{-1/4}\log\beta\|f\|_2.$$

Consequently, by (5.11.4), (5.4.6), (5.11.30), (5.11.31), and (4.6.5), it holds under the assumptions and conclusions of Steps 4 and 5 that

$$\begin{aligned}\|\phi''-\alpha^2\phi\|_2 &\leq \|\tilde{v}_{\mathfrak{D}}\|_2 + |\phi''(1)|\,\|\widehat{\psi}\|_2 \\ &\leq C\beta^{-1/8}[x_\nu\beta^{1/4}]^{-1/6}\log(x_\nu\beta^{1/4})\log\beta\|f\|_2 \\ &\leq C\beta^{-1/8}\log\beta\|f\|_2. \end{aligned} \tag{5.11.32}$$

Step 6: We prove (5.11.2) *for* $|\mu|\leq\beta^{-2}$. We begin by writing as in (5.10.47)

$$\mathcal{B}_{\lambda+2\beta^{-2},\alpha}\phi = f + 2\beta^{-1}(\phi''-\alpha^2\phi), \tag{5.11.33}$$

Hence, by (5.11.32) applied to the pair $(\phi, f+2\beta^{-1}(\phi''-\alpha^2\phi))$ with λ replaced by $\lambda+2\beta^{-2}$, we have

$$\|\phi''-\alpha^2\phi\|_2 \leq C(\beta^{-9/8}\log\beta\|\phi''-\alpha^2\phi\|_2 + \beta^{-1/8}\log\beta\|f\|_2).$$

Consequently, we obtain that

$$\|\phi''-\alpha^2\phi\|_2 \leq C\beta^{-1/8}\log\beta\|f\|_2.$$

We now apply (5.11.2) to (5.11.33) to obtain

$$\|\phi'\|_2 \leq C\beta^{-1/2}\log\beta(\|f\|_2 + \beta^{-1}\|\phi''-\alpha^2\phi\|_2).$$

Hence,

$$\|\phi'\|_2 \leq C\beta^{-1/2}\log\beta\|f\|_2.$$

Step 7: We prove (5.11.2) *in the case* $-\mu_0\leq\mu<-\frac{U(0)-\nu}{\kappa_0}$. We note for below that, under the above assumption,

$$|\mu| > \frac{1}{C(\kappa_0)}\nu_2^{1/2}\beta^{-1/2}. \tag{5.11.34}$$

In this case, we can apply (4.4.13), by (5.4.13), to the pair $(\phi''-\alpha^2\phi, i\beta U''\phi+f)$, so that the L^∞ estimate is applied to $i\beta U''\phi$ and the L^2 estimate to f. We obtain

$$|\phi''(1)| \leq C(\beta^{-1/2}|\mu|^{-3/4}\|f\|_2 + \beta^{1/2}|\mu|^{-1/2}\|\phi\|_\infty).$$

Hence, by (2.9.13) applied to the pair $(\check{\phi}, \phi''(1)(U+i\lambda)\widehat{\psi})$ (see (5.10.27)) together with (5.2.2) for $s=1/2$ and (5.11.34), we obtain

$$\begin{aligned}\|\check{\phi}'\|_2 &\leq C(|\mu|^{-3/4}\beta^{-5/4}\|f\|_2 + \beta^{-1/4}|\mu|^{-1/2}\|\phi\|_\infty) \\ &\leq \widehat{C}(\beta^{-7/8}\|f\|_2 + \nu_2^{-1/4}\|\phi'\|_2).\end{aligned}$$

From the above we conclude, using Poincaré's inequality, that for ν_2 large enough

$$\|\check{\phi}'\|_2 + |\check{\phi}(x_\nu)| \leq C(\beta^{-7/8}\|f\|_2 + \nu_2^{-1/4}\|\phi'_{\mathfrak{D}}\|_2). \tag{5.11.35}$$

We next use (2.9.29) for the pair $(\phi_{\mathfrak{D}}, v_{\mathfrak{D}})$ (see (5.4.31b)) to obtain from (5.11.14)

$$|\phi_{\mathfrak{D}}(x_\nu)| \leq C\|v_{\mathfrak{D}}\|_\infty \leq \widehat{C}\big(\beta^{-5/8}\|f\|_2 + \beta^{-1/4}\|\phi'\|_2 + \beta^{-1/8}|\phi(x_\nu)|\big)$$

which implies for $\beta \geq \beta_0$ with β_0 large enough

$$|\phi_{\mathfrak{D}}(x_\nu)| \leq C\big(\beta^{-5/8}\|f\|_2 + \beta^{-1/4}\|\phi'\|_2 + \beta^{-1/8}|\check{\phi}(x_\nu)|\big). \tag{5.11.36}$$

In the sequel the explicit dependence on (κ_0, Υ) is of little concern to us and is therefore omitted.

Similarly, we obtain, using (2.9.1) with $p = +\infty$ for the pair $(\phi_{\mathfrak{D}}, v_{\mathfrak{D}})$ and (5.11.14)

$$\begin{aligned}\|\phi'_{\mathfrak{D}}\|_2 &\leq C|\mu|^{-1/4}\|v_{\mathfrak{D}}\|_\infty \\ &\leq \widehat{C}\nu_2^{-1/8}\beta^{1/8}\big(\beta^{-5/8}\|f\|_2 + \beta^{-1/4}\|\phi'\|_2 + \beta^{-1/8}|\phi(x_\nu)|\big).\end{aligned}$$

Using Sobolev embedding for $\phi_{\mathfrak{D}}$, we obtain for sufficiently large ν_2

$$\|\phi'_{\mathfrak{D}}\|_2 \leq C\nu_2^{-1/8}\big(\beta^{-1/2}\|f\|_2 + \beta^{-1/8}\|\phi'\|_2 + |\check{\phi}(x_\nu)|\big). \tag{5.11.37}$$

We now continue as in the derivation of (5.11.30) to obtain by (5.11.35), (5.11.36), and (5.11.37) for ν_2 and β_0 large enough and $\beta \geq \beta_0$

$$\|\phi'\|_2 \leq \|\check{\phi}'\|_2 + \|\phi'_{\mathfrak{D}}\|_2 \leq C\beta^{-1/2}\|f\|_2. \tag{5.11.38}$$

Step 8: The case $\mu < -\mu_0$. The proof of Step 7 in Proposition 5.10.1 furnishes (5.10.57) without any modification. Making use of Poincaré's inequality we then establish (5.11.2). ■

5.12 Large $d(\Im\lambda, [0, U(0)])$

In the following we consider the case where ν lies outside the interval $[0, U(0)]$. We consider two different regimes.

- We begin with case $\nu - U(0) \gg \beta^{-1/2}$.
- We then continue by assuming $\beta^{1/3}(-\nu) \gg 1$.

We begin by introducing for some positive constants α_0, Υ, β_0 and κ_1 the zone

$$\mathcal{F}_1(\alpha_0, \beta_0, \Upsilon, \kappa_1) := \left\{\begin{matrix}(\lambda, \alpha, \beta) \in \mathbb{C} \times \mathbb{R}_+^2,\ \beta \geq \beta_0,\ 0 \leq \alpha \leq \alpha_0\beta^{1/3}, \\ \mu < \Upsilon\beta^{-1/2}, U(0) + \kappa_1\beta^{-1/2} \leq \nu\end{matrix}\right\}.$$

Proposition 5.12.1. *Let $U \in C^4([0,1])$ satisfy (2.1.3) and α_0 and Υ denote positive constants. Then, there exist $\beta_0 > 0$, $\kappa_1 > 0$, and $C > 0$, such that, for $(\lambda, \alpha, \beta) \in \mathcal{F}_1(\alpha_0, \beta_0, \Upsilon, \kappa_1)$, it holds that*

$$\|(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\| + \Big\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\Big\| \leq C\beta^{-1/2}. \tag{5.12.1}$$

Proof. Step 1 For positive λ_0 and $\tilde{\alpha}_0$, we prove that for κ_1 and β_0 large enough, (5.12.1) holds true under the additional conditions that $|\lambda| \leq \lambda_0$ and $\alpha \leq \tilde{\alpha}_0\beta^{1/4}$.

Let $f \in L^2(0,1)$ and $\phi \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})$ satisfy

$$\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\phi = f. \tag{5.12.2}$$

Taking the scalar product of (5.12.2) with $\frac{\phi}{U+i\lambda}$, and integrating by parts yields for the imaginary part

$$\begin{aligned}-\Im\Big\langle\frac{\phi}{U+i\lambda}, f\Big\rangle &= \Im\Big\langle\Big(\frac{\phi}{U+i\lambda}\Big)'', \phi''\Big\rangle + \alpha^2\Im\Big\langle\Big(\frac{\phi}{U+i\lambda}\Big)', \phi'\Big\rangle \\ &\quad + \beta\Big(\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 + \Re\Big\langle\frac{U''\phi}{U+i\lambda}, \phi\Big\rangle\Big). \end{aligned} \tag{5.12.3}$$

By (2.1.3) we obtain the following auxiliary estimate, which we repeatedly use in the sequel

$$\Big\|\frac{(U')^n}{(U+i\lambda)^m}\Big\|_\infty \leq C\Big\|\frac{x^n}{(x^2+\check{\nu})^m}\Big\|_\infty \leq C\,\check{\nu}^{-(m-n/2)} \quad \forall m \in \mathbb{R}_+,\ \forall n \in [0,2m], \tag{5.12.4}$$

where

$$\check{\nu} = \nu - U(0) > 0.$$

As

$$\Big(\frac{\phi}{U+i\lambda}\Big)'' = -2\frac{\phi'U'}{(U+i\lambda)^2} + \frac{\phi''}{U+i\lambda} - \frac{\phi U''}{(U+i\lambda)^2} + 2\frac{\phi(U')^2}{(U+i\lambda)^3}, \tag{5.12.5}$$

we need to estimate four different terms to obtain a bound for the first term on the right-hand side of (5.12.3). We begin by writing

$$\Big|\Big\langle\frac{\phi'U'}{(U+i\lambda)^2}, \phi''\Big\rangle\Big| \leq \Big|\Big\langle\frac{\phi'U'}{(U+i\lambda)^2}, \phi'' - \phi''(1)\hat{\psi}\Big\rangle\Big| + \Big|\Big\langle\frac{\phi'U'}{(U+i\lambda)^2}, \phi''(1)\hat{\psi}\Big\rangle\Big|,$$

to obtain by (4.6.5) and (5.12.4)

$$\Big|\Big\langle\frac{\phi'U'}{(U+i\lambda)^2}, \phi''\Big\rangle\Big| \leq C\check{\nu}^{-3/2}\left(\|\phi'' - \phi''(1)\hat{\psi}\|_2 + \beta^{-1/4}|\phi''(1)|\right)\|\phi'\|_2.$$

The contribution of the second term on the right-hand is obtained as follows:

$$\Big|\Big\langle\frac{\phi''}{U+i\lambda},\phi''\Big\rangle\Big| \leq \check{\nu}^{-1}\|\phi''\|_2^2 \leq C\,\check{\nu}^{-1}\big(\|\phi''-\phi''(1)\widehat{\psi}\|_2^2+\beta^{-1/2}|\phi''(1)|^2\big).$$

To obtain an estimate for the contribution of the third term on the right-hand side of (5.12.5) we write

$$\begin{aligned}\Big|\Big\langle\frac{\phi U''}{(U+i\lambda)^2},\phi''\Big\rangle\Big| &\leq C\,\check{\nu}^{-1}\|(U+i\lambda)^{-1}\phi\|\,\|\phi''\| \\ &\leq \widehat{C}\,\check{\nu}^{-1}\|(U+i\lambda)^{-1}\phi\|[\|\phi''-\phi''(1)\widehat{\psi}\|_2+\beta^{-1/4}|\phi''(1)|].\end{aligned}$$

A similar estimate is obtained for the last term on the right-hand side of (5.12.5) by using (5.12.4) with $m=2, n=2$, i.e.,

$$\begin{aligned}&\Big|\Im\Big\langle\Big(\frac{\phi}{U+i\lambda}\Big)'',\phi''\Big\rangle\Big| \\ &\quad\leq C\,\check{\nu}^{-1}\Big(\|\phi''-\phi''(1)\widehat{\psi}\|_2^2+\beta^{-1/2}|\phi''(1)|^2 \\ &\qquad\qquad\times\Big[\Big\|\frac{\phi}{U+i\lambda}\Big\|_2+\check{\nu}^{-1/2}\|\phi'\|_2\Big][\|\phi''-\phi''(1)\widehat{\psi}\|_2+\beta^{-1/4}|\phi''(1)|]\Big),\end{aligned}$$

from which we conclude that

$$\begin{aligned}\Big|\Im\Big\langle\Big(\frac{\phi}{U+i\lambda}\Big)'',\phi''\Big\rangle\Big| \leq C\,\check{\nu}^{-1}\Big(&\|\phi''-\phi''(1)\widehat{\psi}\|_2^2+\beta^{-1/2}|\phi''(1)|^2 \\ &+\Big\|\frac{\phi}{U+i\lambda}\Big\|_2^2+\check{\nu}^{-1}\|\phi'\|_2^2\Big).\end{aligned}\tag{5.12.6}$$

For the second term on the right-hand side (5.12.3), we use (5.12.4) to obtain

$$\alpha^2\Big|\Im\Big\langle\Big(\frac{\phi}{U+i\lambda}\Big)',\phi'\Big\rangle\Big| \leq C\,\alpha^2\check{\nu}^{-1/2}\Big(\Big\|\frac{\phi}{U+i\lambda}\Big\|_2+\check{\nu}^{-1/2}\|\phi'\|_2\Big)\|\phi'\|_2. \tag{5.12.7}$$

Finally, as $U''(x)(U(x)-\nu)>0$, we can deduce that

$$\Re\Big\langle\frac{U''\phi}{U+i\lambda},\phi\Big\rangle \geq \min_{x\in[0,1]}|U''(x)|\check{\nu}\Big\|\frac{\phi}{U+i\lambda}\Big\|_2^2. \tag{5.12.8}$$

Substituting (5.12.6)–(5.12.8) into (5.12.3) yields

$$\begin{aligned}\|\phi'\|_2^2+\alpha^2\|\phi\|_2^2+\check{\nu}\Big\|\frac{\phi}{U+i\lambda}\Big\|_2^2 &\leq C\beta^{-1}\Big[\Big\|\frac{\phi}{U+i\lambda}\Big\|_2\|f\|_2 \\ &+\check{\nu}^{-1}\big(\|\phi''-\phi''(1)\widehat{\psi}\|_2^2+\beta^{-1/2}|\phi''(1)|^2+[\check{\nu}^{-1}+\alpha^2]\|\phi'\|_2^2\big) \\ &+(\alpha^2+\check{\nu}^{-1})\Big\|\frac{\phi}{U+i\lambda}\Big\|_2^2\Big].\end{aligned}$$

From the above inequality we conclude, for sufficiently large κ_1 and β_0,

$$\|\phi'\|_2 + \check{\nu}^{1/2}\Big\|\frac{\phi}{U+i\lambda}\Big\|_2 \\ \leq C\beta^{-1/2}\big[\check{\nu}^{1/2}\|f\|_2 + \check{\nu}^{-1/2}(\|\phi''-\phi''(1)\widehat{\psi}\|_2 + \beta^{-1/4}|\phi''(1)|)\big]. \quad (5.12.9)$$

Clearly,

$$(\mathcal{L}_\beta^{\mathfrak{z}_\alpha} - \beta\lambda)(-\phi''+\alpha^2\phi) = f + i\beta U''\phi,$$

where $\mathfrak{z}_\alpha$ is given by (4.7.1). For sufficiently large κ_1, (4.2.1) and (4.2.2) hold and since $\mu < \Upsilon\beta^{-1/2}$, we may use (4.2.4) here to obtain

$$|\phi''(1)| \leq C(\beta^{-1/3}\check{\nu}^{-5/12}\|f\|_2 + \beta^{1/2}\check{\nu}^{-1/2}\|\phi'\|_2). \quad (5.12.10)$$

Substituting (5.12.10) into (5.12.9) yields for sufficiently κ_1 and β_0

$$\|\phi'\|_2 + \check{\nu}^{1/2}\Big\|\frac{\phi}{U+i\lambda}\Big\|_2 \leq C\beta^{-1/2}\big[\check{\nu}^{1/2}\|f\|_2 + \check{\nu}^{-1/2}\|\phi''-\phi''(1)\widehat{\psi}\|_2\big]. \quad (5.12.11)$$

To estimate $\|\phi''-\phi''(1)\widehat{\psi}\|_2$ we set (see also (5.4.2)–(5.4.7))

$$\hat{v}_{\mathfrak{D}} := -\phi'' + \alpha^2\phi + \frac{U''\phi}{U+i\lambda} + \phi''(1)\widehat{\psi} = \frac{v_{\mathfrak{D}}}{U+i\lambda}. \quad (5.12.12)$$

A simple computation yields

$$(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)\hat{v}_{\mathfrak{D}} = h, \quad (5.12.13a)$$

where

$$h = -f - \Big(\frac{U''\phi}{U+i\lambda}\Big)'' + \phi''(1)\hat{g}. \quad (5.12.13b)$$

By [16, Theorem 1.3], which can be applied to the even extension of $\hat{v}_{\mathfrak{D}}$ and h to $(-1,1)$, it holds that

$$\|\hat{v}_{\mathfrak{D}}\|_2 \leq C\beta^{-1/2}\|h\|_2.$$

Hence, using (5.12.4) and (4.6.5) yields, as $\|\phi''\|_2 \leq \|\phi''-\phi''(1)\widehat{\psi}\|_2 + |\phi''(1)|\|\widehat{\psi}\|_2$,

$$\|\hat{v}_{\mathfrak{D}}\|_2 \leq C\beta^{-1/2}\Big(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2 + \check{\nu}^{-1}\|\phi''-\phi''(1)\widehat{\psi}\|_2 \\ + \beta^{-1/4}|\phi''(1)| + \check{\nu}^{-3/2}\|\phi'\|_2 + \check{\nu}^{-1}\Big\|\frac{\phi}{U+i\lambda}\Big\|_2\Big). \quad (5.12.14)$$

Consequently, by (5.4.19) and (5.12.10) it holds that

$$\|\hat{v}_{\mathfrak{D}}\|_2 \leq C\beta^{-1/2}\Big(\|f\|_2 + \check{\nu}^{-1}\|\phi''-\phi''(1)\widehat{\psi}\|_2 \\ + (\check{\nu}^{-3/2} + \check{\nu}^{-1/2}\beta^{1/4})\|\phi'\|_2 + \check{\nu}^{-1}\Big\|\frac{\phi}{U+i\lambda}\Big\|_2\Big). \quad (5.12.15)$$

By (5.10.18), it holds that

$$\|\phi'' - \phi''(1)\widehat{\psi}\|_2^2 + \alpha^2\|\phi'\|_2^2 \leq C(\|\tilde{v}_{\mathfrak{D}}\|_2^2 + \alpha^2|\phi''(1)|\,\|\phi'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_1).$$

Here, we recall that

$$\tilde{v}_{\mathfrak{D}} = \hat{v}_{\mathfrak{D}} - \frac{U''\phi}{U + i\lambda}. \tag{5.12.16}$$

Using (5.2.2) and (5.12.10), we may conclude that for sufficiently large κ_1, it holds that

$$\|\phi'' - \phi''(1)\widehat{\psi}\|_2^2 + \alpha^2\|\phi'\|_2^2 \leq \widehat{C}\left(\|\tilde{v}_{\mathfrak{D}}\|_2^2 + \alpha^2\beta^{-13/6}\check{\nu}^{-5/6}\|f\|_2^2\right).$$

Then, we obtain from (5.12.12) and (5.12.16) that

$$\|\phi'' - \phi''(1)\widehat{\psi}\|_2 \leq C\Big(\|\hat{v}_{\mathfrak{D}}\|_2 + \Big\|\frac{\phi}{U + i\lambda}\Big\|_2 + \beta^{-5/4}\|f\|_2\Big).$$

By (5.12.15) we then obtain for sufficiently large κ_1

$$\begin{aligned}&\|\phi'' - \phi''(1)\widehat{\psi}\|_2\\ &\quad\leq C\Big(\beta^{-1/2}\|f\|_2 + (\check{\nu}^{-3/2}\beta^{-1/2} + \check{\nu}^{-1/2}\beta^{-1/4})\|\phi'\|_2 + \Big\|\frac{\phi}{U + i\lambda}\Big\|_2\Big).\end{aligned}$$

Substituting the above into (5.12.11) yields

$$\begin{aligned}&\|\phi'\|_2 + \check{\nu}^{1/2}\Big\|\frac{\phi}{U + i\lambda}\Big\|_2\\ &\qquad\leq C\beta^{-1/2}\Big[\check{\nu}^{1/2}\|f\|_2 + (\check{\nu}^{-2}\beta^{-1/2} + \check{\nu}^{-1}\beta^{-1/4})\|\phi'\|_2\\ &\qquad\qquad + \check{\nu}^{-1/2}\Big\|\frac{\phi}{U + i\lambda}\Big\|_2\Big].\end{aligned}$$

For sufficiently large κ_1 and β_0 we then obtain (5.12.1).

Step 2. We prove that there exists $\lambda_0 > 0$ and β_0 such that (5.12.1) holds under the additional condition $|\lambda| \geq \lambda_0$.

Clearly, we must have either $\mu \leq -\lambda_0/2$ or $\nu > \lambda_0/2$.

Consider first the case where $\mu \leq -\lambda_0/2$. As in (5.8.15) we write

$$\Re\langle\phi, \mathcal{B}_{\lambda,\alpha,\beta}\phi\rangle = \|\phi''\|_2^2 + |\mu|\beta\,[\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2] + \beta\Im\langle U'\phi, \phi'\rangle. \tag{5.12.17}$$

Consequently, using Poincaré's inequality, we obtain

$$\|\phi'\|_2 \leq \frac{C}{\lambda_0}(\beta^{-1}\|f\|_2 + \|\phi'\|_2). \tag{5.12.18}$$

For sufficiently large λ_0 and β_0 we can then conclude (5.12.1).

For $\nu > \lambda_0/2$ we write

$$\Im\langle \phi, \mathcal{B}_{\lambda,\alpha,\beta}\phi\rangle = \beta(-\langle (U-\nu)\phi', \phi'\rangle + \alpha^2\langle (U-\nu)\phi', \phi'\rangle - \Re\langle U'\phi, \phi'\rangle - \langle U''\phi, \phi\rangle). \tag{5.12.19}$$

Using Poincaré's inequality we then obtain

$$\|\phi'\|_2 \leq \frac{C}{\lambda_0 - U(0)}(\beta^{-1}\|f\|_2 + \|\phi'\|_2),$$

which validates (5.12.1) for sufficiently large λ_0 and β_0.

Step 3. We prove that there exist positive β_0, $\widetilde{\alpha}_0$, and α_0 such that (5.12.1) holds for $\widetilde{\alpha}_0\beta^{1/4} \leq \alpha \leq \alpha_0\beta^{1/3}$.

An integration by parts yields, in view of (5.4.6) (see also Step 2 of the proof of Proposition 5.11.1)

$$\langle \phi, \tilde{v}_{\mathfrak{D}}\rangle = \|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 - \phi''(1)\langle \phi, \widehat{\psi}\rangle.$$

Since by (5.2.2)

$$|\langle \phi, \widehat{\psi}\rangle| \leq \|\phi\|_\infty\|\widehat{\psi}\|_1 \leq C\beta^{-1/2}\|\phi\|_\infty,$$

we obtain

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq \|\phi\|_2\|\tilde{v}_{\mathfrak{D}}\|_2 + C\beta^{-1/2}\|\phi\|_\infty|\phi''(1)|.$$

We can now conclude that

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq C(\alpha^{-2}\|\tilde{v}_{\mathfrak{D}}\|_2^2 + \beta^{-1/2}|\phi''(1)|\,\|\phi\|_\infty), \tag{5.12.20}$$

yielding

$$\begin{aligned}\|\phi\|_\infty^2 \leq \|\phi'\|_2\|\phi\|_2 &\leq \frac{1}{2\alpha}(\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2)\\ &\leq C(\alpha^{-3}\|\tilde{v}_{\mathfrak{D}}\|_2^2 + \alpha^{-1}\beta^{-1/2}|\phi''(1)|\,\|\phi\|_\infty).\end{aligned}$$

We may then infer that

$$\|\phi\|_\infty \leq C(\alpha^{-3/2}\|\tilde{v}_{\mathfrak{D}}\|_2 + \alpha^{-1}\beta^{-1/2}|\phi''(1)|).$$

By (5.10.13) and (5.10.15) (both remain valid in the present case) we obtain

$$\|\phi\|_\infty \leq C\widetilde{\alpha}_0^{-1}(\|\phi\|_\infty + \beta^{-1/8}\|\phi'\|_2 + \beta^{-7/8}\|f\|_2).$$

For sufficiently large $\tilde{\alpha}_0$ it follows that

$$\|\phi\|_\infty \leq C\widetilde{\alpha}_0^{-1}(\beta^{-1/8}\|\phi'\|_2 + \beta^{-7/8}\|f\|_2). \tag{5.12.21}$$

Consequently, using (5.10.13) and (5.10.15) once again, it holds that

$$|\phi''(1)| + \beta^{3/8}\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C(\beta^{-1/8}\|f\|_2 + \beta^{5/8}\|\phi'\|_2). \tag{5.12.22}$$

Substituting (5.12.21) into (5.12.20) then leads to

$$\|\phi'\|_2^2 \leq C(\alpha^{-2}\|\tilde{v}_{\mathfrak{D}}\|_2^2 + \tilde{\alpha}_0^{-1}\beta^{-5/8}|\phi''(1)|\,[\|\phi'\|_2 + \beta^{-3/4}\|f\|_2]).$$

Making use of (5.12.22) we then obtain that

$$\|\phi'\|_2 \leq C\tilde{\alpha}_0^{-1/2}(\|\phi'\|_2 + \beta^{-3/4}\|f\|_2).$$

For sufficiently large $\tilde{\alpha}_0$ we can then conclude (5.12.1) under the conditions of this step.

The proposition is proved. ∎

We continue by introducing, for some positive constants α_0, Υ, β_0 and κ_2, the zone

$$\begin{aligned}&\mathcal{F}_2(\alpha_0, \beta_0, \Upsilon, \kappa_2)\\ &\quad := \{(\lambda, \alpha, \beta) \in \mathbb{C}\times\mathbb{R}_+^2, \beta \geq \beta_0, 0 \leq \alpha \leq \alpha_0\beta^{1/3}, \mu < \Upsilon\beta^{-1/2}, \nu \leq -\kappa_2\beta^{-1/3}\}.\end{aligned}$$

Proposition 5.12.2. *Let $U \in C^4([0,1])$ satisfy (2.1.3). Let further α_0 and Υ denote positive constants. Then, there exist $\beta_0 > 0$, $\kappa_2 > 0$, and $C > 0$, such that for all $(\lambda, \alpha, \beta) \in \mathcal{F}_2(\alpha_0, \beta_0, \Upsilon, \kappa_2)$ it holds*

$$\big\|(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})^{-1}\big\| + \Big\|\frac{d}{dx}(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})^{-1}\Big\| \leq C\beta^{-1}|\nu|^{-1}\log\beta. \tag{5.12.23}$$

Proof. Taking the scalar product of (5.12.2) with $w = (U-\nu)^{-1}\phi$, and integrating by parts yields for the imaginary part (see (5.12.3))

$$\begin{aligned}-\Im\langle w, f\rangle &= \Im\langle w'', \phi''\rangle + \alpha^2\Im\langle w', \phi'\rangle\\ &\quad + \beta\Big(\|(U-\nu)w'\|_2^2 + \alpha^2\|\phi\|_2^2\alpha^2\|\phi\|_2^2 - |\mu|^2\Big\langle\frac{\phi}{U-\nu}, \frac{U''\phi}{|U+i\lambda|^2}\Big\rangle\Big).\end{aligned}$$

Hence, since $(U-\nu)^{-1}U'' < 0$ we obtain that for any $\delta > 0$ there exists $C > 0$ such that

$$\begin{aligned}&\frac{1}{2}(\|(U-\nu)w'\|_2^2 + \nu^2\|w'\|_2^2) + \alpha^2\|\phi\|_2^2\\ &\leq C\big(\delta^{-1}\beta^{-2}\|f\|_2^2 + \delta\|w\|_2^2 + \alpha^2[\beta^{-4/3}\|w'\|_2^2 + \beta^{-2/3}\|\phi'\|_2^2] + \beta^{-1}|\Im\langle w'', \phi''\rangle|\big).\end{aligned}$$

Step 1. With $\lambda_0 > 0$, we prove (5.12.23) for $(\lambda, \alpha, \beta) \in \mathcal{F}_2(\alpha_0, \beta_0, \Upsilon, \kappa_2)$ satisfying $|\lambda| \leq \lambda_0$.

Step 1a. We estimate $\hat{v}_{\mathfrak{D}} = \frac{v_{\mathfrak{D}}}{U+i\lambda}$ (as defined by (5.12.12)). By (5.12.13) and (3.1.3) it holds that

$$\|\hat{v}_{\mathfrak{D}}\|_2 \leq C\,\beta^{-2/3}\,\|h\|_2, \tag{5.12.24}$$

where h is given by (5.12.13b). Hence,

$$\|h\|_2 \le C\Big(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2 + \Big\|\frac{\phi}{(U+i\lambda)^3}\Big\|_2 + \Big\|\frac{\phi'}{(U+i\lambda)^2}\Big\|_2 + \Big\|\frac{\phi''}{(U+i\lambda)}\Big\|_2\Big)$$
$$\le C\Big(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2 + |\lambda|^{-1}\|\phi''\|_2 + \Big\|\frac{\phi}{(U+i\lambda)^3}\Big\|_2 + \Big\|\frac{\phi'}{(U+i\lambda)^2}\Big\|_2\Big). \tag{5.12.25}$$

To estimate the last two terms we use a decomposition of the interval of integration. (Note that for $\nu \le -1$ we have $\|\cdot\|_{L^2(0,1-|\nu|^{1/2})} = 0$ and $\|\cdot\|_{L^2(1-|\nu|^{1/2},1)} = \|\cdot\|_2$.) In addition, we need the bounds

$$\|(U+i\lambda)^{-m}\|_{L^\infty(0,1-|\nu|^{1/2})} \le C|\nu|^{-m/2}$$

and

$$\|(U+i\lambda)^{-m}\|_{L^\infty(0,1)} \le C|\lambda|^{-m}.$$

We now write

$$\|h\|_2 \le C\Big(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2 + |\lambda|^{-1}\|\phi''\|_2 + |\lambda|^{-2}\|\phi'\|_{L^2(1-|\nu|^{1/2},1)}$$
$$+ |\nu|^{-1}\|\phi'\|_{L^2(0,1-|\nu|^{1/2})} + |\lambda|^{-2}\Big\|\frac{\phi}{U+i\lambda}\Big\|_{L^2(1-|\nu|^{1/2},1)}$$
$$+ |\nu|^{-1}\Big\|\frac{\phi}{U+i\lambda}\Big\|_{L^2(0,1-|\nu|^{1/2})}\Big).$$

By Hardy's inequality (2.2.8) it holds that

$$\Big\|\frac{\phi}{U+i\lambda}\Big\|_{L^2(0,1-|\nu|^{1/2})} \le \Big\|\frac{\phi}{U}\Big\|_{L^2(0,1-|\nu|^{1/2})}$$
$$\le C\Big\|\frac{\phi}{1-x}\Big\|_{L^2(0,1-|\nu|^{1/2})} \le \widehat{C}\,\|\phi'\|_{L^2(0,1-|\nu|^{1/2})}.$$

On the interval $(1-|\nu|^{1/2},1)$, we have again by (2.2.8)

$$\Big\|\frac{\phi}{U+i\lambda}\Big\|_{L^2(1-|\nu|^{1/2},1)} \le C\|\phi'\|_2.$$

Combining the above yields

$$\|h\|_2 \le C\big(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2 + |\lambda|^{-1}\|\phi''\|_2 + |\lambda|^{-2}\|\phi'\|_{L^2(1-|\nu|^{1/2},1)} + |\nu|^{-1}\|\phi'\|_2\big). \tag{5.12.26}$$

By (5.4.15) and (5.4.19) it holds that

$$|\phi''(1)|\,\|\hat{g}\|_2 \le C|\lambda|^{-3/4}(\beta^{1/4}\|\phi'\|_2 + \beta^{-7/12}\|f\|_2). \tag{5.12.27}$$

Substituting (5.12.26) and (5.12.27) into (5.12.25) we obtain

$$\|h\|_2 \leq C\big(\|f\|_2 + |\lambda|^{-1}\|\phi''\|_2 + [|\nu|^{-1} + \beta^{1/4}|\lambda|^{-3/4}]\|\phi'\|_2 + |\lambda|^{-2}\|\phi'\|_{L^2(1-|\nu|^{1/2},1)}\big). \quad (5.12.28)$$

To bound $\|\phi''\|_2$ we use (5.10.54)–(5.10.55) and (5.12.16) to obtain

$$\|\phi''\|_2 \leq \|\phi'' - \alpha^2\phi\|_2 \leq \|\hat{v}_{\mathfrak{D}}\|_2 + \|U''\|_\infty \Big\|\frac{\phi}{U + i\lambda}\Big\|_2 + |\phi''(1)|\,\|\widehat{\psi}\|_2.$$

By (4.6.5), (5.12.24), (5.12.28), (2.2.8), and (5.4.15) (repeatedly using the lower bound $|\nu| \geq \kappa_2\beta^{-1/3}$) it holds that

$$\|\phi''\|_2 \leq C(|\lambda|^{1/4}\beta^{-7/12}\|f\|_2 + \beta^{-2/3}|\lambda|^{-1}\|\phi''\|_2 + |\lambda\beta|^{1/4}\|\phi'\|_2).$$

For sufficiently large β_0 we then obtain

$$\|\phi''\|_2 \leq C(|\lambda|^{1/4}\beta^{-7/12}\|f\|_2 + |\lambda\beta|^{1/4}\|\phi'\|_2). \quad (5.12.29)$$

Substituting (5.12.29) into (5.12.28) yields

$$\|h\|_2 \leq C\big(\|f\|_2 + [|\nu|^{-1} + \beta^{1/4}|\lambda|^{-3/4}]\|\phi'\|_2 + |\lambda|^{-2}\|\phi'\|_{L^2(1-|\nu|^{1/2},1)}\big). \quad (5.12.30)$$

To use (2.7.1b) we must provide an estimate for

$$N(v_{\mathfrak{D}}, \lambda) = \|(1-x)^{1/2}(U + i\lambda)^{-1}v_{\mathfrak{D}}\|_1 = \|(1-x)^{1/2}\hat{v}_{\mathfrak{D}}\|_1.$$

Thus, we use (3.1.75) and (5.12.13) to obtain

$$\|(1-x)^{1/2}\hat{v}_{\mathfrak{D}}\|_1 \leq C\|(U-\nu)^{1/2}\hat{v}_{\mathfrak{D}}\|_1 \leq C\,\|(U-\nu)^{-1/2}\|_2\,\|(U-\nu)\hat{v}_{\mathfrak{D}}\|_2 \leq \hat{C}\,\frac{\log\beta}{\beta}\|h\|_2.$$

Hence, we may conclude from (5.12.30) that

$$\|(1-x)^{1/2}\hat{v}_{\mathfrak{D}}\|_1 \leq C\,\frac{\log\beta}{\beta}\big(\|f\|_2 + [|\nu|^{-1} + \beta^{1/4}|\lambda|^{-3/4}]\|\phi'\|_2 + |\lambda|^{-2}\|\phi'\|_{L^2(1-|\nu|^{1/2},1)}\big). \quad (5.12.31)$$

Step 1b: We prove (5.12.23). As in (5.4.31), we let $\phi = \phi_{\mathfrak{D}} + \check{\phi}$, where

$$\phi_{\mathfrak{D}} = \mathcal{A}_{\lambda,\alpha}^{-1}([U + i\lambda]\hat{v}_{\mathfrak{D}}) = \mathcal{A}_{\lambda,\alpha}^{-1}v_{\mathfrak{D}}$$

and

$$\check{\phi} = \mathcal{A}_{\lambda,\alpha}^{-1}(\phi''(1)[U + i\lambda]\widehat{\psi}).$$

By (2.7.2) applied to the pair $(\check{\phi}, \phi''(1)[U + i\lambda]\widehat{\psi})$ it holds that

$$|\check{\phi}(x)| \leq C(1-x)^{1/2}[1 + \nu^{-1/2}(1-x)^{1/2}]|\phi''(1)\langle\check{\phi}, \widehat{\psi}\rangle|^{1/2}.$$

Hence, integrating over (0, 1),

$$|\langle\check{\phi}, \widehat{\psi}\rangle| \leq C(\|(1-x)^{1/2}\widehat{\psi}\|_1 + |\nu|^{-1/2}\|(1-x)\widehat{\psi}\|_1)|\langle\check{\phi}, \widehat{\psi}\rangle|^{1/2}|\phi''(1)|^{1/2},$$

which implies

$$|\langle\check{\phi}, \widehat{\psi}\rangle| \leq C(\|(1-x)^{1/2}\widehat{\psi}\|_1 + |\nu|^{-1/2}\|(1-x)\widehat{\psi}\|_1)^2|\phi''(1)|.$$

By (5.2.2) and (5.4.15) we then obtain

$$|\langle\check{\phi}, \widehat{\psi}\rangle| \leq C\,|\lambda\beta|^{-1}(\|\phi'\|_2 + \beta^{-5/6}\|f\|_2). \tag{5.12.32}$$

Using (2.7.3) applied to the pair $(\check{\phi}, \phi''(1)[U+i\lambda]\widehat{\psi})$, together with (2.7.6), (5.12.32), and (5.4.15) yields

$$\|\check{\phi}'\|_2^2 \leq C\,|\nu|^{-1}|\lambda|^{-1/2}\beta^{-1/2}(\|\phi'\|_2 + \beta^{-5/6}\|f\|_2)^2.$$

For sufficiently large κ_2 (and $(\lambda, \alpha, \beta) \in \mathcal{F}_2(\alpha_0, \beta_0, \Upsilon, \kappa_2)$) we then conclude that

$$\|\check{\phi}'\|_2 \leq C\,|\nu|^{-3/4}\beta^{-1/4}(\|\phi_{\mathfrak{D}}'\|_2 + \beta^{-5/6}\|f\|_2). \tag{5.12.33}$$

By (2.7.8) and (2.7.3), applied again to the pair $(\check{\phi}, \phi''(1)[U + i\lambda]\widehat{\psi})$, we then obtain that

$$\|\check{\phi}'\|^2_{L^2(1-|\nu|^{1/2},1)} \leq C\,|\nu|^{-1/2}\,|\phi''(1)|\,|\langle\check{\phi}, \widehat{\psi}\rangle|.$$

Using (5.12.32), we then conclude that

$$\|\check{\phi}'\|_{L^2(1-|\nu|^{1/2},1)} \leq C\,|\nu|^{-1/2}\beta^{-1/4}(\|\phi_{\mathfrak{D}}'\|_2 + \beta^{-5/6}\|f\|_2). \tag{5.12.34}$$

By (2.7.1a) applied to the pair $(\phi_{\mathfrak{D}}, (U+i\lambda)\hat{v}_{\mathfrak{D}})$, (5.12.31), (5.12.33), and (5.12.34), it holds, as $\phi = \phi_{\mathfrak{D}} + \check{\phi}$, that

$$\begin{aligned}\|\phi_{\mathfrak{D}}\|_{1,2} &\leq C|\nu|^{-1}\|(1-x)^{1/2}\hat{v}_{\mathfrak{D}}\|_1 \\ &\leq \hat{C}\frac{\log\beta}{\beta}|\nu|^{-1}\big(\|f\|_2 + [|\nu|^{-1} + \beta^{1/4}|\nu|^{-3/4} + |\nu|^{-5/2}\beta^{-1/4}]\|\phi_{\mathfrak{D}}'\|_2 \\ &\qquad + |\lambda|^{-2}\|\phi_{\mathfrak{D}}'\|_{L^2(1-|\nu|^{1/2},1)}\big).\end{aligned} \tag{5.12.35}$$

For sufficiently large β_0 we obtain from (5.12.35)

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\frac{\log\beta}{\beta}|\nu|^{-1}\big(\|f\|_2 + |\lambda|^{-2}\|\phi_{\mathfrak{D}}'\|_{L^2(1-|\nu|^{1/2},1)}\big). \tag{5.12.36}$$

For $\nu < -1/2$, we immediately obtain from (5.12.36) that for sufficiently large β_0

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\frac{\log\beta}{\beta}|\nu|^{-1}\|f\|_2. \tag{5.12.37}$$

For $\nu \geq -1/2$, we substitute (5.12.36) into (5.12.31) to conclude, with the aid of (5.12.33) and (5.12.34)

$$\|(1-x)^{1/2}\hat{v}_{\mathfrak{D}}\|_1 \leq C\frac{\log\beta}{\beta}\big(\|f\|_2 + |\lambda|^{-2}\|\phi'_{\mathfrak{D}}\|_{L^2(1-|\nu|^{1/2},1)}\big).$$

We can now use (2.7.1b) applied to the pair $(\phi_{\mathfrak{D}}, (U + i\lambda)\hat{v}_{\mathfrak{D}})$ to obtain

$$\|\phi'_{\mathfrak{D}}\|_{L^2(1-|\nu|^{1/2},1)} \leq C\frac{\log\beta}{\beta}|\nu|^{-3/4}\big(\|f\|_2 + |\lambda|^{-2}\|\phi'_{\mathfrak{D}}\|_{L^2(1-|\nu|^{1/2},1)}\big).$$

For sufficiently large β_0 we obtain that

$$\|\phi'_{\mathfrak{D}}\|_{L^2(1-|\nu|^{1/2},1)} \leq C\frac{\log\beta}{\beta}|\nu|^{-3/4}\|f\|_2,$$

which, when substituted into (5.12.36), yields in the case $-1/2 \leq \nu$

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\frac{\log\beta}{\beta}|\nu|^{-1}\,\|f\|_2.$$

The above inequality, together with inequalities (5.12.33), (5.12.37) yields (5.12.23) for $|\lambda| < \lambda_0$.

Step 2. We prove that there exists $\lambda_0 > 0$ such that (5.12.23) holds true for any $(\lambda, \alpha, \beta) \in \mathcal{F}_2(\alpha_0, \beta_0, \Upsilon, \kappa_2)$ satisfying $|\lambda| > \lambda_0$.

The proof is almost identical with Step 2 of Proposition 5.12.1. If $\mu < -\lambda_0/2$ we obtain (5.12.18) from (5.12.17) and then (5.12.23) for sufficiently large λ_0. If $\nu < -\lambda_0/2$ we use (5.12.19) to obtain (5.12.18) once again. ∎

Remark 5.12.3. Note that there exists $\mu_1 > 0$ such that for all $\mu < -\mu_1\beta^{-1/3}$ equation (5.12.33) remains valid even in the case where κ_2 is not necessarily large. Thus, we may conclude that under the conditions of Proposition 5.12.2 for all $\kappa_2 > 0$, there exist $\beta_0 > 0$, $\mu_1 > 0$, and $C > 0$, such that for all $(\lambda, \alpha, \beta) \in \mathcal{F}_2(\alpha_0, \beta_0, \Upsilon, \kappa_2)$ satisfying $\mu < -\mu_1\beta^{-1/3}$ (5.12.23) holds true.

Chapter 6

Proof of the main theorems

6.1 Proofs of Theorems 1.1.1 and 1.1.2

The proofs of Theorems 1.1.1 and 1.1.2 rely on a combination of the relevant results in Chapter 5.

Proof of Theorem 1.1.1. We present the proof in the following table, which gives the precise range of parameters where each estimate is valid together with the estimate itself.

α	ν	μ	Stated in	Estimate
$\alpha \lesssim \beta^{1/3}$	$-\nu \gg \beta^{-1/3}$	$\mu \le \beta^{-1/2}$	Prop. 5.12.2	$\beta^{-1}\lvert\nu\rvert^{-1}\log\beta$
$\alpha \ll \beta^{-1/6}$	$\lvert\nu\rvert \le \nu_0$	$\mu < \beta^{-1/2}$	Prop. 5.8.2	$\beta^{-2/3}$
$\alpha \lesssim \beta^{1/3}$	$\beta^{-1/2} \ll U(0)-\nu$, $\nu \ge \nu_0$	$\mu \ll \beta^{-1/2}$	Prop. 5.11.1	$\beta^{-1/2}\log\beta$
$\alpha \lesssim \beta^{1/3}$	$\lvert U(0)-\nu\rvert \lesssim \beta^{-1/2}$	$\mu \ll \beta^{-1/2}$	Prop. 5.10.1	$\min(\beta^{-5/8}\lvert\mu\rvert^{-1/4}, \beta^{-3/8})$
$\alpha \lesssim \beta^{1/3}$	$\nu - U(0) \gg \beta^{-1/2}$	$\mu \lesssim \beta^{-1/2}$	Prop. 5.12.1	$\beta^{-1/2}$
$\alpha \gtrsim \beta^{1/3}$	$\nu \in \mathbb{R}$	$\mu \le \mu^*$	Prop. 5.7.1	$\beta^{-1/2}$

In the above, $\mu^* = \min(\Upsilon\beta^{-1/2}, [\mu_m - \hat{\Upsilon}]\beta^{-1/3} - \alpha^2\beta^{-1}/2)$ for some sufficiently small $\Upsilon > 0$ and any $\hat{\Upsilon} > 0$.

From the table we learn that there exist positive α_L, β_0 and Υ such that for all $\beta > \beta_0$,

$$\sup_{\substack{0\le\alpha\le\alpha_L\beta^{-1/6}\\ \Re\lambda\le\Upsilon\beta^{-1/2}}} \left\|(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{D},\mathrm{sym}})^{-1}\right\| + \left\|\frac{d}{dx}(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{D},\mathrm{sym}})^{-1}\right\| \le C\beta^{-3/8}. \tag{6.1.1}$$

Furthermore, for $\mu = \Upsilon\beta^{-1/2}$ it holds for all $\beta > \beta_0$ that

$$\sup_{\substack{0\le\alpha\le\alpha_L\beta^{-1/6}\\ \Re\lambda=\Upsilon\beta^{-1/2}}} \left\|(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{D},\mathrm{sym}})^{-1}\right\| + \left\|\frac{d}{dx}(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{D},\mathrm{sym}})^{-1}\right\| \le C\beta^{-1/2}\log\beta. \tag{6.1.2}$$

By (6.1.1) $\mathcal{B}_{\lambda,\alpha}^{\mathfrak{D},\mathrm{sym}}$ depends holomorphically on λ for all $\mu \le \Upsilon\beta^{-1/2}$, and hence we can use (6.1.2) together with the Phragmén–Lindelöf theorem to obtain (1.1.12). ∎

Proof of Theorem 1.1.2. As in the proof of Theorem 1.1.1, we use the following table, which gives the precise range of parameters where each estimate is valid together with the estimate itself.

α	ν	μ	Stated in	Estimate
$\alpha \lesssim \beta^{1/3}$	$-\nu \gg \beta^{-1/3}$	$\mu \le \beta^{-1/2}$	Prop. 5.12.2	$\beta^{-1}\|\nu\|^{-1}\log\beta$
$\alpha \lesssim 1$	$\beta^{-1/5+\delta} \le \nu < \nu_0$	$\mu < \beta^{-2/5-\delta}$	Prop. 5.4.1	$\beta^{-1/2+\delta}$
$1 \ll \alpha \ll \beta^{1/3}$	$\|\nu\| < \nu_0$	$\mu \ll \beta^{-1/3}$	Prop. 5.6.1	$\beta^{-5/6}$
$\beta^{-1/10+\delta} \le \alpha \lesssim 1$	$\|\nu\| \le \beta^{-1/5+\delta}$	$\mu < \beta^{-1/3-\delta}$	Prop. 5.5.1	$\beta^{-1/2+\delta}$
$\alpha \lesssim \beta^{1/3}$	$\beta^{-1/2} \ll U(0)-\nu$, $\nu \ge \nu_0$	$\mu \ll \beta^{-1/2}$	Prop. 5.11.1	$\beta^{-1/2}\log\beta$
$\alpha \lesssim \beta^{1/3}$	$\|U(0)-\nu\| \lesssim \beta^{-1/2}$	$\mu \ll \beta^{-1/2}$	Prop. 5.10.1	$\min\left(\beta^{-5/8}\|\mu\|^{-1/4}, \beta^{-3/8}\right)$
$\alpha \lesssim \beta^{1/3}$	$\nu - U(0) \gg \beta^{-1/2}$	$\mu \lesssim \beta^{-1/2}$	Prop. 5.12.1	$\beta^{-1/2}$
$\alpha \gtrsim \beta^{1/3}$	$\nu \in \mathbb{R}$	$\mu \lesssim \beta^{-1/2}$	Prop. 5.7.1	$\beta^{-1/2}$

From the table we learn that there exist positive β_0 and Υ such that for all $\beta > \beta_0$,

$$\sup_{\substack{\beta^{-1/10+\delta}\le\alpha \\ \Re\lambda\le\Upsilon\beta^{-1/2}}} \left\|(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\right\| + \left\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\right\| \le C\beta^{-3/8}. \tag{6.1.3}$$

Furthermore, for $\mu = \Upsilon\beta^{-1/2}$ it holds for any $\delta > 0$ and $\beta > \beta_0$ that

$$\sup_{\substack{\beta^{-1/10+\delta}\le\alpha \\ \Re\lambda=\Upsilon\beta^{-1/2}}} \left\|(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\right\| + \left\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\right\| \le C\beta^{-1/2+\delta}. \tag{6.1.4}$$

By (6.1.3) $B^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta}$ depends holomorphically on λ for all $\mu \le \Upsilon\beta^{-1/2}$, and hence we can use (6.1.4) together with the Phragmén–Lindelöf theorem to obtain (1.1.13). ■

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On the Stability of Symmetric Flows in a Two-Dimensional Channel

We consider the stability of symmetric flows in a two-dimensional channel (including the Poiseuille flow). In 2015 Grenier, Guo, and Nguyen have established instability of these flows in a particular region of the parameter space, affirming formal asymptotics results from the 1940's. We prove that these flows are stable outside this region in parameter space. More precisely we show that the Orr–Sommerfeld operator

$$\mathcal{B} = \Big(-\frac{d^2}{dx^2} + i\beta(U + i\lambda)\Big)\Big(\frac{d^2}{dx^2} - \alpha^2\Big) - i\beta U'' ,$$

which is defined on

$$D(\mathcal{B}) = \{u \in H^4(0,1)\,,\ u'(0) = u^{(3)}(0) = 0 \text{ and } u(1) = u'(1) = 0\}$$

is bounded on the half-plane $\Re\lambda \geq 0$ for $\alpha \gg \beta^{-1/10}$ or $\alpha \ll \beta^{-1/6}$.

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